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§ Quasi-abelian subcategory of an abelian category $\mathcal{A}$

An additive category $\mathcal{E}$ is quasi-abelian if

(1) $\mathcal{E}$ has kernels and cokernels

A morphism $e: E_1 \to E_2$ is strict if $\operatorname{coker} \ker e \xrightarrow{\sim} \ker \operatorname{coker} e$

(2a) pull-back of a strict epimorphism is a strict epimorphism

(2b) push-out of a strict monomorphism is a strict monomorphism

Equivalently:

(2') kernel-cokernel sequences $E_1 \xrightarrow{i} E \xrightarrow{d} E_2$ with i & d strict yield an exact structure on $\mathcal{E}$. (the canonical exact structure).

Recall: Exact structure on an additive category $\mathcal{E}$ is a family $\mathcal{S}$ of conflations, i.e. kernel-cokernel sequences

$$E_1 \xrightarrow{i} E \xrightarrow{d} E_2 \qquad i\text{-inflation} \quad d\text{-deflation}$$

closed under isomorphisms and such that

(1) For any $E \in \mathcal{E}$ $\quad 0 \to E \xrightarrow{id} E \in \mathcal{S}$

(2) Composition of two deflations is a deflation

(3) Pull-back of a deflation along an arbitrary morphism exists and is a deflation

(4) Push-out of an inflation along an arbitrary morphism exists and is an inflation

$$\begin{array}{ccccc} E_1 & \to & E & \to & E_2 \\ \| & & \uparrow & & \uparrow \\ E_1 & \to & E' & \to & E_2' \end{array}, \qquad \begin{array}{ccccc} E_1' & \to & E' & \to & E_2 \\ \uparrow & & \uparrow & & \| \\ E_1 & \to & E & \to & E_2 \end{array}$$

Example: (i) $\mathcal{A}$- additive, $\mathcal{S}_{triv}$ - split sequences $\quad (\mathcal{A}, \mathcal{S}_{triv})$-exact

(ii) $\mathcal{A}$-abelian $\quad \mathcal{S}_{can}$ - short exact sequences $\quad (\mathcal{A}, \mathcal{S}_{can})$-exact

(iii) $\mathcal{A}$-abelian $\quad \mathcal{E} \subset \mathcal{A}$ closed under extensions $\quad \mathcal{S}_{ind}$ $\quad E_1 \to E \to E_2$ s.t. $0 \to E_1 \to E \to E_2 \to 0$

$(\mathcal{E}, \mathcal{S}_{ind})$-exact

Gabriel-Quillen: Any $(\mathcal{E}, \mathcal{S})$ is of the form (iii) for $\mathcal{A} = \mathrm{Lex}(\mathcal{E}^{op}, Ab)$.

Example of quasi-abelian categories: $(\mathcal{T}, \mathcal{F})$ torsion pair on an abelian category $\mathcal{A}$

$\Rightarrow$ $\mathcal{T}, \mathcal{F}$ - quasi-abelian subcategories, canonical conflations: $F_1 \to F \to F_2$ s.t.

$0 \to F_1 \to F \to F_2 \to 0$ exact in $\mathcal{A}$ and $F, F_1, F_2 \in \mathcal{F}$

$f: F_1 \to F_2$ $\quad \ker_{\mathcal{F}} f = \ker_{\mathcal{A}} f \quad \mathrm{coker}_{\mathcal{F}} f = \mathcal{F}(\mathrm{coker}_{\mathcal{A}} f)$.

$(\mathcal{T}, \mathcal{F})$ is cotilting if any object A is a quotient of an object of $\mathcal{F}$.

$(\mathcal{T}, \mathcal{F})$ cotilting $\Rightarrow$ $\mathcal{A} \simeq \mathcal{A}_r(\mathcal{F})$ is the right abelian envelope of $(\mathcal{F}, \mathcal{S}_{can})$.

$\mathcal{E}$ - exact, $\mathcal{A}$-abelian additive $F: \mathcal{E} \to \mathcal{A}$ is right exact if $\forall$ $E_1 \to E \to E_2 \in \mathcal{S}$

$$F(E_1) \to F(E) \to F(E_2) \to 0 \text{ is exact.}$$

$\mathrm{Rex}(\mathcal{E}, \mathcal{A})$ - right exact functors + natural transformations.

