DERIVED CATEGORIES OF CUBIC AND V_{14} THREEFOLDS

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In memory of Andrei Nikolaevich Tyurin

1. Introduction

This paper is devoted to the description of several aspects of a relation of the following two families of Fano threefolds. The first is the family of cubic threefolds, smooth hypersurfaces of degree 3 in $\mathbb{P}^4$. The second, is the family of V_{14} Fano threefolds, smooth complete intersections $\mathbb{P}^9 \cap \mathsf{Gr}(2,6) \subset \mathbb{P}^{14}$.

The fact that geometry of Fano threefolds from these two families is related was known for a long time. The history of the question goes back to Fano himself, who found a birational isomorphism from a V_{14} threefold to a cubic threefold [Fa, Is]. Another birational isomorphism was found later on by Tregub and Takeuchi [Tr, Ta].

The paper [IM] has brought a new character into the story, an instanton bundle on a cubic threefold. An instanton bundle on a cubic threefold Y is a rank 2 stable vector bundle $\mathcal{E}$ such that $c_1(\mathcal{E}) = 0$ and $H^1(Y, \mathcal{E}(-1)) = 0$. A topological charge of $\mathcal{E}$ is defined as the second Chern class, $c_2(\mathcal{E}) \in H^4(Y, \mathbb{Z}) \cong \mathbb{Z}$. It was shown in [IM] that for any V_{14} threefold X there exists a unique cubic threefold Y birational to X and that for generic Y the set of X birational to Y is isomorphic to an open subset of the moduli space $M_0(Y)$ of instanton bundles on Y of topological charge 2. Moreover, the association $X \mapsto (Y, \mathcal{E})$ is constructed rather explicit.

The goal of the present paper is to show how the above relation is reflected on the level of the derived categories. We start however with a more accurate treatment of geometry. First of all, we remove some genericity coniditions having been imposed in [IM] and show that the map $X \mapsto (Y, \mathcal{E})$ is actually an isomorphism of moduli stacks. Further, we show that if $(Y, \mathcal{E})$ is the pair, corresponding to X, then we have the following diagram:

$$\begin{array}{ccccc} \mathbb{P}_Y(\mathcal{E}) & & \overset{\theta}{\dashrightarrow} & & \mathbb{P}_X(\mathcal{U}) \\ {\scriptstyle p_Y}\downarrow & \searrow^{\psi} & & {}^{\phi}\swarrow & \downarrow{\scriptstyle p_X} \\ Y & & Q & & X \end{array} \qquad (*)$$

where $\mathcal{U}$ is the restriction of the tautological rank 2 subbundle from the Grassmannian $\mathsf{Gr}(2,6)$ to $X \subset \mathsf{Gr}(2,6)$; p_Y and p_X are the projectivizations of bundles $\mathcal{E}$ and $\mathcal{U}$ over Y and X respectively; ψ and ϕ are small birational contractions onto a singular quartic hypersurface $Q \subset \mathbb{P}^5$; and $\theta = \phi^{-1} \cdot \psi$ is a flop. The bundle $\mathcal{U}$ on X is an exceptional bundle. In other words, the projectivization of the exceptional bundle on a V_{14} threefold after some natural flop turns into the projectivization of an instanton bundle on a cubic threefold.

A very similar picture was found in [K] in another situation. It was shown there that the projectivization of the exceptional bundle on a V_{22} Fano threefold after a very similar flop turns into the projectivization of an instanton bundle on the projective space $\mathbb{P}^3$. We guess that pictures of this sort should exist for a lot of another pairs of Fano manifolds and that they are of ultimate importance both for the geometry of involved manifolds, and for understanding of Fano manifolds in general.

In the second part of the paper we turn our attention to the derived categories of coherent sheaves on Y and X, $\mathcal{D}^b(Y)$ and $\mathcal{D}^b(X)$ respectively. We show that these categories have a similar structure. First of all, both $\mathcal{D}^b(Y)$ and $\mathcal{D}^b(X)$ contain an exceptional pair of vector bundles. Explicitly, the pair $(\mathcal{O}_Y, \mathcal{O}_Y(1))$ in $\mathcal{D}^b(Y)$, and the pair $(\mathcal{O}_X, \mathcal{U}^*)$ in $\mathcal{D}^b(X)$. As usually in such a situation we obtain semiorthogonal decompositions

$$\mathcal{D}^b(Y) = \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{A}_Y \rangle, \qquad \mathcal{D}^b(X) = \langle \mathcal{O}_X, \mathcal{U}^*, \mathcal{A}_X \rangle,$$

where $\mathcal{A}_Y$ (resp. $\mathcal{A}_X$) is the left orthogonal to the exceptional pair in $\mathcal{D}^b(Y)$ (resp. $\mathcal{D}^b(X)$). In fact, we use slightly another decomposition of $\mathcal{D}^b(Y)$, however this change affects only the embedding functor of $\mathcal{A}_Y$ into $\mathcal{D}^b(Y)$ and doesn't affect the intrinsic structure of $\mathcal{A}_Y$. Now assume that Y is the cubic threefold corresponding to a V_{14} threefold X as above. Then we prove that the categories $\mathcal{A}_Y$ and $\mathcal{A}_X$ are equivalent as triangulated categories. This is the main result of the paper. The functor, giving the equivalence is constructed explicitly (see 12), using diagram $(*)$.

One of implications of the equivalence is the following. Since all V_{14} threefolds contained within a fixed birational class correspond to the same cubic threefold Y it follows that the categories $\mathcal{A}_{X_1}$ and $\mathcal{A}_{X_2}$ are equivalent if X_1 and X_2 are birational. Thus $\mathcal{A}_X$ turns into a *birational invariant* of X. In fact, we conjecture that $\mathcal{A}_X$ allows to distinguish the birational type of X, or equivalently, that $\mathcal{A}_Y$ allows to distinguish the isomorphism class of Y. To give some justification we construct a family of objects in $\mathcal{A}_Y$ parameterized by the Fano surface of lines on Y. If one would be able to describe such a family in intrinsic terms of the category $\mathcal{A}_Y$ (e.g. as a moduli space), then it would be possible to reconstruct the intermediate Jacobian of Y (as the Albanese variety of the Fano surface) from $\mathcal{A}_Y$, and hence, due to the Torelli theorem [CG, T], the isomorphism class of Y. The main difficulty arising on this way is a construction of a "stability data" on the category $\mathcal{A}_Y$ [Br1], with respect to which one could consider a moduli space.

We would like to emphacise that the above results can be considered as a first step to the construction of birational invariants of algebraic varieties from their derived categories. We hope this approach might prove useful when dealing with the problem of rationality of a cubic fourfold.

The paper is organised as follows. In section 2 we introduce a definition of the Pfaffian cubic Y and of the theta-bundle E, corresponding to a V_{14} threefold X and state a theorem on a reconstruction of X from Y and E, which is proved in Appendix A in a greater generality. After that we introduce instanton bundles on Y and show that E is a theta-bundle iff $E(-1)$ is an instanton of charge 2. After that we consider the projectivizations $\mathbb{P}_X(\mathcal{U})$ and $\mathbb{P}_Y(E^*)$, construct their cointractions $\phi : \mathbb{P}_X(\mathcal{U}) \to Q \leftarrow \mathbb{P}_Y(E^*) : \psi$ onto a common (singular) quartic hypersurface $Q \subset \mathbb{P}^5$, da Palatini quartic, and check that $\theta = \phi^{-1} \circ \psi$ is a flop. In conclusion we prove some technical results concerning the fiber product $W = \mathbb{P}_Y(E^*) \times_Q \mathbb{P}_X(\mathcal{U})$.

We start section 3 with reminding some definitions and important properties of semiorthogonal decompositions, mutations, kernel functors, etc. We state also a reformulation of a result of Bridgeland on flops, which we will need afterwards. The remaining part of the section is devoted to the proof of the main theorem, saying that the categories $\mathcal{A}_X$ and $\mathcal{A}_Y$ are equivalent.

In section 4 we discuss some properties of the category $\mathcal{A}_Y$. First of all, we show that for any Fano hypersurface Y is a projective space a power of the Serre functor $\mathsf{S}_{\mathcal{A}_Y}$ of the category $\mathcal{A}_Y$, defined as the orthogonal to the maximal exceptional collection formed by bundles $\mathcal{O}_Y(i)$ is isomorphic to a shift functor. In particular case of a three-dimensional cubic we get $\mathsf{S}^3_{\mathcal{A}_Y} \cong [5]$. It follows that $\mathcal{A}_Y$ is not equivalent to the derived category of a variety. Further, we give some examples of objects in $\mathcal{A}_Y$ and among them an example of a family of objects, parameterized by the Fano surface of line on Y.

In Appendix A we give a general definition of a Pfaffian hypersurface and of a theta-bundle and describe some of their properties. In Appendix B we give a definition of instanton bundles on Fano threefolds of index 2 and compute cohomology groups of their twists.

Notation. We assume the base field k to be an algebraically closed field of characteristic 0. We will use the following notation:

- $V = \mathsf{k}^6$;
- $A = \mathsf{k}^5$;
- $f \in \mathsf{Hom}(A, \Lambda^2 V^*)$ is an A-net of skew-forms on V;
- $X = X_f = \mathbb{P}(f(A)^\perp) \cap \mathsf{Gr}(2, V)$ is a smooth V_{14} Fano threefold;
- $Y = Y_f = \{\mathsf{Pf}(f(a)) = 0\} \subset \mathbb{P}(A)$, the Pfaffian cubic threefold;
- $\alpha : Y \to \mathbb{P}(A)$ is the embedding;
- $E = E_f$ is the theta-bundle on Y;
- $\mathcal{E}$ is an instanton of charge 2 on Y, $\mathcal{E} = E(-1)$;
- $\mathcal{U}$ is a restriction of the tautological vector bundle from $\mathsf{Gr}(2, V)$ to X;
- $p_X : \mathbb{P}_X(\mathcal{U}) \to X$ is the projectivization of $\mathcal{U}$ on X;
- $p_Y : \mathbb{P}_Y(E^*) \to Y$ is the projectivization of E^* on Y;
- $\phi : \mathbb{P}_X(\mathcal{U}) \to \mathbb{P}(V)$ is the map, induced by embedding $\mathbb{P}_X(\mathcal{U}) \subset \mathsf{Fl}(1, 2; V)$;
- $\psi : \mathbb{P}_Y(E^*) \to \mathbb{P}(V)$ is the map, induced by embedding $\mathbb{P}_Y(E^*) \subset \mathsf{Fl}(1, 2; V)$;
- $Q = \phi(\mathbb{P}_X(\mathcal{U})) = \psi(\mathbb{P}_Y(E^*)) \subset \mathbb{P}(V)$ is the quartic da Palatini;
- $C = \mathsf{sing}(Q)$ is a curve, $\deg C = 25$, $p_a(C) = 26$;
- $S_X \subset \mathbb{P}_X(\mathcal{U})$ is a ruled surface, contracted by ϕ to C;
- $S_Y \subset \mathbb{P}_Y(E^*)$ is a ruled surface, contracted by ψ to C;
- $\theta : \mathbb{P}_Y(E^*) \to \mathbb{P}_X(\mathcal{U})$ is the flop in S_Y, $\theta = \phi^{-1} \circ \psi$;
- $W = \mathbb{P}_Y(E^*) \times_Q \mathbb{P}_X(\mathcal{U})$ is the fiber product;
- $\eta : W \to \mathbb{P}_Y(E^*)$, $\xi : W \to \mathbb{P}_X(\mathcal{U})$ and $\chi : W \to Q$ are the projections;
- $i : W \to \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U})$, $j : W \to Y \times X$ and $\lambda : W \to \mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U})$ are the embeddings.

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2. Geometry

Consider a five-dimensional vector space $A = \mathsf{k}^5$, a six-dimensional vector space $V = \mathsf{k}^6$, and a linear map $f : A \to \Lambda^2 V^*$. Such map will be called an *A-net of skew-forms on V*.

Pfaffian cubic and theta-bundle. For any such f let $f(A)^\perp \subset \Lambda^2 V$ denote the annihilator of $f(A) \subset \Lambda^2 V^*$. Denote also $X = X_f = \mathbb{P}(f(A)^\perp) \cap \mathsf{Gr}(2, V) \subset \mathbb{P}(\Lambda^2 V)$. When f is generic X is a smooth Fano threefold of index 1 with $\mathsf{Pic}\, X = \mathbb{Z}$ and of genus 8. Such threefolds are known as *V_{14} Fano threefolds* [Is1, Is1]. Moreover, any V_{14} threefold can be realized as X_f for some f [Mu].

An A-net f is called *regular* if $\mathsf{rank} f(a) \geq 4$ for any $0 \neq a \in A$.

Lemma 2.1. *If $X = X_f \subset \mathsf{Gr}(2, V)$ is a smooth V_{14} threefold then the A-net f is regular.*

Proof: Assume that the A-net $f : A \to \Lambda^2 V^*$ isn't regular. Then the rank of a skew-form $f(a) \in \Lambda^2 V^*$ is less or equal than 2 for some $0 \neq a \in A$. Let $K_a = \operatorname{Ker} f(a) \subset V$ be the kernel of this form. Then $\dim K_a \geq 4$ and the Grassmannian $\mathsf{Gr}(2, K_a) \subset \mathsf{Gr}(2, V)$ has nonempty intersection with X, because $X \cap \mathsf{Gr}(2, K_a)$ is a plane section of $\mathsf{Gr}(2, K_a)$ of codimension ≤ 4, and $\dim \mathsf{Gr}(2, K_a) \geq 4$. But it is easy to check that any point in $X \cap \mathsf{Gr}(2, K_a)$ is singular in X (see the proof of proposition A.4). □

Any A-net f can be considered as an element of $\mathsf{Hom}(V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1), V^* \otimes \mathcal{O}_{\mathbb{P}(A)})$, the space of homomorphisms of coherent sheaves on $\mathbb{P}(A)$. If f is regular then this homomorphism is injective, and its cokernel E_f is a sheaf supported on a cubic hypersurface $Y_f \subset \mathbb{P}(A)$ with equation $\mathsf{Pf}(f(a)) = 0$ (where Pf stands for the Pfaffian of a skew-form), the *Pfaffian cubic* of f. Thus we have an exact sequence of coherent sheaves on $\mathbb{P}(A)$:

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{f} V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to \alpha_* E_f \to 0, \tag{1}$$

where $\alpha : Y_f \to \mathbb{P}(A)$ is the embedding. We call E_f *the theta-bundle* of the A-net f (see Appendix A). The map $V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to \alpha_* E_f$ induces an isomorphism $\gamma_f : V^* = H^0(\mathbb{P}(A), V^* \otimes \mathcal{O}_{\mathbb{P}(A)}) \to H^0(\mathbb{P}(A), \alpha_* E_f) = H^0(Y_f, E_f)$.

Theorem 2.2. *Associating to an A-net f the triple (Y_f, E_f, γ_f) gives a $\mathsf{GL}(A) \times \mathsf{GL}(V)$-equivariant isomorphism between*

- *the subset of $\mathbb{P}(A^* \otimes \Lambda^2 V^*)$ formed by all regular A-nets of skew-forms on V, and*
- *the set of triples (Y, E, γ), where Y is a cubic hypersurface in $\mathbb{P}(A)$, E is a rank 2 locally free sheaf on Y, and γ is an isomorphism $V^* \to H^0(Y, E)$, such that*

$$c_1(E) = 2[h], \quad c_2(E) = 5[l], \quad \text{and} \quad H^{\cdot}(Y, E(t)) = 0 \quad \text{for } -3 \leq t \leq -1, \tag{2}$$

  where $[h] \in H^2(Y, \mathbb{Z})$ and $[l] \in H^4(Y, \mathbb{Z})$ are the classes of a hyperplane section and of a line respectively.

