Conference in honour of Alexey Bondal's 60th birthday, Dec. 15-17, 2021

Group actions on cluster categories from Ginzburg morphisms

Mostly based on joint work with Chris Fraser.

Plan: 1. Overview: the problem and its solution

2. A geometric example: the rel. cluster category of the Grassmannian

3. Generalization

1. Overview: the problem and its solution

Facts: ① Many cluster algebras / cluster varieties carry braid (sub-)group actions.

② One would expect them to come from braid (sub-) actions on the corresponding triangulated cluster categories.

③ The braid (sub-)group should act by (compos. of) spherical twists.

Problem: Cluster categories typically do not contain any spherical objects.

Solution: Use relative cluster categories (Yilin Wu, Ph. D. 12/2021).

Details: Relative cluster categories typically do not contain spherical objects either but each is a quotient of the target of a Ginzburg functor

$$
\begin{tikzcd}
\mathrm{per}\,\Pi \arrow[r, "G"] & \mathrm{per}\,\Gamma \arrow[r, "\text{quotient}"] & \mathcal{C}^{rel} \\
\circlearrowright & \circlearrowright & \circlearrowright \\
Br_\Pi & & \\
\rotatebox{90}{$\subseteq$} & & \\
U \arrow[r, squiggly, "\text{subtle}"'] & U \arrow[r, squiggly, "\text{easy}"'] & U
\end{tikzcd}
$$

contains a spherical coll. (S_i)

associated braid group

relative cluster cat.

2. A geometric example: the Grassmannian rel. cluster category

We follow Jensen-King-Su (2016). Fix integers $0 < k < n$.

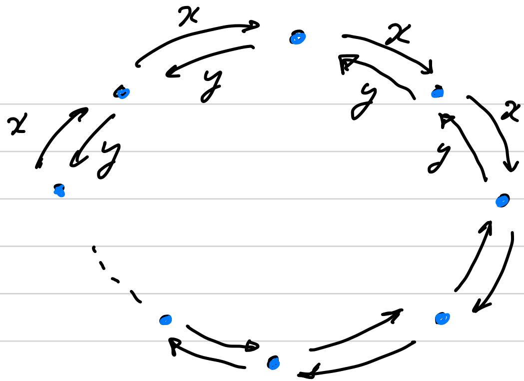
Consider $\quad S = \mathbb{C}[[x,y]] \longrightarrow R = \mathbb{C}[[x,y]]/(x^k - y^{n-k})$.

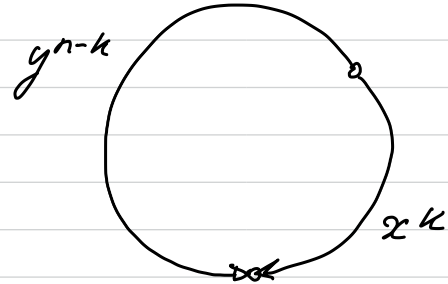
$\mu_n = \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$ acts on S and R via $\zeta.(x,y) = (\zeta x, \zeta^{-1} y)$.

We get a $\mathbb{Z}/n\mathbb{Z}$-graded singularity

$$\begin{tikzcd} S * \mu_n \arrow[r] \arrow[d, phantom, "\wr\|"] & R * \mu_n \arrow[d, phantom, "\|"] \\ \Pi & B \end{tikzcd}$$

completed preproj. algebra of type $\tilde{A}_{n-1}$ $\qquad$ "boundary algebra"





n vertices

preproj. rel.: $xy = yx$

$$\Pi \twoheadrightarrow B = \Pi/(x^k - y^{n-k})$$

On the preprojective algebra Π

Π is noetherian, of gldim. 2 and 2-Calabi-Yau as a bimodule (Ginzburg): $\mathrm{RHom}_{\Pi^e}(\Pi, \Pi^e) \simeq \Sigma^{-2}\Pi$ in $\mathcal{D}(\Pi^e)$

Thus, the simple Π-modules S_i, $0 \le i \le n-1$, form a spherical collection in

$$\mathrm{per}(\Pi) = (\text{perfect der. cat. of } \Pi) = \mathrm{thick}(\Pi) \subset \mathcal{D}\Pi$$

and the associated braid group $Br_\Pi = \langle \sigma_i \mid 0 \le i \le n-1 \rangle$ acts on $\mathrm{per}\,\Pi$ via the spherical twists tw_{S_i} (Seidel-Thomas 2001).