$i_R \in \mathrm{Rex}(\mathcal{E}, \mathcal{A}_r(\mathcal{E}))$ is the right abelian envelope if $\forall$ $\mathcal{A}$-abelian

$(-) \circ i_R: \mathrm{Rex}(\mathcal{A}_r(\mathcal{E}), \mathcal{A}) \xrightarrow{\simeq} \mathrm{Rex}(\mathcal{E}, \mathcal{A})$ is an equivalence.

Schneiders: If $\mathcal{E}$ quasi-abelian then $\mathcal{A}_r(\mathcal{E}, \mathcal{S}_{can})$ exists.

$\mathcal{A}_r(\mathcal{E}, \mathcal{S}_{can})$ - localisation of the full subcategory of $\mathcal{K}(\mathcal{E})$ with objects

$$0 \to E_1 \xrightarrow{\delta_E} E_0 \to 0 \qquad \delta_E \text{ - monomorphism}$$

by the multiplicative system formed by morphisms:

$$\begin{array}{ccccccc} 0 & \to & E_1 & \xrightarrow{\delta_E} & E_0 & \to & 0 \\ & & u_1 \uparrow & & \uparrow u_0 & & \\ 0 & \to & F_1 & \xrightarrow{\delta_F} & F_0 & \to & 0 \end{array}$$

such that the square is both cartesian and cocartesian.

Further examples of quasi-abelian subcategories of $\mathcal{A}$:

torsion (free) part of a torsion pair on a quotient $\mathcal{A}/\mathcal{B}$.

$\mathcal{B} \subset \mathcal{A}$ Serre subcategory $\quad (0 \to A_1 \to A \to A_2 \to 0 \quad A \in \mathcal{B} \Leftrightarrow A_1, A_2 \in \mathcal{B})$

$\mathcal{C} = \mathcal{A}/\mathcal{B} \quad \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{A}$

$$\mathrm{Hom}_{\mathcal{C}}(A_1, A_2) = \left\{ A_1 \xleftarrow{s} A' \xrightarrow{f} A_2 \right\} / \sim \qquad \ker s,\ \mathrm{coker}\, s \in \mathcal{B}$$

$q : \mathcal{A} \longrightarrow \mathcal{C}$ quotient functor

$\mathcal{E} \subset \mathcal{A}$ full subcategory of $\mathcal{B}$-closed objects

$$\mathcal{E} = \{ E \in \mathcal{A} \mid \mathrm{Hom}(\mathcal{B}, E) = 0 = \mathrm{Ext}^1(\mathcal{B}, E) \}.$$

<u>Remark:</u> $\forall A \in \mathcal{A}\ \forall E \in \mathcal{E} \quad q: \mathrm{Hom}_{\mathcal{A}}(A, E) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(q(A), q(E))$

$A \xleftarrow{s} A' \xrightarrow{f} E$

$0 \to \ker s \to A' \to \mathrm{Im}\, s \to 0 \qquad 0 \to \mathrm{Hom}(\mathrm{Im}\, s, E) \to \mathrm{Hom}(A', E) \to \underset{0}{\mathrm{Hom}(\ker s, E)}$

$0 \to \mathrm{Im}\, s \to A \to \mathrm{Coker}\, s \to 0 \qquad \underset{0}{\mathrm{Hom}(\mathrm{Coker}\, s, E)} \to \mathrm{Hom}(A, E) \to \mathrm{Hom}(\mathrm{Im}\, s, E) \to \underset{0}{\mathrm{Ext}^1(\mathrm{Coker}\, s, E)}$

<u>Def:</u> $\mathcal{B} \subset \mathcal{A}$ is weakly localising if $\mathcal{E}$ is the torsion free part of a torsion pair on $\mathcal{A}/\mathcal{B}$.

($\mathcal{B} \subset \mathcal{A}$ is localising if $q : \mathcal{A} \to \mathcal{C}$ admits the right adjoint $\Rightarrow \mathcal{E} \simeq \mathcal{C}$).

