Euler continuants, N-spherical functors and periodic semi-orthogonal decompositions

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Universal continued fractions

$$R_N = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_N}}} \in \mathbb{Q}(x_1, \cdots, x_N)$$

$$R_2 = x_1 - \frac{1}{x_2} = \frac{x_1 x_2 - 1}{x_2}$$

$$R_3 = x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3}} = \frac{x_1 x_2 x_3 - x_1 - x_3}{x_2 x_3 - 1}$$

NB: We can make the x_i noncommutative: $R_n \in$ any skew field containing $\mathbb{Q}\langle x_1, \cdots, x_N \rangle$.

Euler continuants (noncommutative, alternating)

$I \subset \{1, \cdots, N\}$ called cotwinned, if its complement is a (possibly empty) disjoint union of "twins" $\{i, i+1\}$. Depth of I is $\mathrm{dep}(I) = \#(\text{such twins})$. Ordered product x_I of $x_i, i \in I$.

$$E_N(x_1, \cdots x_N) := \sum_{I \subset \{1, \cdots, N\} \text{Cotwinned}} (-1)^{\mathrm{dep}(I)} x_I \in \mathbb{Z}\langle x_1, \cdots, x_N \rangle .$$

$$E_1(x) = x,$$
$$E_2(x_1, x_2) = x_1x_2 - 1,$$
$$E_3(x_1, x_2, x_3) = x_1x_2x_3 - x_1 - x_3,$$
$$E_4(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3 - x_3x_4 + 1, \quad \text{etc.}$$

$\#$ of monomials $= \varphi_N$ (Fibonacci) $1, 2, 3, 5, \cdots$

Continuants and continued fractions

Noncommutative $R_N = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \cfrac{1}{x_N}}}$ is represented as

$$R_N \;=\; P_N Q_N^{-1} \;=\; (Q'_N)^{-1} P'_N, \quad \text{where}$$
$$P_N = E_N(x_1, \cdots, x_N), \quad Q_N = E_{N-1}(x_2, \cdots, x_N),$$
$$P'_N = E_N(x_N, \cdots, x_1), \quad Q'_N = E_{N-1}(x_N, \cdots, x_2).$$

Categorification: continuant complexes of adjoints

$\mathcal{A}, \mathcal{B}$ pre-triangulated dg-categories, $F : \mathcal{A} \to \mathcal{B}$ dg-functor,
$F^* : \mathcal{B} \to \mathcal{A}$ right adjoint (appropriate sense, also ∞-cat. version)

$$\text{counit} : FF^* \Rightarrow \mathrm{Id}_{\mathcal{B}}, \quad \text{unit} : \mathrm{Id}_{\mathcal{A}} \Rightarrow F^*F$$

A chain of iterated adjoints $(F_1, \cdots, F_N)$: $F_i = F_{i-1}^*$. Suppose $\exists$.

$I = \{i_1 < \cdots < i_p\} \subset \{1, \cdots, N\}$ *cotwinned* $\rightsquigarrow F_I = F_{i_1} \cdots F_{i_p}$
Composite functor makes sense!

Nth continuant chain complex of functors

$$\mathcal{E}_N(F_1, \cdots, F_n) = \left\{ F_1 \cdots F_N \to \bigoplus_{\mathrm{dep}(I)=1} F_I \to \bigoplus_{\mathrm{dep}(I)=2} F_I \to \cdots \right\}$$

Differential made of counits.
Dually, the Nth cochain complex $\mathcal{E}^N(F_N, \cdots, F_1)$ (reverse order).
Differential made of units.

Examples

$$\mathcal{E}_2(F, F^*) = \{FF^* \xrightarrow{c} \mathrm{Id}\}, \quad \mathcal{E}^2(F^*, F) = \{\mathrm{Id} \xrightarrow{u} F^*F\}$$

Totalizations (i.e. cones of c, u) = spherical twist, cotwist.
F a spherical functor: when these *Tots* are equivalences.

$$\mathcal{E}_3(F, F^*, F^{**}) = \{FF^*F^{**} \longrightarrow F \oplus F^{**}\}, \quad \mathcal{E}^3: \text{ dually}$$

Kuznetsov's definition of spherical functors : $\operatorname{Tot} \mathcal{E}_3 = 0$ and of $\mathcal{E}^3$.