On the boundary algebra B

$B = \Pi/(x^k - y^{n-k})$ is of infinite global dimension when $2 \le k$ and $2 \le n-k$. Let us assume this.

We have

$$\mathcal{D}^b(\mathrm{mod}\,B) \longrightarrow sg(B) = \mathcal{D}^b(\mathrm{mod}\,B)/\mathrm{per}(B)$$

$$\mathcal{D}^b(\mathrm{mod}\,B) = \mathcal{C}^{rel}, \qquad sg(B) = \mathcal{C}^{abs}$$

relative Grassmannian cluster category (Hom-infinite)

abs. Grassmannian cluster category (Hom-finite, 2-Calabi-Yau)

Summary

$$\begin{tikzcd}
\mathrm{per}\,\Pi \arrow[r, "G"] \arrow[loop below, "Br_\Pi"] \arrow[rr, bend left, "? \otimes_\Pi B"] & \mathrm{per}\,\Gamma \arrow[r] & \mathcal{C}^{rel} \arrow[r] \arrow[d, equal] & \mathcal{C}^{abs} \arrow[d, equal] \\
 & \text{to be defined} & \mathcal{D}^b(\mathrm{mod}\,B) & sg(B)
\end{tikzcd}$$

Γ and the Ginzburg functor

fin. gen. right B-modules

Put $\mathrm{cm}(B) = \{\text{Cohen-Macaulay } B\text{-modules}\} = \{ M \in \mathrm{mod}\, B \mid \mathrm{Ext}^i_B(M,B) = 0 \;\; \forall i > 0 \}$

We have:

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{rel}} = \mathcal{D}^b(\mathrm{mod}\, B) & \longrightarrow & \mathrm{sg}(B) = \mathcal{C}^{\mathrm{abs}} \\ \uparrow \wr & & \uparrow \wr \\ \mathcal{D}^b(\mathrm{cm}\, B) & \longrightarrow & \underline{\mathrm{cm}}(B) = \mathrm{cm}(B)/(\text{proj.-inj.}) \end{array}$$

Thm (Jensen-King-Su 2016 based on Geiss-Leclerc-Schröer 2006):

$\mathrm{cm}(B)$ contains a canonical (basic) cluster-tilting object T, i.e.

a) T is rigid: $\mathrm{Ext}^1_B(T,T) = 0$

b) for each $M \in \mathrm{cm}(B)$, there is a s.e.s. $0 \to T_1 \to T_0 \to M \to 0$

with $T_i \in \mathrm{add}\, T = \{\text{direct summands of finite direct sums of copies of } T\}$

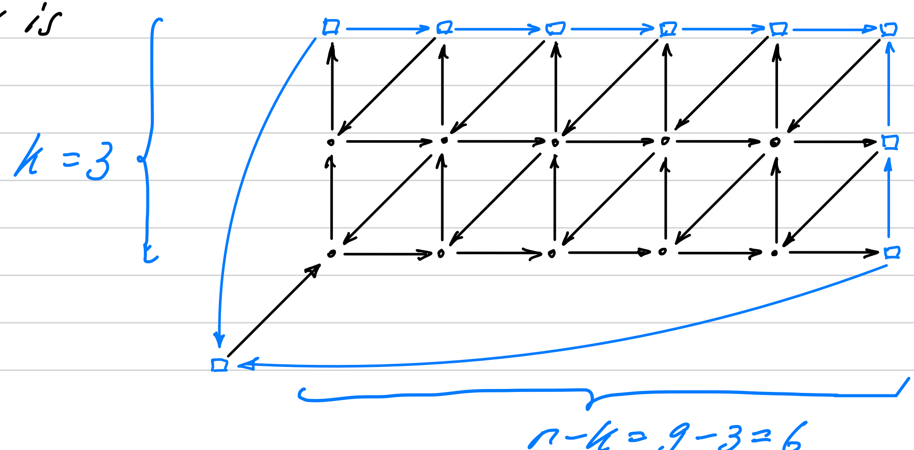
Def.: $\Gamma := \mathrm{End}(T)$.