$\mathcal{B} \subset \mathcal{A}$ weakly localising $\Rightarrow \mathcal{E} \subset \mathcal{A}$ quasi-abelian subcategory.

Canonical exact structure on $\mathcal{E}$:

$E_1 \xrightarrow{i} E \xrightarrow{d} E_2 \in S_{can} \Leftrightarrow 0 \to E_1 \xrightarrow{i} E \xrightarrow{d} E_2 \to B \to 0$ is exact in $\mathcal{A}$

with $B \in \mathcal{B}$.

§ Inducing a torsion pair on $\mathcal{A}/\mathcal{B}$.

$\mathcal{F} \subset \mathcal{A}$ satisfies the embedding condition with respect to $\mathcal{E}$ if

$\forall F \in \mathcal{F}\ \exists E \in \mathcal{E}\ \exists B \in \mathcal{B}$ and a short exact sequence

$0 \to F \to E \to B \to 0$ in $\mathcal{A}$.

<u>Prop:</u> $\mathcal{B} \subset \mathcal{A}$ Serre subcategory, $\mathcal{E} \subset \mathcal{A}$ $\mathcal{B}$-closed objects, $(\mathcal{T}, \mathcal{F})$ torsion pair on $\mathcal{A}$.

If $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{F}$ satisfies the embedding condition w.r.t. $\mathcal{E}$ then

$(q(\mathcal{T}), \mathcal{E})$ is a torsion pair on $\mathcal{A}/\mathcal{B}$.

If $(\mathcal{T}, \mathcal{F})$ is cotilting, so is $(q(\mathcal{T}), \mathcal{E})$.

<u>Thm:</u> If ${}^{\perp}\mathcal{E} = \{A \in \mathcal{A} \mid \mathrm{Hom}(A, \mathcal{E}) = 0\}$ is the torsion part of a torsion pair $({}^{\perp}\mathcal{E}, \mathcal{F})$ on $\mathcal{A}$ (for example is $\mathcal{A}$ is noetherian) then $\mathcal{B} \subset \mathcal{A}$ is weakly localising $\Leftrightarrow$ $\mathcal{F}$ satisfies the embedding condition with respect to $\mathcal{E}$.

$\mathcal{A} \subset \mathcal{D}$ the heart of a non-degenerate t-structure

$(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$ yields a new t-structure. Its heart $R_{\mathcal{T}}\mathcal{A}$ admits a torsion pair $(\mathcal{F}, \mathcal{T}[-1])$

$\mathcal{T}_0 \subset \mathcal{T}_1 \subset \ldots \subset \mathcal{T}_i$ torsion-tilting chain of subcategories if

$R_{\mathcal{T}_0[i]} \ldots R_{\mathcal{T}_{i-1}[-1]}(R_{\mathcal{T}_i}\mathcal{A})$ is well-defined.

$\mathcal{T}_0 \subset \mathcal{T}_1$ torsion-tilting in $\mathcal{A}$

$(\mathcal{T}_1, \mathcal{F}_1)$ - torsion pair $\mathcal{F}_1 \subset R_{\mathcal{T}_1}\mathcal{A}$. The decomposition in the torsion pair

$(\mathcal{T}_0[-1], \mathcal{F}_0)$ is the decomposition from the embedding condition.

<u>Thm:</u> $\mathcal{T}_0 \subset \mathcal{T}_1$ torsion-tilting chain in $\mathcal{A}$, $\mathcal{T} \subset \mathcal{A}$ is a Serre subcategory with the category $\mathcal{E}$ of $\mathcal{T}_0$-closed objects. If $\mathrm{Hom}(\mathcal{T}_1, \mathcal{E}) = 0$ then $\mathcal{F}_1$ satisfies the embedding condition w.r.t. $\mathcal{E}$, in particular $\mathcal{T} \subset \mathcal{A}$ is weakly localising.

A chain $\mathcal{T}_0 \subset \ldots \subset \mathcal{T}_i$ of full subcategories is strongly torsion-tilting if

$\mathcal{T}_0 \subset \ldots \subset \mathcal{T}_k$ is torsion-tilting for any $k \in \{0, \ldots i\}$.