$\mathcal{E}_4 = \{FF^*F^{**}F^{***} \to FF^* \oplus FF^{***} \oplus F^{**}F^{***} \to \mathrm{Id}\}$.

Define the Nth spherical twist and cotwist of a functor F:
$\mathbb{E}_N(F) = \operatorname{Tot} \mathcal{E}_N(F, F^*, \cdots, F^{(N-1)})$
$\mathbb{E}^N(F) = \operatorname{Tot} \mathcal{E}^N(F^{(N-1)}, \cdots, F^*, F)$.
(Co)units allow us to categorify the continuants.

N-spherical functors

Def. A (dg-)functor F (s.t. adjoints $\exists$) is called N-spherical, if $\mathbb{E}_{N-1}(F) = \mathbb{E}^{N-1}(F) = 0$.

Prop. In this case $\mathbb{E}_{N-2}(F)$ and $\mathbb{E}_N(F)$ are equivalences and similarly for $\mathbb{E}^{N-2}, \mathbb{E}^N$.

Reason: Categorification of classical formula ("continued fractions give best approximation")

$$R_{N+1} - R_N = \frac{-1}{Q_N Q'_{N+1}}, \quad \text{or, equivalently}$$

$$Q'_{N+1} P_N - P'_{N+1} Q_N = -1, \quad \text{or, equivalently}$$

(!)
$$E_N(x_1, \cdots, x_N) E_N(x_{N+1}, \cdots, x_2) - \\ - E_{N+1}(x_1, \cdots, x_{N+1}) E_{N-1}(x_N, \cdots, x_2) = 1.$$

Meaning of N-spherical for small N

2-spherical means $F = 0$.

3-spherical means that F is an equivalence.

4-spherical means spherical in the usual sense: Kuznetsov's definition is via $\mathbb{E}_3 = \mathbb{E}^3 = 0$. His arg. categorifies the identity

$$(ab - 1)(cb - 1) - (abc - a - c)b = 1$$

which is an instance of (!).

If N is odd and $F : \mathcal{A} \to \mathcal{B}$ is N-spherical, then $\mathcal{A}$ is equivalent to $\mathcal{B}$ via $\mathbb{E}_{N-2}$ or $\mathbb{E}_N$.

Semi-orthogonal decompositions and gluing functors

[Bondal-K., 1990] $\mathcal{C}$ triangulated $\supset \mathcal{A}, \mathcal{B}$ full triangulated. Said to form an SOD, (notation $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ and $\mathcal{A}$ called left admissible) if

$$\mathcal{A} = \mathcal{B}^{\perp} := \{A : \mathrm{Hom}(B, A) = 0, \forall B \in \mathcal{B}\}, \quad \mathcal{B} = {}^{\perp}\mathcal{A},$$

and any $C \in \mathcal{C}$ includes into into a triangle

$$B \longrightarrow C \longrightarrow A \longrightarrow B[1], \quad A \in \mathcal{A},\ B \in \mathcal{B}.$$

Gluing functor [Bondal, Kuznetsov-Lunts] $F : \mathcal{A} \to \mathcal{B}$ (if $\exists$) s.t.

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{B}}(F(A), B).$$

In dg-setting: can construct an SOD with any F as $\mathcal{C} = S_1(F)$
Ob = data $\big(A, B, \gamma : B \to F(A)$ closed degree 0 morphism$\big)$.
First level of relative Waldhausen S-construction.
For stable ∞: Dyckerhoff-K-Schechtman-Soibelman [2106.02873]

N-Periodic SODs

Iterated orthogonals

$$\cdots {}^{\perp\perp}\mathcal{A} = \mathcal{A}^{(-2)},\ {}^{\perp}\mathcal{A} = \mathcal{A}^{(-1)},\ \mathcal{A} = \mathcal{A}^{(0)},\ \mathcal{A}^{\perp} = \mathcal{A}^{(1)},\ \mathcal{A}^{\perp\perp} = \mathcal{A}^{(2)}, \cdots$$

Can happen that $\mathcal{A}^{(N)} = \mathcal{A}$ (periodic SOD).