Rk: The indecomposable projective B-modules $P_i = e_i B$ are direct summands of $T = \bigoplus_{i=0}^{n-1} P_i \oplus \bigoplus_{j=n}^{N} T_j$. So we get the **Ginzburg morphism** $\Pi \longrightarrow B \subseteq \mathrm{End}(T) = \Gamma$. The algebra Γ is of global dimension 3 but <u>not</u> 3-CY as a bimodule. However, the Ginzburg morphism carries a canonical **left relative 3-CY structure** in the sense of Brav-Dyckerhoff (2016). We get

$$\begin{tikzcd}
\mathrm{per}\,\Pi \arrow[r] \arrow[rr, bend left, "? \overset{L}{\underset{\Pi}{\otimes}} B"] & \mathrm{per}\,\Gamma \arrow[r] & \mathcal{D}(\mathrm{mod}\,B) = \mathcal{C}^{\mathrm{rel}}
\end{tikzcd}$$

$$\begin{array}{ccccccc}
e_i\Pi & \longmapsto & e_i\Gamma & \longmapsto & P_i & & 0 \leq i \leq n-1 \\
 & & e_j\Gamma & \longmapsto & T_j & & n \leq j \leq N .
\end{array}$$

Rk: The cluster-tilting object $T \in \mathrm{cm}(B)$ yields the "non commutative resolution" $\Gamma = \mathrm{End}(T)$ of B. It is canonical but **not unique**. For example, we can *mutate* T at non proj. indec. summands. This corresponds to Fomin-Zelevinsky mutation of the ice quiver of $\Gamma = \mathrm{End}(T)$. E.g. for $Gr(k,n) = Gr(3,9)$, this ice quiver is



frozen vertices = indec. projectives

$\updownarrow$

boundary algebra $B \subseteq \mathrm{End}(T)$

$B = \Pi/(x^k - y^{n-k})$

Braid subgroup action

Recall our aim:

$$\begin{tikzcd}
 & & \mathcal{D}^b(\mathrm{mod}\,B) \arrow[d, equal] \\
\mathrm{per}\,\Pi \arrow[r, "G"] \arrow[loop below] & \mathrm{per}\,\Gamma \arrow[r, "\text{quotient}"] & \mathcal{C}^{\mathrm{rel}} \\
Br_\Pi & & \\
U \arrow[u, phantom, "\cup\!\!\mid" description] \arrow[r, squiggly, "\text{subtle}"'] & U \arrow[uu, dashed, bend right] \arrow[r, squiggly, "\text{easy}"'] & U \arrow[uu, dashed, bend right]
\end{tikzcd}$$

contains a spherical coll. (S_i)

associated braid group

relative cluster cat.

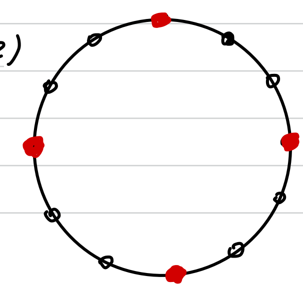
to be defined.

We have $Br_\Pi = \langle \sigma_i \mid 0 \le i \le n-1 \rangle$. Put $d = \gcd(k,n)$. Assume $d > 1$.