§ strongly torsion-tilting chains of length 2.

$\mathcal{D}$ - triangulated category

$(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 1})$ bounded t-structure on $\mathcal{D}$ with the heart $\mathcal{X}$.

$\mathcal{T}_0 \subset \mathcal{T}_1$ strongly torsion-tilting chain on $\mathcal{X}$.

$\mathcal{I} := R_{\mathcal{J}_1}\mathcal{X}$ $\quad\mathcal{U} = R_{\mathcal{J}_0}\mathcal{X}$ $\quad\mathcal{Y} = R_{\mathcal{J}_0[-1]}(R_{\mathcal{J}_1}\mathcal{X})$.

Then the pair $(\mathcal{X},\mathcal{Y})$ satisfies

(i) $\mathcal{X}^{\leq 0} \subset \mathcal{Y}^{\leq 0} \subset \mathcal{X}^{\leq 2}$ of amplitude 2

(ii) $\forall k \in \mathbb{Z}$ $\tau_{\mathcal{X}}^{\leq k}\mathcal{Y}^{\leq 0} \subset \mathcal{Y}^{\leq 0}$ $\quad\tau_{\mathcal{Y}}^{\geq k}\mathcal{X}^{\geq 0} \subset \mathcal{X}^{\geq 0}$ T-consistent.

Thm: A pair $(\mathcal{X},\mathcal{Y})$ is T-consistent of amplitude 2 $\Leftrightarrow$ $\mathcal{X} \cap \mathcal{Y}^{\leq -2} \subset \mathcal{X} \cap \mathcal{Y}^{\leq -1}$ is a strongly torsion tilting chain on $\mathcal{X}$.

$\mathcal{X},\mathcal{Y}$ t-structures on $\mathcal{D}$

intersection: $\mathcal{I}^{\leq 0} \subset \mathcal{X}^{\leq 0} \cap \mathcal{Y}^{\leq 0}$ maximal among those naive if $\mathcal{I}^{\leq 0} = \mathcal{X}^{\leq 0} \cap \mathcal{Y}^{\leq 0}$

union: $\mathcal{U}^{\geq 0} \subset \mathcal{X}^{\geq 0} \cap \mathcal{Y}^{\geq 0}$ maximal among those naive if $\mathcal{U}^{\geq 0} = \mathcal{X}^{\geq 0} \cap \mathcal{Y}^{\geq 0}$

$\mathcal{I} = \mathcal{X}.\mathcal{Y}$ $\qquad \mathcal{U} = \mathcal{X} + \mathcal{Y}$

Bondal: $(\mathcal{X},\mathcal{Y})$ T-consistent pair of amplitude n. All the unions and intersections generated by $\mathcal{X}$ and $\mathcal{Y}$ are naive.

§ Isomorphisms in codimension 1

X – Noetherian separated scheme of finite type $/k$, $\omega_X^{\cdot} \in D^b(X)$ dualizing complex

$(\mathrm{Coh}(X)^{\leq 0}, \mathrm{Coh}(X)^{\geq 1})$ standard t-structure on $D^b(X)$

$\mathrm{Coh}_\omega(X)^{\leq i} = \{ E \in D^b(X) \mid R\mathcal{H}om(E, \omega_X^{\cdot}) \in \mathrm{Coh}(X)^{\geq -i} \}$

$\mathrm{Coh}_\omega(X)^{\geq i} = \{ E \in D^b(X) \mid R\mathcal{H}om(E, \omega_X^{\cdot}) \in \mathrm{Coh}(X)^{\leq -i} \}$

Bondal: If $\dim X = n$ then $(\mathrm{Coh}(X), \mathrm{Coh}_\omega(X)[-n])$ is a T-consistent pair of amplitude n.

$(\mathcal{X},\mathcal{Y})$ T-consistent of amplitude n $\Rightarrow$ $(\mathcal{X}, \mathcal{X}[-2].\mathcal{Y})$ is T-consistent of amplitude 2.

$(\mathcal{X}[-2].\mathcal{Y})^{\leq 0} = \mathcal{X}^{\leq 2} \cap \mathcal{Y}^{\leq 0}$

Cor: X - Noethenian separated scheme of finite type and dimension n.