Further, the theta-bundle E_f of a regular A-net is generated by global sections, $H^0(Y_f, E_f) = V^*$, and induces an embedding $\kappa : Y_f \to \mathsf{Gr}(2, V)$. Finally, $\mathsf{sing}(X_f) = \mathsf{sing}(Y_f) = X_f \cap Y_f \subset \mathsf{Gr}(2, V)$. In particular, Y_f is smooth iff X_f is smooth.

The major part of this theorem is proved in [MT, IM, Beau, Dr] in aslightly more restrictive assumptions. Only the last statement seems to be new. We give a complete proof in Appendix A.

Remark 2.3. It is easy to check that $H^0(X_f, \mathcal{O}_{X_f}(1)) \cong \Lambda^2 V^* / f(A)$. It follows that the A-net f can be reconstructed from X_f up to the action of $\mathsf{GL}(A) \times \mathsf{GL}(V)$, the action of $\mathsf{GL}(V)$ corresponds to a choice of embedding $X_f \to \mathsf{Gr}(2, V)$, and the action of $\mathsf{GL}(A)$ corresponds to a choice of isomorphism $A \to \mathsf{Ker}(H^0(\mathsf{Gr}(2, V), \mathcal{O}_{\mathsf{Gr}(2,V)}(1)) \to H^0(X_f, \mathcal{O}_{X_f}(1)))$.

If X is a smooth V_{14} Fano threefold and f is an A-net of skew-forms on V, such that $X \cong X_f$, then by remark 2.3 and theorem 2.2, the pair (Y_f, E_f) is determined by X up to an isomorphism. We will say that Y_f is the Pfaffian cubic of X and E_f is the corresponding theta-bundle.

Instantons.

Definition 2.4. A sheaf $\mathcal{E}$ on a cubic threefold $Y \subset \mathbb{P}^4$ is an *instanton bundle* if $\mathcal{E}$ is locally free of rank 2, stable (with respect to $\mathcal{O}_Y(1)$) and $c_1(\mathcal{E}) = 0$, $H^1(Y, \mathcal{E}(-1)) = 0$. The *topological charge* of an instanton $\mathcal{E}$ is an integer k, such that $c_2(\mathcal{E}) = k[l]$, where $[l] \in H^4(Y, \mathbb{Z})$ is the class of a line.

This definition is a straightforward generalization of the definition of (mathematical) instanton vector bundle on $\mathbb{P}^3$ [OSS] and admits further generalization to any Fano threefold of index 2.

We introduce such definition and deduce simplest implications in Appendix B. It is shown, in particular, that the smallest possible charge for the instantons on Y is 2, and

Proposition 2.5. *If $\mathcal{E}$ is an instanton vector bundle of charge 2 on Y then*

$$H^p(Y,\mathcal{E}(t)) = \begin{cases} \mathsf{k}^6, & \text{for } (p,t)=(0,1) \text{ and } (p,t)=(3,-3) \\ 0, & \text{for other } (p,t) \text{ with } -3 \le t \le 1 \end{cases}$$

Consider the Gieseker–Maruyama moduli space $M_Y(2;0,2)$ of semistable (with respect to $\mathcal{O}_Y(1)$) rank 2 torsion free sheaves on Y with Chern classes $c_1 = 0$ and $c_2 = 2[l]$ and its Zariski open subset (cf. [MT])

$$\begin{aligned} M_0(Y) = \{[\mathcal{E}] \in M_Y(2;0,2) \mid & (i)\ \mathcal{E} \text{ is } \textit{stable} \text{ and } \textit{locally free}; \\ & (ii)\ H^1(Y,\mathcal{E}(-1)) = H^1(Y,\mathcal{E}(1)) = H^2(Y,\mathcal{E}(1)) = H^2(Y,\mathcal{E}\otimes\mathcal{E}) = 0\}. \end{aligned} \quad (3)$$

Proposition 2.6. *The following conditions are equivalent:*

(i)	$\mathcal{E}$ *is an instanton bundle of charge* 2;
(ii)	$\mathcal{E}(1)$ *satisfies conditions* (2);
(iii)	$\mathcal{E}(1)$ *is a theta-bundle;*
(iv)	$[\mathcal{E}] \in M_0(Y)$.

Proof: The implication $(i) \Rightarrow (ii)$ easily follows from proposition 2.5, $(ii) \Rightarrow (iii)$ is given by theorem 2.2, $(iv) \Rightarrow (i)$ is trivial. Thus it remains to check the implication $(iii) \Rightarrow (iv)$.

Assume that E is a theta-bundle of f and denote $\mathcal{E} = E(-1)$. It follows from theorem 2.2 that it suffices to check that $H^2(Y,\mathcal{E}\otimes\mathcal{E}) = 0$. Restricting (1) to Y and taking into account isomorphism $L^1\alpha^*\alpha_*E \cong E\otimes L^1\alpha^*\alpha_*\mathcal{O}_Y \cong E\otimes\mathcal{O}_Y(-3)$ we get the following exact sequence

$$0 \to E(-3) \to V\otimes\mathcal{O}_Y(-1) \to V^*\otimes\mathcal{O}_Y \to E \to 0.$$

Applying $\mathsf{Hom}(E,-)$ and taking into account isomorphisms

$$\begin{array}{lllll} \mathsf{Ext}^p(E,\mathcal{O}_Y) & \cong & H^p(Y,E^*) & \cong & H^p(Y,E(-2)) = 0, \\ \mathsf{Ext}^p(E,\mathcal{O}_Y(-1)) & \cong & H^p(Y,E^*(-1)) & \cong & H^p(Y,E(-3)) = 0, \end{array}$$

(we used here an isomorphism $\det E^* \cong \mathcal{O}_Y(-2)$ and properties (2)) we obtain isomorphisms

$$\mathsf{Ext}^p(E,E) \cong \mathsf{Ext}^{p+2}(E,E(-3)). \quad (4)$$

In particular, $\mathsf{Ext}^2(E,E) = 0$, but $\mathsf{Ext}^2(E,E) \cong H^2(Y,E^*\otimes E) \cong H^2(Y,\mathcal{E}\otimes\mathcal{E})$. □

Remark 2.7. Note, that by the way we have proved that conditions

$$H^1(Y,\mathcal{E}(1)) = H^2(Y,\mathcal{E}(1)) = H^2(Y,\mathcal{E}\otimes\mathcal{E}) = 0$$

in the definition of the set $M_0(Y)$ in [MT] are redundant.

Remark 2.8. The embedding $E^*(-1) \cong E(-3) \to V\otimes\mathcal{O}_Y(-1)$ identifies the fiber of the bundle E^* over a point $a \in Y$ with the subspace $\mathsf{Ker}\, f(a) \subset V$.

Moduli stacks. Let $\mathcal{N}_{\rm reg,sm}$ denote the open subset of $\mathbb{P}(A^*\otimes\Lambda^2V^*)$, consisting of regular A-nets f, such that X_f (or, equivalently, Y_f) is smooth. Let $\mathcal{M}_X$ denote the moduli stack of V_{14} threefolds. Let $\mathcal{M}_Y$ denote the moduli stack of cubic threefolds. Let $\mathcal{M}_{Y,\mathcal{E}}$ denote the moduli stack of pairs $(Y,\mathcal{E})$, where Y is a cubic threefold and $\mathcal{E}$ is an instanton of charge 2 on Y.

Theorem 2.9. *We have an isomorphism of stacks $\mathcal{M}_X \cong \mathcal{M}_{Y,\mathcal{E}} \cong \mathcal{N}_{\rm reg,sm}/\!/\mathsf{GL}(A)\times\mathsf{GL}(V)$. In particular, the fiber of the projection $\mathcal{M}_X \to \mathcal{M}_Y$ is isomorphic to $M_0(Y)$.*

Proof: It suffices to repeat the arguments of theorem 2.2 and remark 2.3 in relative situation. □

Further properties of theta-bundles.

Lemma 2.10. *If E is a theta-bundle then* $H^p(Y, S^2E(-1)) = \begin{cases} \mathsf{k}, & \text{if } p=1 \\ 0, & \text{otherwise} \end{cases}$

Proof: It follows from (4) that

$$\mathsf{Hom}(E, E(-3)) = \mathsf{Ext}^1(E, E(-3)) = 0 \qquad \text{and} \qquad \mathsf{Ext}^2(E, E(-3)) \cong \mathsf{Hom}(E, E) = \mathsf{k},$$

because E is stable. Applying the Riemann-Roch we deduce that $\mathsf{Ext}^3(E, E(-3)) \cong \mathsf{k}^5$. Further, taking into account an isomorphism $E^*(1) \cong E(-1)$ and applying the Serre duality on Y we obtain

$$H^p(Y, E \otimes E(-1)) \cong \mathsf{Ext}^p(E^*(1), E) \cong \mathsf{Ext}^p(E(-1), E) \cong \mathsf{Ext}^{3-p}(E, E(-3))^* \cong \begin{cases} \mathsf{k}^5, & \text{if } p=0 \\ \mathsf{k}, & \text{if } p=1 \\ 0, & \text{otherwise} \end{cases}$$

But $E \otimes E(-1) \cong \Lambda^2E(-1) \oplus S^2E(-1) \cong \mathcal{O}_Y(1) \oplus S^2E(-1)$. So, it remains to note that $H^0(Y, \mathcal{O}_Y(1)) = H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \cong \mathsf{k}^5$, $H^{>0}(Y, \mathcal{O}_Y(1)) = 0$, and lemma follows. □

$\mathbb{P}^1$-bundle over X. Let $X \subset \mathsf{Gr}(2, V)$ be a smooth V_{14} Fano threefold and let $f : A \to \Lambda^2V^*$ be the corresponding A-net. Let $\mathcal{U}$ denote the restriction to X of the tautological rank 2 subbundle on $\mathsf{Gr}(2, V)$. Then the projectivization $p_X : \mathbb{P}_X(\mathcal{U}) \to X$ is embedded into the partial flag variety $\mathsf{Fl}(1, 2; V)$. Let $\phi : \mathbb{P}_X(\mathcal{U}) \to \mathbb{P}(V)$ denote the restriction of the canonical projection $\mathsf{Fl}(1, 2; V) \to \mathbb{P}(V)$.

Proposition 2.11. *(i) The image $Q = \phi(\mathbb{P}_X(\mathcal{U})) \subset \mathbb{P}(V)$ is a quartic hypersurface, singular along a curve $C \subset Q$, $\deg C = 25$, $p_a(C) = 26$. (ii) The map $\phi : \mathbb{P}_X(\mathcal{U}) - \phi^{-1}(C) \to Q - C$ is an isomorphism, while $\phi : \phi^{-1}(C) \to C$ is a $\mathbb{P}^1$-bundle. (iii) For any point $c \in C$ the curve $L_c = p_X(\phi^{-1}(c)) \subset X$ is a line on X. On the other hand, if L is a line on $X \subset \mathsf{Gr}(2, V)$, then $p_X^{-1}(L)$ is a Hirzebruch surface F_1, and its exceptional section $\widetilde{L}$ coincides with $\phi^{-1}(c)$ for some $c \in C$.*

Proof: (i) It is clear that the image $Q = \phi(\mathbb{P}_X(\mathcal{U})) \subset \mathbb{P}(V)$ is just the set of all points $v \in \mathbb{P}(V)$ which are contained in a 2-dimensional subspace $U \subset V$ isotropic with respect to all skew-forms in the A-net f. Thus $v \in Q$ if and only if the map

$$f_v : A \to V^*, \qquad a \mapsto f(a)(v, -)$$

has image of codimension ≥ 2. In the other words, Q is the determinantal $\{\mathsf{rank}(f) \leq 4\} \subset \mathbb{P}(V)$, where f is considered as a homomorphism of coherent sheaves on $\mathbb{P}(V)$

$$A \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{f} V^* \otimes \mathcal{O}_{\mathbb{P}(V)}.$$

Note that since $f(a)$ is a skew-form we have $f(a)(v, v) = 0$ for all $a \in A$, hence the image of f lies in the annihilator $v^\perp \subset V^*$. In the other words, the above homomorphism of sheaves factors as

$$A \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{f'} \Omega_{\mathbb{P}(V)}(1) \subset V^* \otimes \mathcal{O}_{\mathbb{P}(V)}$$

Note that $\mathsf{rank}(A \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)) = \mathsf{rank}(\Omega_{\mathbb{P}(V)}(1)) = 5$, hence Q is the zero locus of

$$\det f' \in \mathsf{Hom}(\det(A \otimes \mathcal{O}_{\mathbb{P}(V)}(-1)), \det(\Omega_{\mathbb{P}(V)}(1))) = \mathsf{Hom}(\mathcal{O}_{\mathbb{P}(V)}(-5), \mathcal{O}_{\mathbb{P}(V)}(-1)).$$

Thus Q is a quartic hypersurface in $\mathbb{P}(V)$.

By the general properties of determinantals the singular locus $C = \mathsf{sing}(Q)$ is the determiantal $\{\mathsf{rank}(f') \leq 3\} \subset \mathbb{P}(V)$. It will be shown in (iii) below that C parameterizes lines on X, hence

C is 1-dimensional [Is1, Is1]. Thus C is of expected dimension and the standard methods can be applied to compute $\deg(C) = 25$, $p_a(C) = 26$.

(ii) It is clear that for a point $v \in Q$ the fiber $\phi^{-1}(v)$ is isomorphic to the set of all 2-dimensional subspaces $U \subset V$, such that $v \in U$ and $U \subset (\mathsf{Im}\, f_v)^\perp$. But $\dim(\mathsf{Im}\, f_v)^\perp = 2$ for $v \in Q - C$, hence ϕ is an isomorphism over the complement of C. On the other hand, for any point $v \in C$ we have $(\mathsf{Im}\, f_v)^\perp \cong \mathsf{k}^3$ and $\phi^{-1}(v) \cong \mathbb{P}((\mathsf{Im}\, f_v)^\perp/\mathsf{k}v) \cong \mathbb{P}^1$.

(iii) The arguments in *(ii)* show that $L_c := p_X(\phi^{-1}(c))$ is a line on $X \subset \mathsf{Gr}(2, V)$ for any $c \in C$. On the other hand, if L is a line on $X \subset \mathsf{Gr}(2, V)$, then $\mathcal{U}_{|L} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1)$. Thus

$$p_X^{-1}(L) = \mathbb{P}_L(\mathcal{U}_{|L}) = \mathbb{P}_L(\mathcal{O}_L \oplus \mathcal{O}_L(-1)) \cong F_1.$$

It is clear that the map ϕ contracts the exceptional section $\widetilde{L}$ of $p_X^{-1}(L)$, hence $\widetilde{L} = \phi^{-1}(c)$ for some $c \in C$ and $L = L_c$. □

Remark 2.12. It is clear that for any $0 \neq v \in V$ we have $\mathbb{P}(\mathsf{Ker}\, f_v) \subset Y_f$. On the other hand Y_f is a smooth cubic in $\mathbb{P}(A)$ by theorem 2.2, hence it cannot contain a $\mathbb{P}^2$. This means that $\dim \mathsf{Ker}\, f_v \leq 2$ and $\mathsf{rank}(f_v) \geq 3$ for any $0 \neq v \in V$.