Define $\overline{\sigma_j} = \prod_{i \equiv j\,(d)} \sigma_i$, $0 \le j \le d-1$.

$(k,n) = (3,12)$

$\overline{\sigma_0} = \prod_{i \equiv 0\,(3)} \sigma_i$



Put $U = \langle \sigma_j \mid 0 \leq j \leq d-1 \rangle \subseteq Br_{\Pi}$ (coincidence: $U \cong Br_{\tilde{A}_{d-1}}$).

Thm (Fraser-K): Given $u \in U$, there is a unique morphism $\tilde{u} : \operatorname{per}\Gamma \xrightarrow{\sim} \operatorname{per}\Gamma$ such that the square

$$\begin{tikzcd} \operatorname{per}\Pi \arrow[r] \arrow[d, "u"', "\wr"] & \operatorname{per}\Gamma \arrow[d, "\tilde{u}", "\wr"'] \\ \operatorname{per}\Pi \arrow[r] & \operatorname{per}\Gamma \end{tikzcd}$$

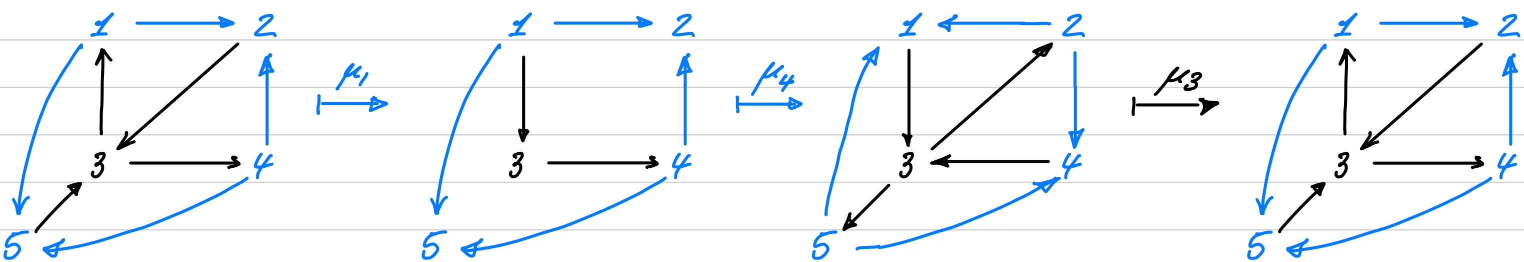
commutes (in the Morita homotopy category of dg categories).

Cor.: The group $U \cong Br_{\tilde{A}_{d-1}}$ acts on the rel. Grassmannian cluster cat. $\mathcal{D}^b(\operatorname{mod} B)$, $B = \Pi/(x^k - y^{n-k})$.

= quasi-cluster

Prop. (Fraser–K): Each equivalence $\tilde{\sigma}_j$ is quasi-reachable, i.e. a composition of equivalences associated with mutations at non projective (= non frozen) and at projective (= frozen) indec. summands of T. It categorifies Fraser's quasi-cluster automorphism $\tilde{\sigma}_j : \widetilde{Gr}(k,n) \xrightarrow{\sim} \widetilde{Gr}(k,n)$ of the Plücker cone $\widetilde{Gr}(k,n)$.

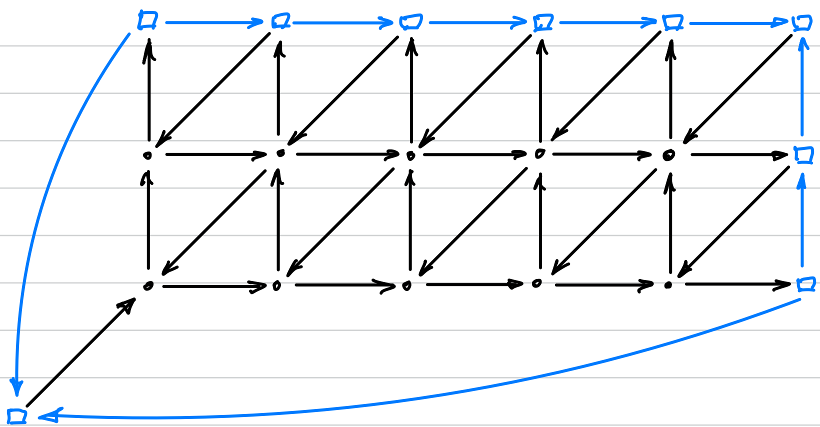
Example: $Gr(2,4)$: $\overline{\sigma_0} = \sigma_2 \sigma_4$, $\tilde{\sigma}_0$: composition of 2 frozen, 1 non fr. mut.