$\Rightarrow$ $Coh_{\leq n-2}(X) \subset Coh_{\leq n-1}(X)$ is a strongly torsion-tilting chain on $Coh(X)$.

$Coh_{\leq \ell}(X) = \{ E \in Coh(X) \mid \dim \operatorname{Supp} E \leq \ell \} =$

$= \{ E \in Coh(X) \mid R\mathcal{H}om(E, \omega_X^\cdot) \in Coh(X)^{\geq -\ell} \}.$

$\mathcal{E}_{n-2}(X) := \{ E \in Coh(X) \mid \mathcal{H}om(Coh_{\leq n-2}(X), E) = 0 = \mathcal{E}xt^1(Coh_{\leq n-2}(X), E) \} =$

Thm: X - Noethenian separated scheme of finite type and dimension n.

The quotient $Coh^{\geq n-1}(X) := Coh(X) / Coh_{\leq n-2}(X)$ admits a torsion pair $(q(Coh_{\leq n-1}(X)), \mathcal{E}_{n-2}(X))$.

If X is equidimensional with no embedded points then this torsion pair is cotilting, i.e. $Coh^{\geq n-1}(X) \simeq \mathcal{A}_r(\mathcal{E}_{n-2}(X), \mathcal{S}_{can})$.

Calabrese-Pinsi: X, Y - schemes. X, Y are isomorphic outside of dimension $k-1$ $\Leftrightarrow$ $Coh^{\geq k}(X) \simeq Coh^{\geq k}(Y)$.

X, Y isomorphic outside of dimension $k-1$ $\Leftrightarrow$ $\exists\, U \subset X$ $V \subset Y$ isomorphic open subsets s.t. $\forall\, p \in X$ $\dim p \geq k$ $\forall\, q \in Y$ $\dim q \geq k$ $p \in U$ $q \in V$.

Cor: X and Y are isomorphic outside of dimension $n-2$ $\Leftrightarrow$ $\mathcal{E}_{n-2}(X) \simeq \mathcal{E}_{n-2}(Y)$.

§ The final model of a normal surface

X-normal surface (irreducible, separated k-scheme of finite type, k-alg. closed)

$\mathcal{E}_0(X) = Ref(X)$ determines X up to open subset with complement of dim 0.

$Coh^{\geq 1}(X)$ admits a torsion pair $\left(\bigoplus_{D \in X^1} Coh_D^{\geq 1}(X), \mathcal{E}_0(X) \right)$

X^1 - set of 1-dim'l points of X

$Coh_D^{\geq 1}(X) := Coh_D(X)/Coh_D^{\leq 0}(X)$ - extension closure of $\mathcal{O}_D$ $\quad End(\mathcal{O}_D)$ = residue field of $D \in X^1$

$\bigoplus Coh_D^{\geq 1}(X)$ - the Serre subcategory generated by simple objects in $Coh^{\geq 1}(X)$.

$U \subset X$ open $\quad D = X \setminus U$ - divisor $\quad D = \bigcup_{i \in I} D_i$

$\bigoplus_{i \in I} Coh_{D_i}^{\geq 1}(X) \subset Coh^{\geq 1}(X)$ Serre subcategory; $\quad Coh^{\geq 1}(X) / \bigoplus_{i \in I} Coh_{D_i}^{\geq 1}(X) \simeq Coh^{\geq 1}(U)$.

Schröer - Vezzosi: $Coh(X)$ has enough locally free sheaves

$A_r(Ref(X), S_{geom}) = Coh(X) \qquad A_r(Ref(X), S_{can}) = Coh^{\geq 1}(X)$

$\Rightarrow$ the centers are equivalent

$Z(Coh^{\geq 1}(X)) = Z(Ref(X)) = Z(Coh(X)) = H^0(X, \mathcal{O}_X)$ - integrally closed k-algebra of finite type and $\dim \leq 2$.

$X^{aff} = \mathrm{Spec}\ H^0(X, \mathcal{O}_X)$

X is CM-affine if $\quad Hom(\mathcal{O}_X, -): Ref(X) \xrightarrow{\simeq} Ref(X^{aff})$.