Remark 2.13. Using description of C as a determinantal one can show that $\mathcal{O}_{\mathbb{P}(V)}(1)_{|C}$ is a degenerate even theta-characteristic on C with $\dim H^0(C, \mathcal{O}_{\mathbb{P}(V)}(1)_{|C}) = 6$.

Corollary 2.14. *The curve C parameterizes lines on X.*

$\mathbb{P}^1$-bundle over Y. Now let Y be the Pfaffian cubic of X and let E be the theta-bundle of X. By theorem 2.2 the bundle E induces an embedding $\kappa : Y \to \mathsf{Gr}(2, V)$. Then we obtain an embedding of the projectivization $p_Y : \mathbb{P}_Y(E^*) \to Y$ into the partial flag variety $\mathsf{Fl}(1, 2; V)$. Let $\psi : \mathbb{P}_Y(E^*) \to \mathbb{P}(V)$ denote the restriction of the canonical projection $\mathsf{Fl}(1, 2; V) \to \mathbb{P}(V)$.

Proposition 2.15. *(i) We have $\psi(\mathbb{P}_Y(E^*)) = Q$. (ii) The map $\psi : \mathbb{P}_Y(E^*) - \psi^{-1}(C) \to Q - C$ is an isomorphism, while $\psi : \psi^{-1}(C) \to C$ is a $\mathbb{P}^1$-bundle. (iii) For any point $c \in C$ the curve $M_c = p_Y(\psi^{-1}(c)) \subset Y$ is a line on Y such that $E^*_{|M_c} \cong \mathcal{O}_{M_c} \oplus \mathcal{O}_{M_c}(-2)$. On the other hand, if M is a line on Y such that $E^*_{|M} \cong \mathcal{O}_M \oplus \mathcal{O}_M(-2)$, then $p_Y^{-1}(M)$ is a Hirzebruch surface F_2, and its exceptional section $\tilde{M}$ coincides with $\psi^{-1}(c)$ for some $c \in C$.*

Proof: *(i)* The fiber of E^* over a point $a \in Y$ is the kernel of the skew-form $f(a) \in \Lambda^2 V^*$. Hence $v \in \psi(\mathbb{P}_Y(E^*))$ iff $f(a)(v, -) = 0$, that is iff $f_v(a) = 0$ for some $0 \neq a \in A$. Thus $\psi(\mathbb{P}_Y(E^*)) = Q$.

(ii) Note that the fiber of ψ over $v \in \mathbb{P}(V)$ coincides with $\mathbb{P}(\mathsf{Ker}\, f_v) \subset \mathbb{P}(A)$. For $v \in Q - C$ we have $\mathsf{rank}(f_v) = 4$, hence $\dim \mathsf{Ker}\, f_v = 1$. Thus ψ is an isomorphism over $Q - C$. On the other hand, for $v \in C$ we have $\mathsf{rank}(f_v) = 3$, hence $\dim \mathsf{Ker}\, f_v = 2$ and $\psi^{-1}(v) \cong \mathbb{P}^1$.

(iii) The arguments in *(ii)* show that $M_c = p_Y(\psi^{-1}(c))$ is a line on the cubic $Y \subset \mathbb{P}(A)$. Note that $\det(E^*_{|M_c}) = \det(E^*)_{|M_c} = \mathcal{O}_Y(-2)_{|M_c} = \mathcal{O}_{M_c}(-2)$, and the point $c \in C \subset \mathbb{P}(V)$ gives a nonvanishing section of $E^*_{|M_c} \subset V \otimes \mathcal{O}_{M_c}$, hence $E^*_{|M_c} \cong \mathcal{O}_{M_c} \oplus \mathcal{O}_{M_c}(-2)$. On the other hand, if M is a line on Y such that $E^*_{|M} \cong \mathcal{O}_M \oplus \mathcal{O}_M(-2)$, then

$$p_Y^{-1}(M) = \mathbb{P}_{M_c}(E^*_{|M}) = \mathbb{P}_M(\mathcal{O}_M \oplus \mathcal{O}_M(-2)) \cong F_2.$$

It is clear that the map ψ contracts the exceptional section $\tilde{M}$ of $p_Y^{-1}(M)$, hence $\tilde{M} = \psi^{-1}(c)$ for some $c \in C$ and $M = M_c$. □

Corollary 2.16. *The curve C parameterizes jumping lines of the instanton $\mathcal{E} = E(-1)$ on Y.*

The flop. Denote $S_X = \phi^{-1}(C) \subset \mathbb{P}_X(\mathcal{U})$ and $S_Y = \psi^{-1}(C) \subset \mathbb{P}_Y(E^*)$. Thus S_X and S_Y are ruled surfaces over the curve C. It is proved in propositions 2.11 and 2.15 that ϕ contracts S_X onto C and ψ contracts S_Y onto C. Hence the rational map $\theta = \phi^{-1} \circ \psi : \mathbb{P}_Y(E^*) \to \mathbb{P}_X(\mathcal{U})$ is a birational isomorphism.

Theorem 2.17. *The map θ is a flop in the surface S_Y. The map θ^{-1} is a flop in the surface S_X.*

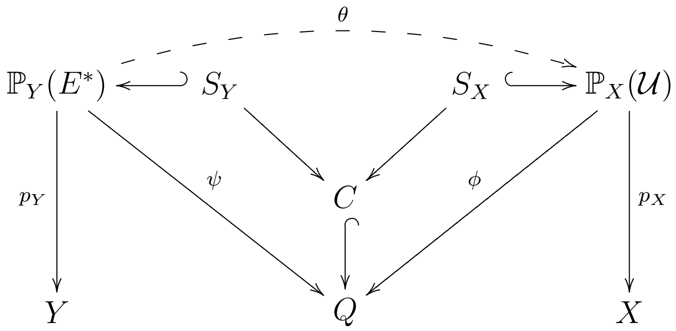
Proof: Since ϕ and ψ are small contractions by propositions 2.11 and 2.15, it remains to check that the canonical classes of $\mathbb{P}_Y(E^*)$ and $\mathbb{P}_X(\mathcal{U})$ are pull-backs from Q. But it is easy to see that the canonical classes equal $\psi^*\mathcal{O}_Q(-2)$ and $\phi^*\mathcal{O}_Q(-2)$ respectively. Indeed,

$$\begin{aligned} \omega_{\mathbb{P}_Y(E^*)} &= p_Y^*\omega_Y \otimes \omega_{\mathbb{P}_Y(E^*)/Y} = p_Y^*\mathcal{O}_Y(-2) \otimes \left(\psi^*\mathcal{O}_Q(-2) \otimes p_Y^* \det E\right) \cong \psi^*\mathcal{O}_Q(-2), \\ \omega_{\mathbb{P}_X(\mathcal{U})} &= p_X^*\omega_X \otimes \omega_{\mathbb{P}_X(\mathcal{U})/X} = p_X^*\mathcal{O}_X(-1) \otimes \left(\phi^*\mathcal{O}_Q(-2) \otimes p_X^* \det \mathcal{U}^*\right) \cong \phi^*\mathcal{O}_Q(-2). \end{aligned}$$

since $\psi^*\mathcal{O}_Q(1)$ and $\phi^*\mathcal{O}_Q(1)$ are the Grothendieck relatively ample line bundles on $\mathbb{P}_Y(E^*)$ and $\mathbb{P}_X(\mathcal{U})$ respectively by definition of ψ and ϕ. □

Summarizing, we get the following.

Theorem 2.18. *Let X be a smooth V_{14} Fano threefold. Let Y be its Pfaffian cubic and let E be the theta-bundle of X on Y. Then we have the following diagram*



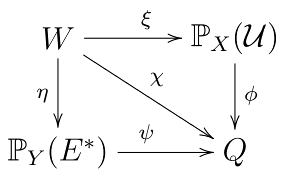
where

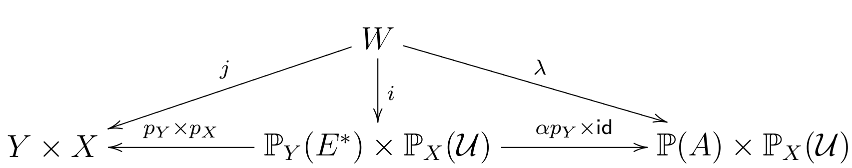
- *Q is a quartic hypersurface in $\mathbb{P}(V)$, singular along a curve C;*
- *$S_X \subset \mathbb{P}_X(\mathcal{U})$ and $S_Y \subset \mathbb{P}_Y(E^*)$ are ruled surfaces over the curve C, ruled by exceptional sections over lines on X and by exceptional sections over jumping lines on Y respectively;*
- *ϕ and ψ contract ruled surfaces S_X and S_Y onto C and bijective elsewhere;*
- *$\theta = \phi^{-1} \cdot \psi$ is a flop in S_Y.*

The quartic Q is known as da Palatini quartic.

Remark 2.19 ([IM]). If H is a hyperplane in $\mathbb{P}(V)$ then it is easy to see that $p_X \circ \phi^{-1} : Q \cap H \to X$ and $p_Y \circ \psi^{-1} : Q \cap H \to Y$ are birational isomorphisms. In particular, the Pfaffian cubic Y of a smooth V_{14} Fano threefold X is birational to X. Moreover, the Torelly theorem [CG, T] implies that cubic threefolds Y_1 and Y_2 are birational if and only if they are isomorphic. It follows that the fibers of the map of the moduli stacks $\mathcal{M}_X \to \mathcal{M}_Y$ are birational classes of V_{14} threefolds.

The fiber product. Consider the fiber product $W = \mathbb{P}_Y(E^*) \times_Q \mathbb{P}_X(\mathcal{U})$ and denote the embedding $W \to \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U})$ by i. Let $\xi : W \to \mathbb{P}_X(\mathcal{U})$, $\eta : W \to \mathbb{P}_Y(E^*)$, $\chi : W \to Q$ denote the projections. Put $j = (p_Y \times p_X) \cdot i : W \to Y \times X$ and $\lambda = (\alpha p_Y \times \mathsf{id}) \cdot i : W \to \mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U})$.





Proposition 2.20. (i) *j is a closed embedding and we have the following exact sequence on $Y \times X$:*

$$0 \to E^*(-1) \boxtimes \mathcal{O}_X \to \mathcal{O}_Y(-1) \boxtimes V/\mathcal{U} \longrightarrow \mathcal{O}_Y \boxtimes \mathcal{U}^* \longrightarrow j_*\chi^*\mathcal{O}_Q(1) \to 0, \tag{5}$$

(ii) *λ is a closed embedding and we have the following exact sequence on $\mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U})$:*

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}(A)}(-4) \boxtimes \Lambda^4(\phi^*\mathcal{O}_Q(-1) \otimes p_X^* V/\mathcal{U}) \to \mathcal{O}_{\mathbb{P}(A)}(-3) \boxtimes \Lambda^3(\phi^*\mathcal{O}_Q(-1) \otimes p_X^* V/\mathcal{U}) \to \\ \to \mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \Lambda^2(\phi^*\mathcal{O}_Q(-1) \otimes p_X^* V/\mathcal{U}) \to \mathcal{O}_{\mathbb{P}(A)}(-1) \boxtimes \Lambda^1(\phi^*\mathcal{O}_Q(-1) \otimes p_X^* V/\mathcal{U}) \to \\ \to \mathcal{O} \to \lambda_*\mathcal{O}_W \to 0. \end{aligned} \tag{6}$$

Proof: (i) By definition of X the composition $\mathcal{U} \to V \otimes \mathcal{O}_X \xrightarrow{f(a)} V^* \otimes \mathcal{O}_X \to \mathcal{U}^*$ vanishes for any $a \in A$. Therefore, f induces a morphism of vector bundles $\mathcal{O}_Y(-1) \boxtimes V/\mathcal{U} \xrightarrow{\tilde{f}} \mathcal{O}_Y \boxtimes \mathcal{U}^*$. Moreover, in the following commutative diagram

$$\begin{array}{ccccccc} \mathsf{Ker}\, f(a) & \longrightarrow & V \otimes \mathcal{O}_X & \xrightarrow{f(a)} & V^* \otimes \mathcal{O}_X & \longrightarrow & \mathsf{Coker}\, f(a) \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & V/\mathcal{U} & \xrightarrow{\tilde{f}_a} & \mathcal{U}^* & \longrightarrow & \mathsf{Coker}\, \tilde{f}_a \end{array}$$

the upper row is a complex, hence the sequence

$$E^*(-1) \boxtimes \mathcal{O}_X \to \mathcal{O}_Y(-1) \boxtimes V/\mathcal{U} \xrightarrow{\tilde{f}} \mathcal{O}_Y \boxtimes \mathcal{U}^*, \tag{7}$$

in which the first morphism is the canonical embedding $E^*(-1) \boxtimes \mathcal{O}_X \to \mathcal{O}_Y(-1) \boxtimes V/\mathcal{U}$ (corresponding to the diagonal arrow in the diagram, see remark 2.8), is a complex.

For any point $(a, U) \in Y \times X \subset \mathbb{P}(A) \times \mathsf{Gr}(2, V)$ the composition $\mathsf{Ker}\, f(a) \to V \to V/U$ is an embedding unless $\mathsf{Ker}\, f(a) \cap U \neq 0$. Similarly, the map $V/U \xrightarrow{\tilde{f}_a} U^*$ is a surjection unless $\mathsf{Ker}\, f(a) \cap U \neq 0$. Any $0 \neq v \in \mathsf{Ker}\, f(a) \cap U$ specifies a point $(a, v) \in \mathbb{P}_Y(E^*) \subset \mathbb{P}(A) \times \mathbb{P}(V)$ and a point $(U, v) \in \mathbb{P}_X(\mathcal{U}) \subset \mathsf{Gr}(2, V) \times \mathbb{P}(V)$ such that $\psi(a, v) = v = \phi(U, v) \in \mathbb{P}(V)$. This means that the degeneration sets of both morphisms of (7) coincide with $j(W)$.

Let W' denote the degeneration subscheme of the morphism $\tilde{f}$ on $Y \times X$. We already have shown that j is a set-theoretical bijection $W \to W'$. Let us show that j is a scheme-theoretical isomorphism.

Indeed, the pullback of the morphism $\tilde{f}$ via j is $\eta^* p_Y^* \mathcal{O}_Y(-1) \otimes \xi^* p_X^* V/\mathcal{U} \to \xi^* p_X^* \mathcal{U}^*$. It is clear that its composition with the surjection $(\xi^* p_X^* \mathcal{U}^* \to \chi^*\mathcal{O}_Q(1)) = \xi^*(p_X^*\mathcal{U}^* \to \phi^*\mathcal{O}_Q(1))$ vanishes, hence $j : W \to Y \times X$ factors through the subscheme $W' \subset Y \times X$.