3. Generalization

We start "from the resolution", i.e. from an ice quiver (Q, F) with potential $W \in HH_0(\mathbb{C}Q)$, e.g.

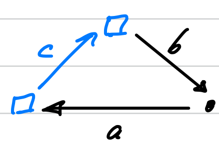
Q:



F = frozen subquiver

$$W = \sum \text{clockwise cordless cycles} - \sum \text{counterclockwise c.c.}$$

Smaller example:

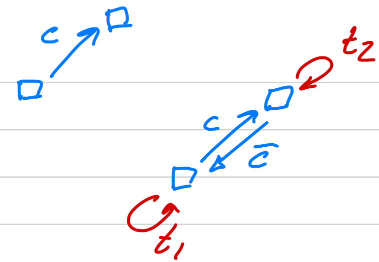


, $W = abc$

$$\left(\text{Cluster structure on } \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \subset SL_3(\mathbb{C}) \right)$$

Geiss–Leclerc–Schröer

Put F = frozen subquiver of Q, e.g.



Π = 2-CY completion of kF, e.g.

= derived 2-preprojective alg. of F

$|t_i| = -1$

$|c| = |\bar{c}| = 0$

$d(t_1) = -\bar{c}c$

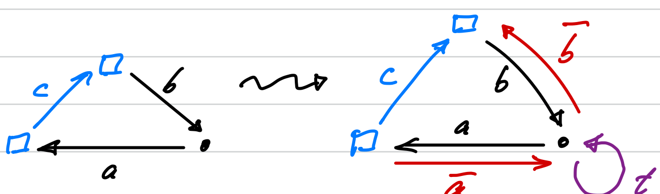
$d(t_2) = c\bar{c}$

Notice: $H^0\Pi$ = ordinary preproj. alg. of F

F connected non ADE quiver $\Longrightarrow$ $\Pi \xrightarrow[\text{qis}]{} H^0\Pi$.

Π is smooth and 2-CY as a bimodule.

Put Γ = relative Ginzburg algebra of (Q, F, W), e.g.



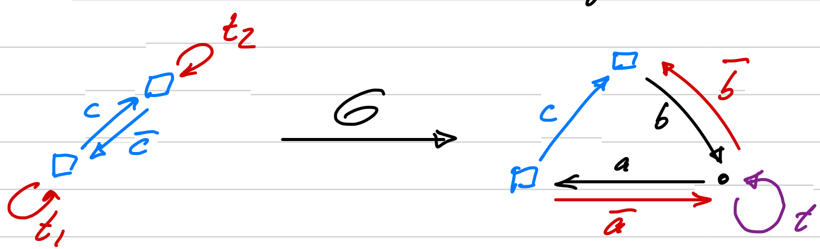
$W = abc$

$|\bar{a}| = |\bar{b}| = -1$, $d(\bar{a}) = \partial_a W = bc$, $d(\bar{b}) = \partial_b W = ca$

$|t| = -2$, $d(t) = b\bar{b} - \bar{a}a$.

Rk: Surprisingly, Γ is often qis to $H^0\Gamma$! (much more often than the absolute Ginzburg algebra). This holds for the above example and for the Grassmannian ice quiver with potential!