Calabrese - Pirisi: Up to an autoequivalence of $Coh^{\geq 1}(X)$, $\mathcal{O}_X$ is the object E s.t.

(i) for any $P \in Coh^{\geq 1}(X)$ simple $\quad Hom(E, P)$ - 1-dim'l over $End(P)$

(ii) E is locally maximal with this property: E' satisfies (i) $\forall E' \twoheadrightarrow E$ is an iso.

Prop: X is CM-affine $\iff g: X \to X^{aff}$ open embedding with complement of codimension 2.

Define

$$\hat{X} = \underset{\substack{U \subset X \text{ CM-affine}, \\ X \setminus U - \text{divisor}}}{\mathrm{colim}} U^{aff}$$

Thm: $\hat{X}$ - separated normal surface, $U \hookrightarrow U^{aff}$ glue to an open immersion $j_X: X \hookrightarrow \hat{X}$ with complement of codimension 2.

Remark: $\hat{X}$ can be constructed directly from the additive category $\mathrm{Ref}(X)$.

X is final if $j_X : X \hookrightarrow \hat{X}$ is an isomorphism.

($\mathrm{Coh}^{\geq 1}(U)$ is a quotient of $\mathrm{Coh}^{\geq 1}(X) = \mathcal{A}_r(\mathcal{E}_0(X))$ by Serre subcat, $\mathcal{O}_U \in \mathrm{Coh}^{\geq 1}(U)$ has categorical char $\Rightarrow \mathrm{End}(\mathcal{O}_U) \rightsquigarrow U^{aff}$ $\mathcal{E}_0(U) \subset \mathrm{Coh}^{\geq 1}(U)$ the torsion-free class of a canonical torsion pair)

Thm: X is final

$\Leftrightarrow$ any open embedding $X \hookrightarrow X'$ with complement of codim 2 is an isomorphism

$\Leftrightarrow \forall\, i : U \hookrightarrow X$ open quasi-affine U^{aff} admits an open embedding $i' : U^{aff} \to X$ $i'|_U = i$

$\Leftrightarrow \exists$ proper normal surface Y and an open embedding $X \hookrightarrow Y$ s.t.

(a) $Y \setminus X$ is a divisor D

(b) the contraction of any negative definite connected component D' of D yields a proper algebraic space.

Ex: X - proper over affine is final (can be reconstructed from $\mathrm{Ref}(X)$).

Hironaka: Y - blow-up of $\mathbb{P}^2$ in 10 points in general position on an elliptic curve C

$D \subset Y$ the strict transform of C

$X = Y \setminus D$ is final, not proper over $X^{aff} = \mathrm{Spec}\, k$.

Sketch of proof of properties of $\hat{X}$.

$U \subset V$ open embedding U, V - CM - affine $\Rightarrow U^{af} \subset V^{af}$ open embedding $\Rightarrow \hat{X}$ - scheme

$U \subset V \subset V^{af}$ $V \setminus V^{af} = Z$ $\dim Z = 0$

$V^{af} \setminus U = D \cup Z'$

<u>Nagata:</u> $V^{af} \setminus D$ is affine $\Rightarrow U^{af} = V^{af} \setminus D$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \bar{X} \arrow[d] \\ & X^{aff} \end{tikzcd}$$ - Nagata's compactification

$f : \bar{X} \dashrightarrow U^{aff}$ $\quad f|_U = id|_U$ $\quad f : X_u \to U^{aff}$ is proper (Zariski's main thm)

$\hat{f} : \bar{X} \dashrightarrow \hat{X}$ proper on its domain $\quad \hat{X} = \cup U^{aff}$ $\quad \hat{f}^{-1}(U^{aff})$ has finite subcovering

$\Rightarrow \hat{X} = U_1^{aff} \cup \ldots \cup U_n^{aff}$ $\qquad U_i^{aff} \setminus U_i$ - artinian scheme

$\Rightarrow X \hookrightarrow \hat{X}$ open embedding with artinian complement.

Valuative criteria $\Rightarrow$ $\hat{X} \longrightarrow X^{aff}$ is separated $\Rightarrow$ $\hat{X}$ separated.