On the other hand, the rank of $\tilde{f}$ restricted to W' equals 1 identically (if $\tilde{f} = 0$ at a point (a, U) then $U \subset \mathsf{Ker}\, f_a$, hence a is a singular point of Y, see proposition A.4). Hence, the cokernel of $\tilde{f}$ is a line bundle on W', denote it by $\mathcal{L}_{W'}$. The composition of the canonical projection $V^* \otimes \mathcal{O}_{W'} \to (\mathcal{O}_Y \boxtimes \mathcal{U}^*)_{|W'}$ and of the cokernel morphism $(\mathcal{O}_Y \boxtimes \mathcal{U}^*)_{|W'} \to \mathcal{L}_{W'}$ specifies a map $W' \to Y \times X \times \mathbb{P}(V)$. Furthermore, since this morphism factors through $(\mathcal{O}_Y \boxtimes \mathcal{U}^*)_{|W'}$, the map factors through $Y \times \mathbb{P}_X(\mathcal{U})$. Similarly, it is easy to show that the morphism $V^* \otimes \mathcal{O}_{W'} \to \mathcal{L}_{W'}$ factors through $(E \boxtimes \mathcal{O}_X)_{|W'}$, hence the map $W' \to Y \times X \times \mathbb{P}(V)$ factors through $\mathbb{P}_Y(E^*) \times X$. Therefore, we obtain a pair of maps $W' \to \mathbb{P}_Y(E^*)$ and $W' \to \mathbb{P}_X(\mathcal{U})$, such that the compositions $W' \to \mathbb{P}_Y(E^*) \to Q \subset \mathbb{P}(V)$ and $W' \to \mathbb{P}_X(\mathcal{U}) \to Q \subset \mathbb{P}(V)$ coincide. Thus we obtain a map $W' \to \mathbb{P}_Y(E^*) \times_Q \mathbb{P}_X(\mathcal{U}) = W$. It is easy to see that this map is inverse to the map j above.

Thus we have proved that $j : W \to W'$ is a scheme-theoretical isomorphism. Moreover, the above arguments show that the cokernel of $\tilde{f}$, $\mathcal{L}_{W'}$, is isomorphic to $j_*\chi^*\mathcal{O}_Q(1)$. Therefore the sequence (5) is exact at least at the right two terms. Furthermore, the above arguments also prove exactness at the left term. It remains to check that the embedding $E^*(-1)\boxtimes\mathcal{O}_X\to\operatorname{\mathsf{Ker}}\tilde f$ is an isomorphism. Indeed, this is true because both sheaves are reflexive of rank 2 and

$$\begin{aligned}\det\operatorname{\mathsf{Ker}}\tilde f\cong\det(\mathcal{O}_Y(-1)\boxtimes V/\mathcal{U})\otimes\det(\mathcal{O}_Y\boxtimes\mathcal{U}^*)^{-1}\otimes\det j_*\chi^*\mathcal{O}_Q(1)\cong\\ \cong(\mathcal{O}_Y(-4)\boxtimes\mathcal{O}_X(1))\otimes(\mathcal{O}_Y\boxtimes\mathcal{O}_X(1))^{-1}\cong\mathcal{O}_Y(-4)\boxtimes\mathcal{O}_X\cong\det(E^*(-1)\boxtimes\mathcal{O}_X).\end{aligned}$$

Here $\det j_*\chi^*\mathcal{O}_Q(1)\cong\mathcal{O}_{Y\times X}$ because $\operatorname{\mathsf{codim}}\operatorname{\mathsf{supp}}(j_*\chi^*\mathcal{O}_Q(1))=2$.

(ii) Let $\hat f$ denote the composition of the following morphisms on $\mathbb{P}(A)\times\mathbb{P}_X(\mathcal{U})$:

$$\mathcal{O}_{\mathbb{P}(A)}(-1)\boxtimes p_X^*V/\mathcal{U}\to\mathcal{O}_{\mathbb{P}(A)}\boxtimes p_X^*\mathcal{U}^*\to\mathcal{O}_{\mathbb{P}(A)}\boxtimes\phi^*\mathcal{O}_Q(1),$$

where the first morphism is defined similarly to the morphism $\tilde f$ in (i), and the second morphism is the canonical projection. Let $W''\subset\mathbb{P}(A)\times\mathbb{P}_X(\mathcal{U})$ denote the zero scheme of $\hat f$. In the other words W'' is the zero scheme of a section of the vector bundle $\mathcal{O}_{\mathbb{P}(A)}(1)\boxtimes(\phi^*\mathcal{O}_Q(1)\otimes p_X^*(V/\mathcal{U})^*)$, corresponding to $\hat f$. We are going to prove that the map $\mathbb{P}(A)\times\mathbb{P}_X(\mathcal{U})\xrightarrow{\mathsf{id}\times p_X}\mathbb{P}(A)\times X$ induces an isomorphism of $W''\subset\mathbb{P}(A)\times\mathbb{P}_X(\mathcal{U})$ to $W'\subset Y\times X\subset\mathbb{P}(A)\times X$.

Indeed, the definition of $\hat f$ shows that the pullback under $\mathsf{id}\times p_X$ of $(\alpha\times\mathsf{id}_X)_*\tilde f$ degenerates on W'', hence $(\mathsf{id}\times p_X)(W'')\subset W'$. Similarly, the map $(\mathcal{O}_Y\boxtimes\mathcal{U}^*)_{|W'}\to\chi^*\mathcal{O}_Q(1)$ from (5) specifies an embedding $W'\to Y\times\mathbb{P}_X(\mathcal{U})\subset\mathbb{P}(A)\times\mathbb{P}_X(\mathcal{U})$, and it is clear that the pullback of $\hat f$ under this embedding vanishes. Thus we obtain the iverse map $W'\to W''$.

Further, it is easy to see that the composition of $j:W\to W'$, and of the above isomorphism $W'\to W''$ coincides with λ. Finally, $\operatorname{\mathsf{codim}}W''=\operatorname{\mathsf{rank}}(\mathcal{O}_{\mathbb{P}(A)}(1)\boxtimes(\phi^*\mathcal{O}_Q(1)\otimes p_X^*(V/\mathcal{U})^*))$, hence the structure sheaf $\lambda_*\mathcal{O}_W$ admits a Koszul resolution (6). □

3. Derived categories

Preliminaries. Let $\mathcal{D}$ be a triangulated category [V, GM]. An important example of a triangulated category is $\mathcal{D}^b(M)$, the bounded derived category of coherent sheaves on a smooth projective variety M. We briefly remind some definitions and results from [BK, B, BO, Or] and [Br].

Definition 3.1 ([B])**.** An object $F\in\mathcal{D}$ is called *exceptional* if $\operatorname{\mathsf{Hom}}(F,F)=\mathsf{k}$ and $\operatorname{\mathsf{Ext}}^p(F,F)=0$ for $p\neq 0$. A collection of exceptional objects $(F_1,\dots,F_k)$ is called *exceptional* if $\operatorname{\mathsf{Ext}}^p(F_i,F_j)=0$ for $i>j$ and all $p\in\mathbb{Z}$.

Definition 3.2 ([B])**.** A strictly full triangulated subcategory $\mathcal{A}\subset\mathcal{D}$ is *admissible* if the embedding functor $\mathcal{A}\to\mathcal{D}$ admits the left and the right adjoint functors $\mathcal{D}\to\mathcal{A}$.

Proposition 3.3 ([B])**.** *Let $(F_1,\dots,F_k)$ be an exceptional collection in $\mathcal{D}$. The triangulated subcategory $\langle F_1,\dots,F_k\rangle\subset\mathcal{D}$ generated by objects F_1, ... , F_k is admissible.*

If $\mathcal{A}$ is a full triangulated subcategory of $\mathcal{D}$ then the *right orthogonal* to $\mathcal{A}$ in $\mathcal{D}$ is the full subcategory $\mathcal{A}^\perp\subset\mathcal{D}$ consisting of all objects $G\in\mathcal{D}$ such that $\operatorname{\mathsf{Hom}}(F,G)=0$ for all $F\in\mathcal{A}$. Similarly, the *left orthogonal* to $\mathcal{A}$ in $\mathcal{D}$ is the full subcategory ${}^\perp\mathcal{A}\subset\mathcal{D}$ consisting of all objects $G\in\mathcal{D}$ such that $\operatorname{\mathsf{Hom}}(G,F)=0$ for all $F\in\mathcal{A}$.

Definition 3.4 ([BO])**.** A sequence of admissible subcategories $(\mathcal{A}_1,\dots,\mathcal{A}_n)$ in $\mathcal{D}$ is *semiorthogonal* if $\mathcal{A}_j\subset\mathcal{A}_i^\perp$ for $i>j$. Triangulated subcategory of $\mathcal{D}$ generated by subcategoires $\mathcal{A}_1$, ... , $\mathcal{A}_n$ is denoted by $\langle\mathcal{A}_1,\dots,\mathcal{A}_n\rangle$. A semiorthogonal collection $(\mathcal{A}_1,\dots,\mathcal{A}_n)$ is *full* if $\langle\mathcal{A}_1,\dots,\mathcal{A}_n\rangle=\mathcal{D}$. A full semiorthogonal collection in $\mathcal{D}$ is called a *semiorthogonal decomposition* of $\mathcal{D}$.

Definition 3.5 ([BK]). A covariant additive functor $\mathsf{S}_\mathcal{D} : \mathcal{D} \to \mathcal{D}$ is a *Serre functor* if it is a category equivalence and for all objects $F, G \in \mathcal{D}$ there are given bi-functorial isomorphisms $\varphi_{F,G} : \mathsf{Hom}(F,G) \to \mathsf{Hom}(G, \mathsf{S}_\mathcal{D}(F))^*$ such that the composition

$$(\varphi^{-1}_{G,\mathsf{S}_\mathcal{D}(F)})^* \circ \varphi_{F,G} : \mathsf{Hom}(F,G) \to \mathsf{Hom}(G,\mathsf{S}_\mathcal{D}(F))^* \to \mathsf{Hom}(\mathsf{S}_\mathcal{D}(F), \mathsf{S}_\mathcal{D}(G))$$

coincides with the isomorphism induced by $\mathsf{S}_\mathcal{D}$.

Proposition 3.6 ([BK]). *If a Serre functor exists then it is unique up to a canonical functorial isomorphism. If $\mathcal{D} = \mathcal{D}^b(M)$ then $\mathsf{S}_\mathcal{D}(F) := F \otimes \omega_M[\dim M]$ is a Serre functor.*

Proposition 3.7 ([B]). *If $\mathcal{D}$ admits a Serre functor and $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a semiorthogonal sequence of admissible subcategories, then $\mathcal{D} = \langle \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ and $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{A}_{n+1} \rangle$ are semiotrhogonal decompositions, where $\mathcal{A}_0 = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle^\perp$ and $\mathcal{A}_{n+1} = {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$.*

Proposition 3.8 ([B]). *If $\mathcal{D}$ admits a Serre functor and $\mathcal{A} \subset \mathcal{D}$ is admissible then there exist exact functors $L_\mathcal{A} : \mathcal{D} \to \mathcal{A}^\perp$ and $R_\mathcal{A} : \mathcal{D} \to {}^\perp\mathcal{A}$ inducing equivalences ${}^\perp\mathcal{A} \to \mathcal{A}^\perp$, $\mathcal{A}^\perp \to {}^\perp\mathcal{A}$, such that $L_\mathcal{A}(\mathcal{A}) = 0$, $R_\mathcal{A}(\mathcal{A}) = 0$, $(L_\mathcal{A})_{|{}^\perp\mathcal{A}} = \mathsf{S}_\mathcal{D} \circ \mathsf{S}^{-1}_{{}^\perp\mathcal{A}}$, and $(R_\mathcal{A})_{|\mathcal{A}^\perp} = \mathsf{S}^{-1}_\mathcal{D} \circ \mathsf{S}_{\mathcal{A}^\perp}$. Moreover, such functors are unique up to a canonical functorial isomorphism.*

Proposition 3.9 ([B, BO]). *Let $\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semiorthogonal decmposition. If $\mathcal{D}$ admits a Serre functor then for any $1 \le k \le n-1$ we have semiorthogonal decompositions*

$$\begin{aligned} \mathcal{D} &= \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, R_{\mathcal{A}_{k+1}}\mathcal{A}_k, \mathcal{A}_{k+2}, \dots, \mathcal{A}_n \rangle, \\ \mathcal{D} &= \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, L_{\mathcal{A}_k}\mathcal{A}_{k+1}, \mathcal{A}_k, \mathcal{A}_{k+2}, \dots, \mathcal{A}_n \rangle, \end{aligned}$$

and $R_{\mathcal{A}_{k+1}} : \mathcal{A}_k \to R_{\mathcal{A}_{k+1}}\mathcal{A}_k$, $L_{\mathcal{A}_k} : \mathcal{A}_{k+1} \to L_{\mathcal{A}_k}\mathcal{A}_{k+1}$ are equivalences. If additionally $\mathcal{A}_{k+1} \subset \mathcal{A}_k^\perp$ (i.e. $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$ are completely orthogonal), then $L_{\mathcal{A}_k}\mathcal{A}_{k+1} = \mathcal{A}_{k+1}$, $R_{\mathcal{A}_{k+1}}\mathcal{A}_k = \mathcal{A}_k$.

We will call these operations on semiorthogonal decompositions the (right) *mutation of $\mathcal{A}_k$ through $\mathcal{A}_{k+1}$* and the (left) *mutation of $\mathcal{A}_{k+1}$ through $\mathcal{A}_k$* respectively. If $\mathcal{A} = \langle F \rangle$ we will denote mutation functors, $L_\mathcal{A}$ and $R_\mathcal{A}$, by L_F and R_F respectively.

Lemma 3.10. *If Φ is an autoequivalence of $\mathcal{D}$ then we have canonical isomorphisms of functors $\Phi \circ L_\mathcal{A} \cong L_{\Phi(\mathcal{A})} \circ \Phi$, $\Phi \circ R_\mathcal{A} \cong R_{\Phi(\mathcal{A})} \circ \Phi$.*

Proposition 3.11 ([B]). *If $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$ are generated by exceptional objects F_k and F_{k+1} respectively, then $L_{\mathcal{A}_k}\mathcal{A}_{k+1}$ and $R_{\mathcal{A}_{k+1}}\mathcal{A}_k$ are generated by exceptional objects $L_{F_k}F_{k+1}$ and $R_{F_{k+1}}F_k$ respectively, defined by the following exact triangles*

$$\mathsf{RHom}(F_k, F_{k+1}) \otimes F_k \xrightarrow{\mathsf{ev}} F_{k+1} \to L_{F_k}F_{k+1}, \qquad R_{F_{k+1}}F_k \to F_k \xrightarrow{\mathsf{ev}^*} \mathsf{RHom}(F_k, F_{k+1})^* \otimes F_{k+1},$$

where ev and ev^* denote the canonical evaluation and coevaluation homomorphisms.

Let M_1, M_2 be smooth projective varieties and let $p_i : M_1 \times M_2 \to M_i$ denote the projections. Take any $K \in \mathcal{D}^b(M_1 \times M_2)$ and define $\Phi_K(F) := p_{2*}(p_1^*F \otimes K)$. Then Φ_K is an exact functor $\mathcal{D}^b(M_1) \to \mathcal{D}^b(M_2)$, the *kernel functor* with kernel K. Kernel functors can be thought of as analogues of correspondences on categorical level.

Lemma 3.12. *If $K \in \mathcal{D}^b(M_2 \times M_3)$, $F_1 \in \mathcal{D}^b(M_1)$, $F_2 \in \mathcal{D}^b(M_2)$, then $\Phi_K \circ \Phi_{F_1 \boxtimes F_2} \cong \Phi_{F_1 \boxtimes \Phi_K(F_2)}$.*

Proposition 3.13. *If M is a smooth projective variety, $\mathcal{D} = \mathcal{D}^b(M)$, and $F \in \mathcal{D}$ is an exceptional object then the mutation functors L_F, R_F are kernel functors given by the kernels ${}_F\mathsf{K}$ and K_F on $M \times M$ defined by the following exact triangles*

$$\mathcal{RH}om(F, \mathcal{O}_M) \boxtimes F \xrightarrow{\mathsf{ev}} \Delta_*\mathcal{O}_M \to {}_F\mathsf{K}, \qquad \mathsf{K}_F \to \Delta_*\mathcal{O}_M \xrightarrow{\mathsf{ev}^*} \mathcal{RH}om(F, \omega_M[\dim M]) \boxtimes F,$$

where ev *and* ev* *are the evaluation and coevaluation homomorphisms, and* $\Delta : M \to M \times M$ *is the diagonal.*

Let M be a smooth projective variety and let $\mathcal{E}$ be a rank r vector bundle on M. Consider its projectivization $\mathbb{P}_M(\mathcal{E})$ and denote by $p : \mathbb{P}_M(\mathcal{E}) \to M$ the projection and by $L = \mathcal{O}_{\mathbb{P}_M(\mathcal{E})/M}(1)$ a Grothendieck relatively ample line bundle.