Thm. In the dg-setting, for a dg-functor F TFAE:

(i) The glued (along F) SOD $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is N-periodic.

(ii) F is N-spherical.

For $N = 4$ this is due to Halpern-Leinster and Shipman.

Rem. For any ∞-admissible chain of orthogonals (each $(\mathcal{A}^{(i)}, \mathcal{A}^{(i-1)})$ is an SOD) we have mutation equivalences $\mathcal{A}^{(i)} \to \mathcal{A}^{(i+2)}$. The equivalences $\mathbb{E}_{N-2}(F)$, $\mathbb{E}_N(F)$ are compositions of such mutations.

Why continued fractions?

Continued Fractions $\sim$ compositions of FLT $\sim$ of 2×2 matrices

$$z \mapsto a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_N - \cfrac{1}{z}}}} \quad \text{is a FLT} \quad \frac{az+b}{cz+d}$$

composition of

$$z \mapsto a_i - \frac{1}{z} = \frac{a_i z - 1}{z}, \quad \text{matrix} = \begin{bmatrix} a_i & -1 \\ 1 & 0 \end{bmatrix}$$

Continuants and continued fractions $\sim$ multiplying such matrices.

Matrix calculus for functors between SODs

(Dg- or stable ∞-context) Suppose:

$\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$, so $\mathcal{A}_i \underset{\text{proj.}}{\overset{\text{emb.}}{\rightleftarrows}} \mathcal{A}$, with gluing functor $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$,

$\mathcal{B} = \langle \mathcal{B}_1, \mathcal{B}_2 \rangle$, so $\mathcal{B}_i \underset{\text{proj.}}{\overset{\text{emb.}}{\rightleftarrows}} \mathcal{B}$, with gluing functor $\psi : \mathcal{B}_1 \to \mathcal{B}_2$,

$F : \mathcal{A} \to \mathcal{B}$: (dg- or exact ∞-) functor $\overset{1:1}{\leftrightsquigarrow}$ "Enhanced matrix", i.e.,

Matrix of functors $\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$, $\quad F_{ij} : \mathcal{A}_j \to \mathcal{B}_i$

$+$ Nat. transformations $\quad \psi F_{1j} \Rightarrow F_{2j}, \quad F_{i1} \Rightarrow F_{i2}\varphi$

such that the two ways to paste a transformation $\psi F_{11} \Rightarrow F_{22}\varphi$ are the same ("commutative tetrahedron").

Such enhanced matrices can be composed.

Mutation coordinate change as a Cont. Fr.-matrix

Suppose $\mathcal{A} \subset \mathcal{C}$ is an admissible subcategory, i.e.,

$$\mathcal{C} = \begin{cases} \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle, & \text{with gluing functor } \varphi : \mathcal{A} \to {}^{\perp}\mathcal{A} \\ \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle, & \text{gluing functor then } \varphi^* M[1]. \end{cases}$$

$M : \mathcal{A}^{\perp} \xrightarrow{\sim} {}^{\perp}\mathcal{A}$ mutation.

(Enhanced) matrix of $\langle \mathcal{A}^{\perp}, \mathcal{A} \rangle \xrightarrow{\mathrm{Id}_{\mathcal{C}}} \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$ is of Cont. Fr. type

	$\mathcal{A}^{\perp}$	$\mathcal{A}$
$\mathcal{A}$	$\varphi^* \circ M[1]$	Id
${}^{\perp}\mathcal{A}$	M	0

This explains the relevance of continued fractions in the theory of SODs

Examples of N-periodic SOD's: quivers

Ex.1: A_n-quiver. $\mathcal{C} = D^b\{V_1 \to \cdots \to V_n\} = \{V_1^\bullet \to \cdots \to V_n^\bullet\}$.

$$\mathcal{A} = \{V^\bullet \to 0 \to \cdots \to 0\}, \quad \mathcal{B} = \{0 \to V_2^\bullet \to \cdots \to V_n^\bullet\}$$

$\langle \mathcal{A}, \mathcal{B} \rangle$ is a $2(n+1)$-periodic SOD.