The **Ginzburg morphism** $G : \Pi \longrightarrow \Gamma$ is defined as in



$$\begin{aligned}
c &\longmapsto c \\
\bar{c} &\longmapsto \boxed{\partial_c W} = ab \\
t_1 &\longmapsto a\bar{a} \\
t_2 &\longmapsto -\bar{b}\, b
\end{aligned}$$

Thm (Yeung '16): G carries a can. rel. left 3-CY structure in the sense of Brav–Dyckerhoff.

The **boundary algebra** is $\mathcal{B} = e\cdot(H^0\Gamma)\cdot e$, $e = \sum_{i\in F_0} e_i$, e.g.

$H^0\Gamma$:

$$\begin{tikzcd} & \boxed{2} \arrow[dr, "b"] & \\ \boxed{1} \arrow[ur, "c"] & & 3 \arrow[ll, "a"] \end{tikzcd}$$

$bc = 0$, $ca = 0$

$e = e_1 + e_2$: $e(H^0\Gamma)e$:

$$\begin{tikzcd} \boxed{1} \arrow[r, shift left, "c"] & \boxed{2} \arrow[l, shift left] \end{tikzcd}$$

$\bar{c} = ab$, $c\bar{c} = 0$, $\bar{c}c = 0$

preproj. alg. of A_2.

We get

$$\begin{tikzcd} \mathrm{per}\,\Pi \arrow[loop below, "Br_\Pi"] \arrow[r, "G^*"] & \mathrm{per}\,\Gamma \arrow[r] & \mathcal{C}^{rel} := \mathrm{per}\,\Gamma / \mathrm{thick}(S_i \mid i \notin F) \end{tikzcd}$$

(usually NOT equiv. to $\mathcal{D}^b(\mathrm{mod}\,\mathcal{B})$!)

Thm (Fraser-K) : Suppose that

a) $\Gamma \xrightarrow[\text{qis}]{} H^0\Gamma$.

b) $H^0G : H^0\Pi \longrightarrow B = e(H^0\Gamma)e$ is surjective (enough: is an epimorphism of $Z(H^0\Pi)$-algebras).

Then for $\sigma \in \Pi$, there is at most one morphism $\tilde{\sigma} : \mathrm{per}\,\Gamma \xrightarrow{\sim} \mathrm{per}\,\Gamma$ s.th. the square

$$\begin{tikzcd}
\mathrm{per}\,\Pi \arrow[r] \arrow[d, "\sigma"', "\wr"] & \mathrm{per}\,\Gamma \arrow[d, "\wr"', "\tilde{\sigma}"] \\
\mathrm{per}\,\Pi \arrow[r] & \mathrm{per}\,\Gamma
\end{tikzcd}$$

commutes (in the Morita homotopy category of dg categories).

Thus, the group $U = \{\sigma \in Br_\Pi \mid \exists\, \tilde{\sigma}\}$ acts on $\mathrm{per}\,\Gamma$ and $\mathcal{C}^{rel}$.

composition of equivalences assoc. with mutations at non frozen and at frozen vertices

Cor.: The group $U_{cl} = \{ \sigma \in \Pi \mid \tilde{\sigma}$ exists and is quasi-cluster$\}$ acts by by quasi-cluster automorphisms on the cluster algebra $\mathcal{A}(Q,F)$ (with invertible coefficients) associated with (Q,F).

Confirmed examples: Grassmannian braiding (Fraser), Positroid "braiding" (Fraser-K)

Expected examples: Actions by

- braid group on unipotent upper triang. matrices (Bondal '04, Chekhov-Shapiro '20)
- finite type Artin braid groups (Fock-Goncharov '06)
- Weyl groups and braid groups (Goncharov-Shen '16, Inoue-Lam-Pylyavskyy '16, Inoue-Ishibashi-Oya '19, Goncharov-Shen '19)
- braid groups on deformed Groth. rings (Kashiwara-Kim-Oh-Park '20)