Proposition 3.14 ([Or]). *If* $\mathcal{D}^b(M) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ *is a semiorthogonal decomposition then*

$$\mathcal{D}^b(\mathbb{P}_M(\mathcal{E})) = \langle L^k \otimes p^*\mathcal{A}_1, \dots, L^k \otimes p^*\mathcal{A}_n, \dots, L^{k+r-1} \otimes p^*\mathcal{A}_1, \dots, L^{k+r-1} \otimes p^*\mathcal{A}_n \rangle$$

is a semiorthogonal decomposition for any $k \in \mathbb{Z}$.

We also will need the following reformulation of results of Bridgeland.

Theorem 3.15 ([Br]). *Let* M *be a smooth projective variety and let* $\psi : M \to M'$ *be a crepant contraction of relative dimension* 1. *Let* $Z \subset M$ *denote the exceptional locus of* ψ. *Assume that* $\psi^+ : M^+ \to M'$ *is a flop of* ψ *with* M^+ *smooth and let* $Z^+ \subset M^+$ *denote the exceptional locus of* ψ^+, *so that* $\psi^{-1} \circ \psi^+ : M^+ - Z^+ \to M - Z$ *is an isomorphism. For any point* $s \in M^+$ *let* $j_s : M \to M \times M^+$ *denote the corresponding embedding. If* $K \in \mathcal{D}^b(M \times M^+)$ *is an object, such that for any point* $s \in M^+$ *we have either*

- $j_s^*K \cong \mathcal{O}_{\psi^{-1}\circ\psi^+(s)}$, *if* $s \in M^+ - Z^+$; *or*
- *we have an exact triangle* $j_s^*K \to \mathcal{O}_L \xrightarrow{\epsilon} \mathcal{O}_L(-1)[2]$ *with* $\epsilon \neq 0$, *where* $L = \psi^{-1} \circ \psi^+(s) \cong \mathbb{P}^1$, *if* $s \in Z^+$.

Then the kernel functor $\Phi_K : \mathcal{D}^b(M) \to \mathcal{D}^b(M^+)$ *is an equivalence.*

Derived categories of Y and X. Let $Y \subset \mathbb{P}(A)$ be a smooth cubic threefold and let X be a smooth V_{14} threefold. To avoid an abuse of notation, let us denote by $\mathcal{O}(y)$ the sheaf $\mathcal{O}_{\mathbb{P}(A)}(1)$ and its pullbacks to Y, $\mathbb{P}_Y(E^*)$, W etc., by $\mathcal{O}(x)$ the sheaf $\mathcal{O}_{\mathsf{Gr}(2,V)}(1)$ and its pullbacks to X, $\mathbb{P}_X(\mathcal{U})$, W etc., and by $\mathcal{O}(e)$ the sheaf $\mathcal{O}_{\mathbb{P}(V)}(1)$ and its pullbacks to Q, $\mathbb{P}_Y(E^*)$, $\mathbb{P}_X(\mathcal{U})$, W etc.

Lemma 3.16. *The pairs* $(\mathcal{O}, \mathcal{O}(y))$ *in* $\mathcal{D}^b(Y)$ *and* $(\mathcal{O}, \mathcal{U}^*)$ *in* $\mathcal{D}^b(X)$ *are exceptional.*

Proof: Straightforward computations using the Koszul resolutions of Y in $\mathbb{P}(A)$ and of X in $\mathsf{Gr}(2, V)$ and Borel–Bott–Weil theorem. □

The subcategories $\langle \mathcal{O}, \mathcal{O}(y) \rangle \subset \mathcal{D}^b(Y)$ and $\langle \mathcal{O}, \mathcal{U}^* \rangle \subset \mathcal{D}^b(X)$ are admissible by proposition 3.3, hence by proposition 3.7 we obtain semiorthogonal decompositions

$$\mathcal{D}^b(X) = \langle \mathcal{O}, \mathcal{U}^*, \mathcal{A}_X \rangle, \qquad \mathcal{D}^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(y) \rangle, \tag{8}$$

where $\mathcal{A}_X = {}^\perp\langle \mathcal{O}, \mathcal{U}^* \rangle \subset D^b(X)$ and $\mathcal{A}_Y = \langle \mathcal{O}, \mathcal{O}(y) \rangle^\perp \subset \mathcal{D}^b(Y)$.

Theorem 3.17. *If* Y *is the Pfaffian cubic of* X *then categories* $\mathcal{A}_X$ *and* $\mathcal{A}_Y$ *are equivalent.*

Corollary 3.18. *If* X *and* X' *are birational then* $\mathcal{A}_X$ *and* $\mathcal{A}_{X'}$ *are equivalent.*

Proof: If X and X' are birational then their Pfaffian cubics Y and Y' are isomorphic by remark 2.19, hence $\mathcal{A}_X \cong \mathcal{A}_Y \cong \mathcal{A}_{Y'} \cong \mathcal{A}_{X'}$. □

Note that a triangulated category generated by an exceptional object is equivalent to the derived category of k-vector spaces. Therefore we have

Corollary 3.19. *If* Y *is the Pfaffian cubic of* X *then derived categories* $\mathcal{D}^b(X)$ *and* $\mathcal{D}^b(Y)$ *admit semiorthogonal decompositions with pairwise equivalent summands.*

The rest of the section is devoted to the proof of Theorem 3.17. We begin with a short plan of the proof. From now on we assume that Y is the Pfaffian cubic of X, E is the corresponding theta-bundle, so that theorem 2.18 holds.

Step 1: First of all, we replace for convenience the decomposition (8) of $\mathcal{D}^b(Y)$ by the decomposition $\mathcal{D}^b(Y) = \langle \mathcal{O}(-y), \mathcal{A}_Y, \mathcal{O} \rangle$. This is done by mutating $\mathcal{O}(y)$ to the left, since $\omega_Y \cong \mathcal{O}(-2y)$. Further, we note that $\mathcal{O}(e)$ is the Grothendieck relatively ample line bundle both for $\mathbb{P}_Y(E^*) \to Y$ and for $\mathbb{P}_X(\mathcal{U}) \to X$. Hence by proposition 3.14 we obtain the following semiorthogonal decompositions

$$\begin{aligned} \mathcal{D}^b(\mathbb{P}_X(\mathcal{U})) &= \langle \mathcal{O}(-e), \mathcal{U}^*(-e), \mathcal{A}_X(-e), \mathcal{O}, \mathcal{U}^*, \mathcal{A}_X \rangle, &(9)\\ \mathcal{D}^b(\mathbb{P}_Y(E^*)) &= \langle \mathcal{O}(-y), \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(e-y), \mathcal{A}_Y(e), \mathcal{O}(e) \rangle, &(10) \end{aligned}$$

where p_X^* and p_Y^* are omitted for brevity.

Step 2: We perform with the decomposition (10) a sequence of mutations (described below) and obtain the following semiorthogonal decomposition

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \left\langle \begin{array}{lll} \mathcal{O}(-e), & L_{\mathcal{O}}\mathcal{O}(2e-y), & R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e), \\ \mathcal{O}, & L_{\mathcal{O}(e)}\mathcal{O}(3e-y), & R_{\mathcal{O}(3e-y)}\mathcal{A}_Y(2e) \end{array} \right\rangle. \qquad (11)$$

Step 3: Let $K = i_*\mathcal{O}_W$, where $i : W = \mathbb{P}_Y(E^*) \times_Q \mathbb{P}_X(\mathcal{U}) \to \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U})$ is the embedding. We show that the kernel K satisfies the conditions of theorem 3.15. It follows that the kernel functor $\Phi_K : \mathcal{D}^b(\mathbb{P}_Y(E^*)) \to \mathcal{D}^b(\mathbb{P}_X(\mathcal{U}))$ is an equivalence. We check also that Φ_K commutes with tensoring by pullbacks of sheaves from Q.

Step 4: We show that

$$\Phi_K(\mathcal{O}) \cong \mathcal{O}, \quad \Phi_K(L_{\mathcal{O}}\mathcal{O}(2e-y)) \cong \mathcal{U}^*(-e), \quad \text{and} \quad p_{X*}\Phi_K(R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e)) = 0.$$

Lemma 3.10 implies that

$$\begin{aligned} L_{\mathcal{O}(e)}\mathcal{O}(3e-y) &\cong (L_{\mathcal{O}}\mathcal{O}(2e-y)) \otimes \mathcal{O}(e), \\ R_{\mathcal{O}(3e-y)}\mathcal{A}_Y(2e) &\cong (R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e)) \otimes \mathcal{O}(e), \end{aligned}$$

hence the second line of the decomposition (11) coincides with the twist by $\mathcal{O}(e)$ of the first line. Since the functor Φ_K commutes with the twist by $\mathcal{O}(-e)$, we obtain $\Phi_K(\mathcal{O}(-e)) \cong \mathcal{O}(-e)$, hence Φ_K takes the first line of the decomposition (11) into the subcategory $p_X^*\mathcal{D}^b(X) \otimes \mathcal{O}(-e)$. Therefore, the second line is mapped into the subcategory $p_X^*\mathcal{D}^b(X)$. Since Φ_K is an equivalence, it follows that $p_{X*} \circ \Phi_K$ is an equivalence of the second line onto $\mathcal{D}^b(X)$. Finally, since

$$\begin{aligned} (p_{X*} \circ \Phi_K)(\mathcal{O}) &\cong p_{X*}\mathcal{O} \cong \mathcal{O}, \\ (p_{X*} \circ \Phi_K)(L_{\mathcal{O}(e)}\mathcal{O}(3e-y)) &\cong p_{X*}(\Phi_K(L_{\mathcal{O}}\mathcal{O}(2e-y)) \otimes \mathcal{O}(e)) \cong \\ &\cong p_{X*}(\mathcal{U}^*(-e) \otimes \mathcal{O}(e)) \cong p_{X*}(\mathcal{U}^*) \cong \mathcal{U}^*, \end{aligned}$$

and $\mathcal{D}^b(X) = \langle \mathcal{O}, \mathcal{U}^*, \mathcal{A}_X \rangle$, we deduce that $p_{X*} \circ \Phi_K$ is an equivalence of the category $R_{\mathcal{O}(3e-y)}\mathcal{A}_Y(2e)$ onto $\mathcal{A}_X$. Summarizing, we see that

$$\Phi(A) = p_{X*}(\Phi_K(R_{\mathcal{O}(3e-y)}(p_Y^*(A) \otimes \mathcal{O}(2e)))), \qquad K = i_*\mathcal{O}_W \qquad (12)$$

is an equivalence $\mathcal{A}_Y \to \mathcal{A}_X$.

Now we start implementing above steps. Step 1 is already quite clear, so we can pass to Step 2.

Mutations. First of all, we note that $\omega_{\mathbb{P}_Y(E^*)} \cong \mathcal{O}(-2e)$ (see the proof of theorem 2.17). Further, we will need the following

Lemma 3.20. *In* $\mathcal{D}^b(\mathbb{P}_Y(E^*))$ *we have*
(*i*) $\mathsf{Ext}^p(\mathcal{O}, \mathcal{O}(e-y)) = 0$ *for all* $p \in \mathbb{Z}$.
(*ii*) $\mathsf{Ext}^p(\mathcal{O}, \mathcal{O}(2e-y)) = \begin{cases} \mathsf{k}, & \text{if } p = 1 \\ 0, & \text{if } p \neq 1 \end{cases}$
(*iii*) $\mathsf{Ext}^p(\mathcal{O}(-e), R_{\mathcal{O}(e-y)}F) = 0$ *for any* $p \in \mathbb{Z}$ *and any* $F \in \mathcal{A}_Y$.

Proof: (*i*) $\mathsf{Ext}^{\cdot}(\mathcal{O}, \mathcal{O}(e-y)) = H^{\cdot}(\mathbb{P}_Y(E^*), \mathcal{O}(e-y)) = H^{\cdot}(Y, p_{Y*}(\mathcal{O}(e-y))) = H^{\cdot}(Y, E(-1)) = 0$ by theorem 2.2.

(*ii*) Similarly, we have $\mathsf{Ext}^{\cdot}(\mathcal{O}, \mathcal{O}(2e-y)) = H^{\cdot}(Y, p_{Y*}(\mathcal{O}(2e-y))) = H^{\cdot}(Y, S^2E(-1))$ and it remains to apply lemma 2.10.

(*iii*) Using lemma 3.10 and theorem 3.21 we deduce that

$$\mathsf{Ext}^{\cdot}(\mathcal{O}(-e), R_{\mathcal{O}(e-y)}F) \cong \mathsf{Ext}^{\cdot}(\mathcal{O}, R_{\mathcal{O}(2e-y)}F(e)) \cong \mathsf{Ext}^{\cdot}(\Phi_K(\mathcal{O}), \Phi_K(R_{\mathcal{O}(2e-y)}F(e))).$$

But $\Phi_K(\mathcal{O}) \cong \mathcal{O}$ by proposition 3.23, hence

$$\mathsf{Ext}^{\cdot}(\mathcal{O}(-e), R_{\mathcal{O}(e-y)}F) \cong H^{\cdot}(\mathbb{P}_X(\mathcal{U}), \Phi_K(R_{\mathcal{O}(2e-y)}F(e))) \cong H^{\cdot}(X, p_{X*}\Phi_K(R_{\mathcal{O}(2e-y)}F(e)))$$

and it remains to note that $p_{X*}\Phi_K(R_{\mathcal{O}(2e-y)}F(e)) = 0$ by proposition 3.24. □

Now, we explain the sequence of transformations. We start with semiortogonal deocmposition $\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle \mathcal{O}(-y), \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(e-y), \mathcal{A}_Y(e), \mathcal{O}(e) \rangle$.