NB: Here $\mathcal{C}$ is fractional CY: $\text{Serre}^{n+1} = [-2]$. So any SOD is $2(n+1)$ periodic, as $\mathcal{A}^{\perp\perp} = \text{Serre}(\mathcal{A})$.

Similarly for other quivers, e.g., $\mathcal{C}$ consist of

$$V_1^\bullet \to V_2^\bullet \to \cdots \to V_{n-2}^\bullet \begin{array}{l} \nearrow V_n^\bullet \\ \searrow V_{n-1}^\bullet \end{array} \qquad \mathcal{A} = \{\text{only } V_1^\bullet \neq 0\}$$

Example: Waldhausen S-construction

Ex.2. $f : \mathcal{A} \to \mathcal{B}$ usual (4-)spherical functor $\rightsquigarrow$ $S_n(f)$ nth Waldhausen category. Ob $= \{B_1 \to \cdots \to B_n \to f(A)\}$. Has SOD

$$\langle \mathcal{B}, \cdots, \mathcal{B}, \mathcal{A} \rangle = \langle \mathcal{D}, \mathcal{A} \rangle, \quad \mathcal{D} = \langle \mathcal{B}, \cdots, \mathcal{B} \rangle$$

It is $2(n+1)$-periodic.

This is because $S_\bullet(f) = (S_n(f))_{n \geq 0}$ is a paracyclic object, see [DKSS 2106.02873]. Paracyclic rotation τ_n acts on $S_n(f)$ with $\tau_n^{n+1} =$ "monodromy of the schober". Also the SOD

$$\langle \text{first } \mathcal{B}, \mathcal{E} \rangle, \quad \mathcal{E} = \langle \text{second } \mathcal{B}, \cdots, \mathcal{B}, \mathcal{A} \rangle.$$

N-spherical objects

X smooth projective, $\omega = \Omega_X^n[n]$, $n = \dim X$, $E \in D^b\mathrm{Coh}_X$ object.

$$\mathcal{A} = D^b(\mathrm{Vect}) \xrightarrow{F = - \otimes E} \mathcal{B} = D^b\mathrm{Coh}_X$$
$$\xleftarrow{F^* = \mathrm{Hom}(E,-)}$$
$$\xrightarrow{F^{**} = - \otimes E \otimes \omega}$$
$$\xleftarrow{F^{(3)} = \mathrm{Hom}(E \otimes \omega, -)}$$
$$\xrightarrow{F^{(4)} = - \otimes E \otimes \omega^{\otimes 2}}$$

E is called an *N*-spherical object, if $F = - \otimes E$ is an N-spherical functor.
Examples related to $X = \mathrm{CY}/\mathbb{Z}_n$ (generalized Enriques mflds).
[In progress].

Relation to other work

T. Kuwagaki [1902.04269]: N-periodic SOD are categorical analogs of irregular connections near $\infty \in \mathbb{C}$ with exponential data (Lissajous figure) being a $2:1$ covering of S^1_∞ with N switches.

Such as for $\mathbb{C}$- Schrödinger $\psi'' = P(z)\psi$, $P(z) \in \mathbb{C}[z]$, $\deg = N$.

P. Boalch [1501.00930] Moduli space of Stokes data for $\mathbb{C}$-Schrödinger (following Shibuya) related to Euler continuants.

M. Fairon, D. Fernandez [2105.04858] Continuants as group valued moment maps for some multiplicative quiver varieties. NB: by [Bezrukavnikov-Kapranov 1506.07050] these varieties parametrize microlocal sheaves on the nodal curve which is the complexification of the Lissajous figure above ($\mathbb{CP}^1$'s instead of circles).