(1) We mutate $\mathcal{O}(-y)$ to the right; it is get twisted by $\mathcal{O}(2e)$, the anticanonical class of $\mathbb{P}_Y(E^*)$:

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(e-y), \mathcal{A}_Y(e), \mathcal{O}(e), \mathcal{O}(2e-y) \rangle.$$

(2) We mutate $\mathcal{O}$ through $\mathcal{O}(e-y)$ and $\mathcal{O}(e)$ through $\mathcal{O}(2e-y)$; lemma 3.20 (i) and proposition 3.11 imply that $R_{\mathcal{O}(e-y)}\mathcal{O} = \mathcal{O}$, $R_{\mathcal{O}(2e-y)}\mathcal{O}(e) = \mathcal{O}(e)$, and we get

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle \mathcal{A}_Y, \mathcal{O}(e-y), \mathcal{O}, \mathcal{A}_Y(e), \mathcal{O}(2e-y), \mathcal{O}(e) \rangle.$$

(3) We mutate $\mathcal{A}_Y$ through $\mathcal{O}(e-y)$ and $\mathcal{A}_Y(e)$ through $\mathcal{O}(2e-y)$:

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle \mathcal{O}(e-y), R_{\mathcal{O}(e-y)}\mathcal{A}_Y, \mathcal{O}, \mathcal{O}(2e-y), R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e), \mathcal{O}(e) \rangle.$$

(4) We mutate $\mathcal{O}(e-y)$ to the right; it is get twisted by $\mathcal{O}(2e)$:

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle R_{\mathcal{O}(e-y)}\mathcal{A}_Y, \mathcal{O}, \mathcal{O}(2e-y), R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e), \mathcal{O}(e), \mathcal{O}(3e-y) \rangle.$$

(5) We mutate $\mathcal{O}(2e-y)$ through $\mathcal{O}$ and $\mathcal{O}(3e-y)$ through $\mathcal{O}(e)$; lemma 3.20 (ii) and proposition 3.11 imply that $L_{\mathcal{O}}\mathcal{O}(2e-y)$ and $L_{\mathcal{O}(e)}\mathcal{O}(3e-y)$ are the unique nontrivial extensions

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(2e-y) & \to & L_{\mathcal{O}}\mathcal{O}(2e-y) & \to & \mathcal{O} & \to & 0, \\ 0 & \to & \mathcal{O}(3e-y) & \to & L_{\mathcal{O}(e)}\mathcal{O}(3e-y) & \to & \mathcal{O}(e) & \to & 0, \end{array} \tag{13}$$

and we get:

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle R_{\mathcal{O}(e-y)}\mathcal{A}_Y, L_{\mathcal{O}}\mathcal{O}(2e-y), \mathcal{O}, R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e), L_{\mathcal{O}(e)}\mathcal{O}(3e-y), \mathcal{O}(e) \rangle.$$

(6) We mutate $\mathcal{O}(e)$ to the left; it is get twisted by $\mathcal{O}(-2e)$:

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle \mathcal{O}(-e), R_{\mathcal{O}(e-y)}\mathcal{A}_Y, L_{\mathcal{O}}\mathcal{O}(2e-y), \mathcal{O}, R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e), L_{\mathcal{O}(e)}\mathcal{O}(3e-y) \rangle.$$

(7) We mutate $\mathcal{O}(-e)$ through $R_{\mathcal{O}(e-y)}\mathcal{A}_Y$ and $\mathcal{O}$ through $R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e)$; lemma 3.20 (iii) and proposition 3.9 imply that the mutations coincide with transpositions:

$$\mathcal{D}^b(\mathbb{P}_Y(E^*)) = \langle R_{\mathcal{O}(e-y)}\mathcal{A}_Y, \mathcal{O}(-e), L_{\mathcal{O}}\mathcal{O}(2e-y), R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e), \mathcal{O}, L_{\mathcal{O}(e)}\mathcal{O}(3e-y) \rangle.$$

(8) We mutate $R_{\mathcal{O}(e-y)}\mathcal{A}_Y$ to the right; it is get twisted by $\mathcal{O}(2e)$ and once again using lemma 3.10 we get the desired decomposition (11).

This completes Step 2.

The flop. We adopt the notation of proposition 2.20 and of theorem 3.15.

Theorem 3.21. *If $K = i_*\mathcal{O}_W$ then the kernel functor $\Phi_K : \mathcal{D}^b(\mathbb{P}_Y(E^*)) \to \mathcal{D}^b(\mathbb{P}_X(\mathcal{U}))$ is an equivalence. Moreover, Φ_K commutes with tensoring by pullbacks of bundles from Q.*

Proof: We must check the conditions of theorem 3.15. Take an arbitrary point $s \in \mathbb{P}_X(\mathcal{U})$. Then

$$(\alpha p_Y \times \mathsf{id})_* j_{s*} j_s^* K \cong (\alpha p_Y \times \mathsf{id})_* j_{s*} j_s^* i_* \mathcal{O}_W \cong (\alpha p_Y \times \mathsf{id})_* \big(i_*\mathcal{O}_W \otimes j_{s*}\mathcal{O}_{\mathbb{P}_Y(E^*)}\big) \cong$$
$$\cong (\alpha p_Y \times \mathsf{id})_* i_* i^* j_{s*}\mathcal{O}_{\mathbb{P}_Y(E^*)} \cong \lambda_* i^* \pi_2^* \mathcal{O}_s \cong \lambda_*\lambda^*{\pi'_2}^*\mathcal{O}_s \cong {\pi'_2}^*\mathcal{O}_s \otimes \lambda_*\mathcal{O}_W.$$

Here π_2 and π'_2 denote the projections of $\mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U})$ and $\mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U})$ to $\mathbb{P}_X(\mathcal{U})$:

$$\begin{array}{ccccc} & & & W & \\ & & \swarrow^{i} & & \searrow^{\lambda} \\ \mathbb{P}_Y(E^*) & \xrightarrow{j_s} & \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U}) & \xrightarrow{\alpha p_Y \times \mathsf{id}} & \mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U}) \\ & & \searrow_{\pi_2} & & \swarrow_{\pi'_2} \\ & & & \mathbb{P}_X(\mathcal{U}) & \end{array}$$

Computing ${\pi'_2}^*\mathcal{O}_s \otimes \lambda_*\mathcal{O}_W$ with the help of resolution (6) we obtain

- ${\pi'_2}^*\mathcal{O}_s \otimes \lambda_*\mathcal{O}_W \cong \mathcal{O}_{\gamma(\theta^{-1}(s))}$, for $s \notin S_X$;
- $\mathcal{H}^0({\pi'_2}^*\mathcal{O}_s \otimes \lambda_*\mathcal{O}_W) \cong \mathcal{O}_{\gamma(M)}$, $\mathcal{H}^{-1}({\pi'_2}^*\mathcal{O}_s \otimes \lambda_*\mathcal{O}_W) \cong \mathcal{O}_{\gamma(M)}(-1)$, where $M = \psi^{-1}(\phi(s))$, for $s \in S_X$.

Since ${\pi'_2}^*\mathcal{O}_s \otimes \lambda_*\mathcal{O}_W \cong \gamma_* j_s^* K$, the complex j_s^*K is supported on M, and $\gamma : M \to \mathbb{P}(A) \times \mathbb{P}_X(\mathcal{U})$ is a closed embedding, it follows that

- $j_s^*K \cong \mathcal{O}_{\theta^{-1}(s)}$, for $s \notin S_X$;
- $\mathcal{H}^0(j_s^*K) \cong \mathcal{O}_M$, $\mathcal{H}^{-1}(j_s^*K) \cong \mathcal{O}_M(-1)$, for $s \in S_X$.

It remains to check that $j_s^*K \not\cong \mathcal{O}_M \oplus \mathcal{O}_M(-1)[1]$ for $s \in S_X$. In other words, it suffices to check that $\mathsf{Ext}^1(j_s^*K, \mathcal{O}_M(-1)[1]) = 0$. To this end we consider the following diagram

$$\begin{array}{ccccc} \mathbb{P}_Y(E^*) & \xrightarrow{j_s} & \mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U}) & \xleftarrow{i} & W \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi\times\phi} & & \downarrow{\scriptstyle\chi} \\ Q & \xrightarrow{j_c} & Q \times Q & \xleftarrow{\Delta} & Q \end{array}$$

where $c = \phi(s) \in Q$, $j_c(v) = (v, c) \in Q \times Q$ and Δ is the diagonal. In this diagram the right square is Cartesian and Δ is a closed embedding, hence lemma 3.22 implies that there is a functorial morphism $(\psi \times \phi)^*\Delta_* \to i_*\chi^*$, and furthermore, for any $F \in \mathcal{D}^{\le 0}(Q)$ the object F' in the exact triangle

$$F' \to (\psi \times \phi)^*\Delta_* F \to i_*\chi^* F$$

is contained in $\mathcal{D}^{\le -1}(\mathbb{P}_Y(E^*) \times \mathbb{P}_X(\mathcal{U}))$. Applying j_s^* and using $j_s^*(\psi \times \phi)^* = \psi^* j_c^*$ we get an exact triangle

$$F'' \to \psi^* j_c^* \Delta_* F \to j_s^* i_* \chi^* F,$$

with $F'' = j_s^* F' \in \mathcal{D}^{\le -1}(\mathbb{P}_Y(E^*))$ since j_s^* is right exact. Substituting $F = \mathcal{O}_Q$ and using isomorphisms $\chi^*\mathcal{O}_Q \cong \mathcal{O}_W$, we obtain a triangle

$$F'' \to \psi^* j_c^* \Delta_* \mathcal{O}_Q \to j_s^* K.$$

Applying the functor $\mathsf{Hom}(-,\mathcal{O}_M(-1))$ and using

$$\mathsf{Ext}^{\cdot}(\psi^*j_c^*\Delta_*\mathcal{O}_Q,\mathcal{O}_M(-1))\cong\mathsf{Ext}^{\cdot}(j_c^*\Delta_*\mathcal{O}_Q,\psi_*\mathcal{O}_M(-1))=\mathsf{Ext}^{\cdot}(j_c^*\Delta_*\mathcal{O}_Q,0)=0,$$

we deduce that $\mathsf{Ext}^1(j_s^*K,\mathcal{O}_M(-1))=\mathsf{Hom}(F'',\mathcal{O}_M(-1))=0$, since we have $F''\in\mathcal{D}^{\le-1}(\mathbb{P}_Y(E^*))$ and $\mathcal{O}_M(-1)\in\mathcal{D}^{\ge0}(\mathbb{P}_Y(E^*))$.

Now theorem 3.15 implies that the functor $\Phi_K:\mathcal{D}^b(\mathbb{P}_Y(E^*))\to\mathcal{D}^b((\mathbb{P}_X(\mathcal{U}))$ is an equivalence.

Finally, let $\mathcal{V}$ be an arbitrary vector bundle on Q. Then the functor $F\mapsto\Phi_K(F\otimes\psi^*\mathcal{V})$ is a kernel functor with kernel $K\otimes\pi_1^*\psi^*\mathcal{V}=i_*\mathcal{O}_W\otimes\pi_1^*\psi^*\mathcal{V}=i_*i^*\pi_1^*\psi^*\mathcal{V}=i_*\chi^*\mathcal{V}$, and the functor $F\mapsto\Phi_K(F)\otimes\phi^*\mathcal{V}$ is a kernel functor with kernel $K\otimes\pi_2^*\phi^*\mathcal{V}=i_*\mathcal{O}_W\otimes\pi_2^*\phi^*\mathcal{V}=i_*i^*\pi_2^*\phi^*\mathcal{V}=i_*\chi^*\mathcal{V}$, where π_1 and π_2 are projections of $\mathbb{P}_Y(E^*)\times\mathbb{P}_X(\mathcal{U})$ to the factors. The kernels are isomorphic, hence the functors are isomorphic as well. □

Lemma 3.22 (cf. [Sw]). *For any Carthesian square*

$$\begin{array}{ccc} T & \xleftarrow{\ f'\ } & T' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ S & \xleftarrow{\ f\ } & S' \end{array}$$

there is a canonical morphism of functors $g^*f_*\to f'_*g'^*$. Further, if f is affine then for any $F\in\mathcal{D}^{\le0}(S')$ we have $F'\in\mathcal{D}^{\le-1}(T)$, where F' fits into the triangle $F'\to g^*f_*F\to f'_*g'^*F$.

Proof: Using the adjunction morphisms for g and g', and an isomorphis $f_*g'_*\cong g_*f'_*$ we define the morphism of functors as the following composition

$$g^*f_*\to g^*f_*g'_*g'^*\to g^*g_*f'_*g'^*\to f'_*g'^*.$$

Now, assume that f is affine and let us check that $F'\in\mathcal{D}^{\le-1}(T)$. The property is local, so we can assume that S and T are affine, say $S=\mathsf{Spec}\,A$, $T=\mathsf{Spec}\,B$. Then $S'=\mathsf{Spec}\,A'$, $T'=\mathsf{Spec}\,B\otimes_A A'$, where A' is a finitely generated A-algebra. Note that

$$f'_*g'^*(A')\cong B\otimes_A A',\qquad\text{and}\qquad g^*f_*(A')\in\mathcal{D}^{\le0}(T),\quad\mathcal{H}^0(g^*f_*('))\cong B\otimes_A A'.$$

Taking a resolution of $F\in\mathcal{D}^{\le0}(S')$ by free A'-modules we deduce the claim. □

The final step.

Proposition 3.23. *We have*

(i) $\Phi_K(\mathcal{O})\cong\mathcal{O}$, $\Phi_K(\mathcal{O}(-e))\cong\mathcal{O}(-e)$*;*

(ii) $\Phi_K(L_{\mathcal{O}}\mathcal{O}(2e-y))\cong\mathcal{U}^*(-e)$, $\Phi_K(L_{\mathcal{O}(e)}\mathcal{O}(3e-y))\cong\mathcal{U}^*$.

Proof: First of all we note that for any $F\in\mathcal{D}^b(\mathbb{P}(A))$ we have

$$\Phi_K(p_Y^*\alpha^*F)=\pi_{2*}(\pi_1^*p_Y^*\alpha^*F\otimes i_*\mathcal{O}_W)\cong\pi_{2*}i_*i^*\pi_1^*p_Y^*\alpha^*F\cong$$
$$\cong\xi_*\eta^*p_Y^*\alpha^*F=\pi'_{2*}\lambda_*\lambda^*\pi_1'^*F=\pi'_{2*}(\pi_1'^*F\otimes\lambda_*\mathcal{O}_W),$$

where π_1, π_2 are the projections of $\mathbb{P}_Y(E^*)\times\mathbb{P}_X(\mathcal{U})$ to the factors, and π'_1, π'_2 are the projections of $\mathbb{P}(A)\times\mathbb{P}_X(\mathcal{U})$ to the factors.

(i) Taking $F=\mathcal{O}_{\mathbb{P}(A)}$ and applying (6) we get $\Phi_K(\mathcal{O})=\mathcal{O}$. Further, since $\mathcal{O}(-e)$ is a pullback of a line bundle from Q, it follows from theorem 3.21 that $\Phi_K(\mathcal{O}(-e))=\mathcal{O}(-e)$.

(ii) Taking $F=\mathcal{O}_{\mathbb{P}(A)}(-1)$ and applying (6) we get $\Phi_K(\mathcal{O}(-y))\cong R^4\pi'_{2*}\Lambda^4(V/U)(-4e-5y)\cong\mathcal{O}(x-4e)$. Further, since $\mathcal{O}(2e)$ is a pullback of a line bundle from Q, it follows from theorem 3.21 that

$$\Phi_K(\mathcal{O}(2e-y))\cong\mathcal{O}(x-2e).\tag{14}$$

Since $L_{\mathcal{O}}\mathcal{O}(2e-y)$ is the unique nontrivial extension of $\mathcal{O}$ by $\mathcal{O}(2e-y)$ and since Φ_K is an equivalence, it follows that $\Phi_K(L_{\mathcal{O}}\mathcal{O}(2e-y))$ is the unique nontrivial extension of $\mathcal{O}$ by $\mathcal{O}(x-2e)$. On the other hand, it is clear that $\mathcal{U}^*(-e)$ is such an extension. Hence $\Phi_K(L_{\mathcal{O}}\mathcal{O}(2e-y)) \cong \mathcal{U}^*(-e)$. Finally, by lemma 3.10 we have $L_{\mathcal{O}(e)}\mathcal{O}(3e-y) \cong (L_{\mathcal{O}}\mathcal{O}(2e-y)) \otimes \mathcal{O}(e)$ hence by theorem 3.21 $\Phi_K(L_{\mathcal{O}(e)}\mathcal{O}(3e-y)) \cong \mathcal{U}^*$. □

Proposition 3.24. *We have* $p_{X*}\Phi_K(R_{\mathcal{O}(2e-y)}\mathcal{A}_Y(e)) = 0$.

Proof: We are going to show that $\Phi = p_{X*} \circ \Phi_K \circ R_{\mathcal{O}(2e-y)} \circ T(\mathcal{O}(e)) \circ p_Y^* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$, $T(\mathcal{O}(e))$ — $\mathcal{O}(e)$, is a kernel functor and we will compute its kernel explicitly First, we naote that the functor $R_{\mathcal{O}(2e-y)} : \mathcal{D}^b(\mathbb{P}_Y(E^*)) \to \mathcal{D}^b(\mathbb{P}_Y(E^*))$ is given by the kernel $\mathsf{K}_{\mathcal{O}(2e-y)}$, which by proposition 3.13) is determined by the triangle

$$\{\mathsf{K}_{\mathcal{O}(2e-y)} \to \Delta_*\mathcal{O} \xrightarrow{\rho} \mathcal{RH}om(\mathcal{O}(2e-y), \omega_{\mathbb{P}_Y(E^*)}[4]) \boxtimes \mathcal{O}(2e-y)\} =$$
$$= \{\mathsf{K}_{\mathcal{O}(2e-y)} \to \Delta_*\mathcal{O} \xrightarrow{\rho} \mathcal{O}(y-4e) \boxtimes \mathcal{O}(2e-y)[4]\}$$

Composing functors $\Phi_{\Delta_*\mathcal{O}} = \mathsf{id}$ and $\Phi_{\mathcal{O}(y-4e)\boxtimes\mathcal{O}(2e-y)}$ with other functors, entering the definition of Φ and using lemma 3.12, we obtain:

$$\begin{aligned} p_{X*} \circ \Phi_K \circ \mathsf{id} \circ T(\mathcal{O}(e)) \circ p_Y^* &= \Phi_{(p_Y \times p_X)_* K(e)} = \Phi_{(p_Y \times p_X)_* i_* \mathcal{O}_W(e)} = \Phi_{j_*\mathcal{O}_W}, \\ p_{X*} \circ \Phi_K \circ \Phi_{\mathcal{O}(y-4e)\boxtimes\mathcal{O}(2e-y)} \circ T(\mathcal{O}(e)) \circ p_Y^* &= \Phi_{(p_Y \times p_X)_*(\mathcal{O}(y-3e)\boxtimes K(\mathcal{O}(2e-y)))} = \\ &= \Phi_{p_{Y*}\mathcal{O}(y-3e)\boxtimes p_{X*}\mathcal{O}(x-2e)} = \Phi_{E^*(-y)\boxtimes\mathcal{O}[-2]}. \end{aligned}$$

Therefore, $\Phi = \Phi_L$ is a kernel functor, and its kernel is determined by the triangle

$$L \to j_*\mathcal{O}_W(e) \xrightarrow{\rho} E^*(-y) \boxtimes \mathcal{O}[2].$$

Note that $\rho \neq 0$. Indeed, if $\rho = 0$, then $L \cong j_*\mathcal{O}_W(e) \oplus E^*(-y) \boxtimes \mathcal{O}[1]$, hence for all objects $F \in \mathcal{D}^b(Y)$, such that $H^0(Y, F \otimes E^*(-y)) \neq 0$ it follows, that $\Phi_L(F)$ contain the object $\mathcal{O}[1]$ as a direct summand, and in particular $\mathsf{Hom}(\Phi_L F, \mathcal{O}[1]) \neq 0$. We will show however, that $\mathsf{Hom}(\Phi_L(F), \mathcal{O}[1]) = 0$. Indeed, using the Serre duality on X and $\mathbb{P}_X(\mathcal{U})$ we deduce

$$\begin{aligned} \mathsf{Hom}(\Phi_L(F), \mathcal{O}[1]) &= \mathsf{Hom}(p_{X*}\Phi_K R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F), \mathcal{O}[1]) = \\ &= \mathsf{Hom}(\mathcal{O}(x)[-2], p_{X*}(\Phi_K R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F))^* = \\ &= \mathsf{Hom}(\mathcal{O}(x)[-2], \Phi_K R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F)^* = \\ &= \mathsf{Hom}(\Phi_K R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F, \mathcal{O}(x-2e)[2]). \end{aligned}$$

Further, we note that $\mathcal{O}(x-2e) = \Phi_K(\mathcal{O}(2e-y))$, hence

$$\begin{aligned} \mathsf{Hom}(\Phi_K R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F, \mathcal{O}(x-2e)[2]) &= \mathsf{Hom}(\Phi_K R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F, \Phi_K \mathcal{O}(2e-y)[2]) = \\ &= \mathsf{Hom}(R_{\mathcal{O}(2e-y)} T(\mathcal{O}(e)) p_Y^* F, \mathcal{O}(2e-y)[2]) = 0 \end{aligned}$$

by definition of the mutation functor.

Thus $\rho \neq 0$ and it remains to show that $\Phi_L(\mathcal{A}_Y) = 0$. Using resolution (5) we deduce that $\mathsf{Hom}(j_*\mathcal{O}_W(e), E^*(-y)\boxtimes\mathcal{O}[2]) \cong \mathsf{Hom}(E^*(-1), E^*(-1)) = \mathsf{k}$ because E is stable by proposition 2.6. Hence ρ comes from the identity morphism $E^*(-1) \to E^*(-1)$, and L is quasiisomorphic to the complex $\mathcal{O}(-y) \boxtimes V/\mathcal{U} \to \mathcal{O} \boxtimes \mathcal{U}^*$. It remains to note that

$$H^\cdot(Y, F \otimes \mathcal{O}(-y)) = \mathsf{Hom}^\cdot(\mathcal{O}(y), F) = 0, \quad H^\cdot(Y, F \otimes \mathcal{O}) = \mathsf{Hom}^\cdot(\mathcal{O}, F) = 0,$$

for any $F \in \mathcal{A}_Y$ by (8), hence $\Phi_{\mathcal{O}(-y)\boxtimes V/\mathcal{U}}(\mathcal{A}_Y) = \Phi_{\mathcal{O}\boxtimes\mathcal{U}^*}(\mathcal{A}_Y) = 0$, hence $\Phi_L(\mathcal{A}_Y) = 0$. □

4. Some properties of the category $\mathcal{A}_Y$

Serre functor. Take arbitrary $n, d \in \mathbb{Z}$ such that $n+2 > d$. Let for a moment Y be a smooth n-dimensional hypersurface of degree d in $\mathbb{P}^{n+1}$. Then Y is a Fano manifold and it is easy to check that $(\mathcal{O}_Y, \dots, \mathcal{O}_Y(n+1-d))$ is an exceptional collection in $\mathcal{D}^b(Y)$. Consider the category $\mathcal{A}_Y = \langle \mathcal{O}_Y, \dots, \mathcal{O}_Y(n+1-d) \rangle^{\perp} \subset \mathcal{D}^b(Y)$, so that

$$\mathcal{D}^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \dots, \mathcal{O}_Y(n+1-d) \rangle$$

is a semiorthogonal decomposition.

Consider the functor $\mathsf{O} : \mathcal{D}^b(Y) \to \mathcal{D}^b(Y)$ defined as follows:

$$\mathsf{O}(F) = L_{\mathcal{O}}(F \otimes \mathcal{O}_Y(1))[-1].$$

Note that O takes $\mathcal{A}_Y$ to $\mathcal{A}_Y$.

Lemma 4.1. *We have an isomorhism of functors* $\mathsf{O}^{n+2-d}_{|\mathcal{A}_Y} \cong \mathsf{S}^{-1}_{\mathcal{A}_Y}[d-2]$.

Proof: Let $\Phi : \mathcal{D}^b(Y) \to \mathcal{D}^b(Y)$ denote the functor $F \mapsto F \otimes \mathcal{O}_Y(1)$. Then using lemma 3.10, isomorphism $\mathsf{S}_{\mathcal{D}^b(Y)} = \Phi^{d-2-n}[n]$ and proposition 3.8 we get

$$\begin{aligned}\mathsf{O}^{n+2-d}_{|\mathcal{A}_Y} &= (L_{\mathcal{O}_Y} \circ \Phi[-1]) \circ (L_{\mathcal{O}_Y} \circ \Phi[-1]) \circ \dots \circ (L_{\mathcal{O}_Y} \circ \Phi[-1]) \cong \\ &\cong L_{\mathcal{O}_Y} \circ L_{\mathcal{O}_Y(1)} \circ \dots \circ L_{\mathcal{O}_Y(n-2)} \circ \Phi^{n+2-d}[d-2-n] \cong \\ &\cong L_{\langle \mathcal{O}_Y, \dots, \mathcal{O}_Y(n-2) \rangle} \circ \mathsf{S}^{-1}_{\mathcal{D}^b(Y)}[d-2] \cong \mathsf{S}^{-1}_{\mathcal{A}_Y}[d-2].\end{aligned}$$

□

Lemma 4.2. *We have an isomorhism of functors* $\mathsf{O}^{d}_{|\mathcal{A}_Y} \cong [2-d]$.

Proof: Note that $\mathsf{O} = \Phi_{K_1}$ with the kernel K_1 represented by the following complex

$$\mathcal{O}_Y(1) \boxtimes \mathcal{O}_Y \to \Delta_{Y*}\mathcal{O}_Y(1).$$

Iterating, we find that $\mathsf{O}^d = \Phi_{K_d}$ with the kernel K_d represented by the following complex

$$\mathcal{O}_Y(1) \boxtimes \Omega^{d-1}(d-1)_{|Y} \to \dots \to \mathcal{O}_Y(d-1) \boxtimes \Omega^1(1)_{|Y} \to \mathcal{O}_Y(d) \boxtimes \mathcal{O}_Y \to \Delta_{Y*}\mathcal{O}_Y(d).$$

On the other hand, restricting a resolution of the diagonal in $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$ to $Y \times Y$ we see that the complex

$$\begin{aligned}0 \to \mathcal{O}_Y(d-1-n) \boxtimes \Omega^{n+1}(n+1)_{|Y} \to \dots \to \mathcal{O}_Y \boxtimes \Omega^d(d)_{|Y} \to \\ \to \mathcal{O}_Y(1) \boxtimes \Omega^{d-1}(d-1)_{|Y} \to \dots \to \mathcal{O}_Y(d-1) \boxtimes \Omega^1(1)_{|Y} \to \mathcal{O}_Y(d) \boxtimes \mathcal{O}_Y \to \Delta_{Y*}\mathcal{O}_Y(d)\end{aligned}$$

is quasiisomorphic to

$$L_1(\alpha \times \alpha)^* \Delta_* \mathcal{O}_{\mathbb{P}^{n+1}}(d) \cong \Delta_{Y*}\mathcal{O}_Y,$$

where $\alpha : Y \to \mathbb{P}^{n+1}$ is the embedding. Applying the natural morphism between these two complexes we deduce that $\Delta_{Y*}\mathcal{O}_Y[2-d]$ is quasiisomorphic to the complex

$$\mathcal{O}_Y(d-1-n) \boxtimes \Omega^{n+1}(n+1)_{|Y} \to \dots \to \mathcal{O}_Y \boxtimes \Omega^d(d)_{|Y} \to K_d.$$

It remains to note that

$$\Phi_{\mathcal{O}_Y(d-1-n) \boxtimes \Omega^{n+1}(n+1)_{|Y}}(\mathcal{A}_Y) = \dots = \Phi_{\mathcal{O}_Y \boxtimes \Omega^d(d)_{|Y}}(\mathcal{A}_Y) = 0,$$

hence $\Phi_{K_d|\mathcal{A}_Y} \cong \Phi_{\Delta_{Y*}\mathcal{O}[2-d]|\mathcal{A}_Y} \cong [2-d]$. □

Corollary 4.3. *If c is the greatest common divisor of d and $n+2$, then*

$$\mathsf{S}_{\mathcal{A}_Y}^{d/c} \cong [(d-2)(n+2)/c].$$

Corollary 4.4. *If Y is a cubic threefold then $\mathsf{S}_{\mathcal{A}_Y}^{3} \cong [5]$. If Y is a cubic fourfold then $\mathsf{S}_{\mathcal{A}_Y} \cong [2]$.*

Corollary 4.5. *If Y is a cubic threefold then $\mathcal{A}_Y$ is not equivalent to $\mathcal{D}^b(M)$ for any M.*

Objects. Let Y be a smooth cubic threefold. Simplest examples of objects in $\mathcal{A}_Y$ are provided by instantons.

Lemma 4.6. *If $\mathcal{E}$ is an instanton of charge 2 on Y then $\mathcal{E} \in \mathcal{A}_Y$ and $\mathcal{E}(-1) \in \mathcal{A}_Y$.*

Proof: Follows from proposition 2.5. □

Another examples of objects in $\mathcal{A}_Y$ are provided by curves with theta-characteristics.

Lemma 4.7. *Let M be a smooth curve and let $\mathcal{L}$ be a nondegenerate theta-characteristic on M. For any map $\mu : M \to Y$ the natural morphism $H^0(M, \mathcal{L} \otimes \mu^*\mathcal{O}(1)) \otimes \mathcal{O}_Y \to (\mu_*\mathcal{L}) \otimes \mathcal{O}(1)$ is surjective and its kernel $\mathcal{F}_{\mu,\mathcal{L}} \in \mathcal{A}_Y$.*

Proof: Evident. □

Taking $\mathbb{P}^1$ as a curve, $\mathcal{O}_{\mathbb{P}^1}(-1)$ as a theta-characteristic, and considering omly maps μ of degree 1, we obtain a family of objects $\mathcal{F}_L$ in $\mathcal{A}_Y$, parameterized by the Fano surface of lines L on Y (in fact, $\mathcal{F}_L$ is nothing but the sheaf of ideals of $L \subset Y$). It's Albanese variety is well known to be isomorphic to the intermediate jacobian of Y.

So, if one would be able to define a notion of stability in $\mathcal{A}_Y$ in such a way, that any stable object in $\mathcal{A}_Y$ numerically equivalent to some $\mathcal{F}_L$ would be isomorphic to some $\mathcal{F}_{L'}$, then the Fano surface would become a moduli space of stable objects in $\mathcal{A}_Y$, and it would be possible to reconstruct the intermediate jacobian of Y from $\mathcal{A}_Y$. Since Torelly theorem holds for cubic threefolds (see [CG, T]) it would prove that $\mathcal{A}_Y$ and $\mathcal{A}_{Y'}$ are equivalent if and only if $Y \cong Y'$. It would follow also that $\mathcal{A}_X$ and $\mathcal{A}_{X'}$ are equivalent if and only if X and X' are birational.

However, it is quite unclear how such stability notion can be defined.

Appendix A. The Pfaffian hypersurface of a net of skew-forms

Results of this section are the straightforward generalization of the classical results of A.N. Tyurin (see [T1]) on nets of quadrics.

Let $A = \mathsf{k}^n$ and $V = \mathsf{k}^{2m}$. An *A-net of skew-forms on V* is a linear embedding $f : A \to \Lambda^2 V^*$. Then $F(a) = \mathsf{Pf}(f(a))$ is a homogeneous polynomial of degree m on A. Let $Y = Y_f$ be the corresponding hypersurface of degree m in $\mathbb{P}(A)$. We call Y *the Pfaffian hypersurface* of the A-net f.

The A-net f induces a morphism of coherent sheaves on $\mathbb{P}(A)$

$$V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{f} V^* \otimes \mathcal{O}_{\mathbb{P}(A)}.$$

This map is an isomorphism outside of Y. Let $E = E_f$ denote its cokernel. It is a coherent sheaf on $\mathbb{P}(A)$ with support on Y. We call E *the theta-bundle* of the A-net. This terminology is suggested by an analogy with the role of a theta-characteristic on a degeneration curve of a net of quadrics [T1].

Thus for any A-net f we have the following exact sequence

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \to V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to \alpha_* E_f \to 0, \tag{15}$$

where α is the closed embedding $Y \to \mathbb{P}(A)$. Second morphism in this sequence induces an isomorphism

$$\gamma_f : V^* = H^0(\mathbb{P}(A), V^* \otimes \mathcal{O}_{\mathbb{P}(A)}) \to H^0(\mathbb{P}(A), \alpha_* E_f) = H^0(Y_f, E_f).$$

Definition A.1. An A-net f is called *regular* if $\mathsf{rank} f(a) \geq 2m - 2$ for any $0 \neq a \in A$.

Remark A.2. Dimension calculations imply that a regular A-net f may exist only for $\dim A \leq 6$.

Theorem A.3. *Associating to an A-net f the triple (Y_f, E_f, γ_f) gives a $\mathsf{GL}(A) \times \mathsf{GL}(V)$-equivariant isomorphism between*

- *the subset of $\mathbb{P}(A^* \otimes \Lambda^2 V^*)$ formed by all regular A-nets of skew-forms on V, and*
- *the set of triples (Y, E, γ), where Y is a hypersurface of degree m in $\mathbb{P}(A)$, E is a rank 2 locally free sheaf on Y, and γ is an isomorphism $V^* \to H^0(Y, E)$, such that*

$$\begin{array}{c} c_1(E) = (m-1)[h], \quad c_2(E) = \frac{(m-1)(2m-1)}{6} h^2, \\ H^{\cdot}(Y, E(t)) = 0 \quad \text{for } -(n-2) \leq t \leq -1, \end{array} \tag{16}$$

where $[h] \in H^2(Y, \mathbb{Z})$ is the class of a hyperplane section.

Further, the theta-bundle E_f of a regular A-net is generated by global sections, $H^0(Y_f, E_f) = V^*$, and induces an embedding $\kappa : Y_f \to \mathsf{Gr}(2, V)$.

Proof: First of all, we prove that for the theta-bundle of a regular A-net f conditions (16) are satisfied. This is done by a straightforward calculations, based on the exact sequence (15). This sequence also implies that E is generated by global sections. It remains to check that κ is an embedding. Note that Y parameterize degenerate skew-forms in the A-net, and κ takes a degenerate skew-form to its kerenl. If skew-forms corresponding to a pair of points of Y (even a pair of infinitely close points) have the same kernel, then a certain linear combination of these skew-forms has $\mathsf{rank} \leq 2m - 4$, which contradicts the regularity of the A-net.

Now, assume that (Y, E, γ) is a triple, satisfying (16). Then $H^{\cdot}(\mathbb{P}(A), \alpha_* E(t)) = H^{\cdot}(Y, E(t))$, hence it is zero for $-(n-2) \leq t \leq -1$. Now, let us compute $H^{\cdot}(\mathbb{P}(A), \alpha_* E(t))$ for $t = 0$ and $t = -(n-1)$. To this end choose a line $\mathbb{P}^1 \cong L \subset \mathbb{P}(A)$ not lying on Y. Then $L \cap Y$ is a 0-dimensional subscheme in Y of length $\deg Y = m$. The line L is cut out in $\mathbb{P}(A)$ by $(n-2)$ hyperplanes, hence $L \cap Y$ is cut out in Y by $(n-2)$ hyperplanes. Therefore we have the Koszul resolution

$$0 \to E(-(n-2)) \to E(-(n-3))^{\oplus(n-2)} \to \cdots \to E(-1)^{\oplus(n-2)} \to E \to E_{L\cap Y} \to 0.$$

It follows from (16) that

$$H^p(\mathbb{P}(A), \alpha_* E) = H^p(Y, E) = H^p(L \cap Y, E_{L\cap Y}) = \begin{cases} \mathsf{k}^{2m}, & \text{for } p = 0 \\ 0, & \text{for } p > 0 \end{cases}$$

since $E_{L\cap Y}$ is an artinian sheaf of length $\mathsf{rank}(E) \deg Y = 2m$. Twisting the Koszul resolution by $\mathcal{O}_{\mathbb{P}(A)}(-1)$ we see that

$$H^p(\mathbb{P}(A), \alpha_* E(-(n-1))) = H^p(Y, E(-(n-1)) = H^{p-(n-2)}(L \cap Y, E_{L\cap Y}) = \begin{cases} \mathsf{k}^{2m}, & \text{for } p = n-2 \\ 0, & \text{for } p \neq n-2 \end{cases}$$

Let us denote $V' = H^{n-2}(\mathbb{P}(A), \alpha_* E(-(n-1)))$ and recall that we have fixed an isomorphism $\gamma : V^* \cong H^0(\mathbb{P}(A), \alpha_* E)$. Summarizing, we see that

$$H^p(\mathbb{P}(A), \alpha_* E(t)) = \begin{cases} V^*, & \text{for } p = 0,\ t = 0 \\ V', & \text{for } p = n-2,\ t = -(n-1) \\ 0, & \text{for other } (p,t) \text{ with } -(n-1) \le t \le 0 \end{cases} \tag{17}$$

Now we can describe the sheaf $\alpha_* E$ via the Beilinson spectral sequence on $\mathbb{P}(A)$. It follows from (17) that the spectral sequence degenerates in the $(n+1)$-th term and gives

$$0 \to V' \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{f} V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to \alpha_* E \to 0.$$

Dualizing this sequence and twisting it by $\mathcal{O}_{\mathbb{P}(A)}(-1)$ we get

$$0 \to V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{f^*} V'^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to \mathcal{E}xt^1(\alpha_* E, \mathcal{O}_{\mathbb{P}(A)}(-1)) \to 0.$$

But since E is locally free on Y it follows that

$$\mathcal{E}xt^1(\alpha_* E, \mathcal{O}_{\mathbb{P}(A)}(-1)) \cong \alpha_*(E^* \otimes \alpha^* \mathcal{E}xt^1(\alpha_* \mathcal{O}_Y, \mathcal{O}_{\mathbb{P}(A)}(-1))) \cong \\ \cong \alpha_*(E^* \otimes \alpha^* \alpha_* \mathcal{O}_Y(m-1)) \cong \alpha_*(E^*(m-1)).$$

Since $\Lambda^2 E = \det E \cong \mathcal{O}_Y(m-1)$ by (16) it follows that there exists a skew-symmetric isomorphism $\sigma : E \to E^*(m-1)$ and $\alpha_* \sigma : \alpha_* E \to \alpha_* E^*(m-1)$. Since the Beilinson spectral sequence is functorial there exist unique isomorphisms $g : V' \to V$ and $h : V^* \to V'^*$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) & \xrightarrow{f} & V^* \otimes \mathcal{O}_{\mathbb{P}(A)} & \longrightarrow & \alpha_* E & \longrightarrow & 0 \\ & & \Big\downarrow g & & \Big\downarrow h & & \Big\downarrow \alpha_*\sigma & & \\ 0 & \longrightarrow & V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) & \xrightarrow{f^*} & V'^* \otimes \mathcal{O}_{\mathbb{P}(A)} & \longrightarrow & \alpha_*(E^*(m-1)) & \longrightarrow & 0 \end{array}$$

Dualizing this diagram and twisting it by $\mathcal{O}(-1)$ we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) & \xrightarrow{f} & V^* \otimes \mathcal{O}_{\mathbb{P}(A)} & \longrightarrow & \alpha_* E & \longrightarrow & 0 \\ & & \Big\downarrow h^* & & \Big\downarrow g^* & & \Big\downarrow \alpha_*\sigma^* & & \\ 0 & \longrightarrow & V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) & \xrightarrow{f^*} & V'^* \otimes \mathcal{O}_{\mathbb{P}(A)} & \longrightarrow & \alpha_*(E^*(m-1)) & \longrightarrow & 0 \end{array}$$

Now note, that due to skew-symmetry of σ we have $\sigma^* = -\sigma$. It follows that $h^* = -g$ and $g^* = -h$. Identfying V' with V via g and using the commutativity of the second diagram we see that

$$(hf)^* = f^* h^* = g^* f = -hf,$$

hence $hf \in \mathsf{Hom}(V \otimes \mathcal{O}_{\mathbb{P}(A)}(-1), V^* \otimes \mathcal{O}_{\mathbb{P}(A)}) = V^* \otimes V^* \otimes A^*$ is skew-symmetric with respect to V, therefore is given by an A-net of skew-forms $f : A \to \Lambda^2 V^*$. Finally, it is easy to see that the A-net f is regular (because E is locally free of rank 2), that Y is its Pfaffian hypersurface and that E is its theta-bundle. □

For any A-net of skew-forms $f : A \to \Lambda^2 V^*$ let X_f denote the scheme-theoretic intersection of the Grassmannian $\mathsf{Gr}(2, V) \subset \mathbb{P}(\Lambda^2 V)$ with the codimension n linear subspace $\mathbb{P}(f(A)^\perp) \subset \mathbb{P}(\Lambda^2 V)$.

Proposition A.4. *If f is a regular A-net then $\mathsf{sing}(X_f) = \mathsf{sing}(Y_f) = X_f \cap Y_f \subset \mathsf{Gr}(2, V)$. In particular, Y_f is smooth iff X_f is smooth.*

Proof: Let U be a 2-dimensional subspace of V. Then U lies on X_f iff U is isotropic with respect to all skew-forms from the A-net f. The tangent space to $\mathsf{Gr}(2,V)$ at U is $\mathsf{Hom}(U, V/U)$. The normal space of $\mathbb{P}(f(A)^\perp)$ in $\mathbb{P}(\Lambda^2 V)$ at $\Lambda^2 U$ is A^*. The map from the tangent space to the normal space $\mathsf{Hom}(U, V/U) \to A^*$ is dual to the map $A \otimes U \to (V/U)^*$, $(a,u) \mapsto f(a)(u,-)$. Therefore, U is a singular point of X_f iff U lies in the kernel of some skew-form from the A-net. Thus $\mathsf{sing}(X_f) = X_f \cap Y_f$.

On the other hand, let $a \in \mathbb{P}(A)$. Then a lies on Y_f iff $f(a)$ is a degenerate skew-form. Since f is regular, $\mathsf{rank} f(a) = 2m-2$, hence its kernel U is 2-dimensional. The tangent space to $\mathbb{P}(A)$ at a is $A/\mathsf{k}a$. The normal space of the locus of degenerate skew-forms in $\mathbb{P}(\Lambda^2 V)$ at $f(a)$ is $\Lambda^2 U^*$. The map from the tangent space to the normal space $A/\mathsf{k}a \to \Lambda^2 U^*$ is given by $a' \mapsto f(a')_{|U}$. Therefore, a is a singular point of Y_f iff all skew-forms forms from the A-net f vanish on U. Thus, $\mathsf{sing}(Y_f) = X_f \cap Y_f$. □

Appendix B. Instanton bundles on Fano threefolds of index 2

Let Y be a smooth Fano threefold of index 2, so that $\omega_Y = \mathcal{O}_Y(-2)$. Let $d = -c_1(\omega_Y)^3/8$ be the degree of Y.

Definition B.1. A sheaf $\mathcal{E}$ on Y is called *instanton bundle* if $\mathcal{E}$ is locally free of rank 2, stable and

$$c_1(\mathcal{E}) = 0, \qquad H^1(Y, \mathcal{E}(-1)) = 0.$$

The *topological charge* of an instanton $\mathcal{E}$ is an integer k, such that $c_2(\mathcal{E}) = k[l]$, where $[l] \in H^4(Y,\mathbb{Z})$ is the class of a line.

This definition is a straightforward analog of the definition of (mathematical) instanton vector bundle on $\mathbb{P}^3$, see [OSS].

Lemma B.2. *If $\mathcal{E}$ is an instanton vector bundle of charge k on a Fano threefold Y of index 2 then the dimensions of the cohomology spaces of twists of $\mathcal{E}$ are given by the following table:*

t	-3	-2	-1	0	1
$h^3(\mathcal{E}(t))$	$\le 2d$	0	0	0	0
$h^2(\mathcal{E}(t))$	$\le 2k-4$	$k-2$	0	0	0
$h^1(\mathcal{E}(t))$	0	0	0	$k-2$	$\le 2k-4$
$h^0(\mathcal{E}(t))$	0	0	0	0	$\le 2d$

where $h^p(\mathcal{E}(t)) = \dim H^p(Y, \mathcal{E}(t))$, and d is the degree of Y. Moreover,

$$h^3(\mathcal{E}(-3)) = h^0(\mathcal{E}(1)), \quad h^2(\mathcal{E}(-3)) = h^1(\mathcal{E}(1)), \text{ and } h^0(\mathcal{E}(1)) - h^1(\mathcal{E}(1)) = 2d - 2k + 4.$$

Proof: Note, that $h^0(\mathcal{E}(t)) = 0$ for $t \le 0$ by stability and that Serre duality gives

$$h^p(\mathcal{E}(t)) = h^{3-p}(\mathcal{E}^*(-2-t)) = h^{3-p}(\mathcal{E}(-2-t)).$$

Hence $h^3(\mathcal{E}(t)) = 0$ for $t \ge -2$ and $h^2(\mathcal{E}(-1)) = 0$. Choosing a generic codimension 3 plane section L of Y we get a Koszul resolution

$$0 \to \mathcal{E}(-2) \to \mathcal{E}(-1)^{\oplus 3} \to \mathcal{E}^{\oplus 3} \to \mathcal{E}(1) \to \mathcal{E}(1)_{|L} \to 0.$$

Since L is a 0-dimensional subscheme in Y we have

$$H^{>0}(Y, \mathcal{E}(1)_{|L}) = 0, \qquad H^0(Y, \mathcal{E}(1)_{|L}) = \mathsf{rank}(\mathcal{E}) \cdot \deg Y = 2d.$$

Hence the hypercohomology spectral sequence of the Koszul resolution implies $h^1(\mathcal{E}(-2)) = 0$. Then $h^2(\mathcal{E}) = 0$ by Serre duality, hence $h^2(\mathcal{E}(1)) = 0$ by the spectral sequence. Further, we have

$h^1(\mathcal{E}(-3)) = 0$ by Serre duality. And again from the spectral sequence we deduce $h^1(\mathcal{E}(1)) \le 2h^1(\mathcal{E})$. Finally, using the Riemann-Roch we get

$$h^1(\mathcal{E}) = -\chi(\mathcal{E}) = k-2, \quad h^0(\mathcal{E}(1)) - h^1(\mathcal{E}(1)) = \chi(\mathcal{E}(1)) = 2d-2k+4.$$

and lemma follows. □

Corollary B.3. *The minimal possible charge for instantons on a smooth Fano threefold of index* 2 *is* 2*, and if* $\mathcal{E}$ *is an instanton of charge* 2 *on* Y *then*

$$H^p(Y,\mathcal{E}(t)) = \begin{cases} \mathsf{k}^{2d}, & \text{for } (p,t)=(0,1) \text{ and } (p,t)=(3,-3) \\ 0, & \text{for other } (p,t) \text{ with } -3 \le t \le 1 \end{cases}$$

Following the analogy with instanton bundles on $\mathbb{P}^3$ we introduce the following.

Definition B.4. Let $L \subset Y$ be a line. We say that L is *jumping line* for an instanton $\mathcal{E}$ on Y if $\mathcal{E}_{|L} \cong \mathcal{O}_L(t) \oplus \mathcal{O}_L(-t)$ with $t > 0$.

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