Classical R-H corresp.

1° X smooth alg. variety $/\mathbb{C}$

$(\mathcal{E}, \nabla)$ $\overset{\sim}{\longleftrightarrow}$ $\mathbb{C}$. local systems of finite rk on $X(\mathbb{C})$

alg. bundles on X $\quad\quad\quad\quad$ $\wr\wr$

∇ flat conn. $\quad$ reg. sing. $\quad$ Fin. Repr. of $\pi_1(X(\mathbb{C}), x_0)$

2° X, $\quad$ Holonomic $\mathcal{D}$-mod with reg. sing. $\overset{\sim}{\longleftrightarrow}$ Perverse sheaves $/\mathbb{C}$

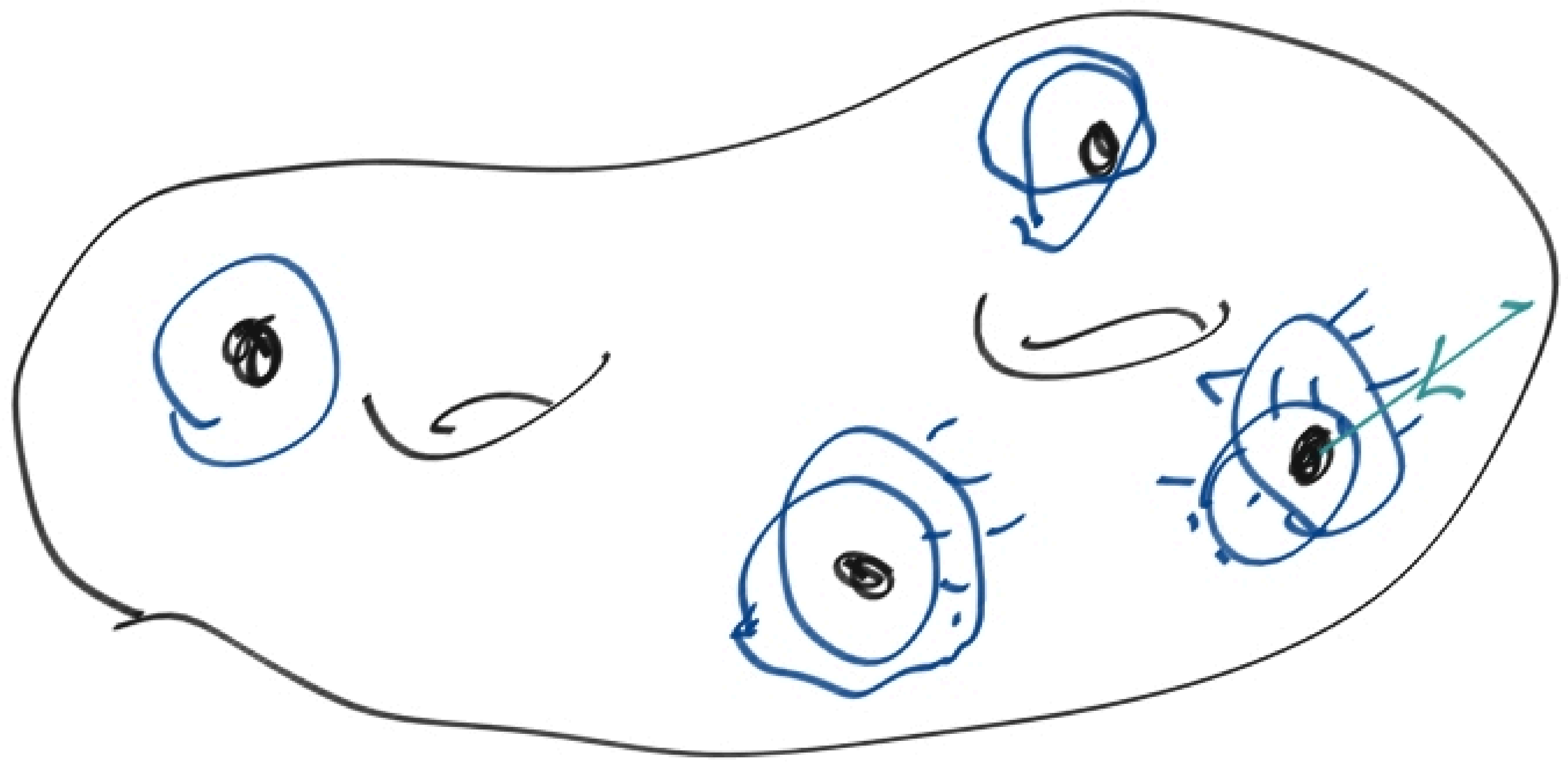
3° $\dim X = 1$ $\quad$ irreg. hol $\mathcal{D}$-mod. (special case of bundle. ∇) $\quad$ Deligne-Malgrange

$X = \overline{X} -$ finite set S

compact curve

$\mathcal{E}$ alg.
$d\nabla$
X

$\left\{ \varepsilon \cdot e^{\mathrm{Re}\, f_\alpha(\varepsilon e^{i\theta})} \cdot e^{i\theta} \in \mathbb{C} \right\}$
$\theta \in \mathbb{R}/2\pi\mathbb{Z}$



local solution around, $x \to x_i$ $\quad \nabla \psi = 0$

$e^{\frac{1}{x-x_i}}$

Puiseux polynom. $(x-x_i)$
$f_\alpha(x-x_i)$

$$e^{\sum_{\text{finite},\ a/b<0} c_{a/b}(x-x_i)^{a/b}}$$

⇝ cooriented curves.

if $\mathrm{Arg}(x-x_i) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\quad \underbrace{\left| e^{\frac{1}{x-x_i}} \right|} \gg 1$

Bundles with conn

$\longleftrightarrow$ Constr. sheaves on $\overline{X}$

$=0$ near x_i

μsupp. $\subset$ zero-sect. $+ \bigcup N^{*}_{+}$ (coor. lines)

Up to now: $\nexists$ satisfactory description of the Betti side of RH corr in higher-dimensional case,

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D-modules $\rightsquigarrow$ modules over quantum torus

$$\mathcal{A}_1^{(q)} = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1} \rangle / XY = qYX \qquad q \in \mathbb{C}^\times$$

$$\mathcal{A}_n^{(q)} = \underbrace{\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1}_{n \text{ copies}} \qquad x_i, y_i \ldots$$

<u>Assumption</u>: $|q| \neq 1$ $\qquad (0 < |q| < 1)$

Holonomic q-$\mathcal{D}$-modules

$\mathcal{M} \in \mathcal{A}_1$-mod

$\exists$ filtration

$\mathcal{M}_{\le 0} \subset \mathcal{M}_{\le 1} \subset \dots$

$$X^{\pm 1}, Y^{\pm 1} \cdot \mathcal{M}_{\le k} \subseteq \mathcal{M}_{\le k+1}$$

$$\dim \mathcal{M}_{\le k} \le \text{const} \cdot k.$$

($\Rightarrow$ fin. generated)

Finite dim. Homs, Ext

numerically CY
(same is true
for hol. D-modules)

$$\chi(\mathrm{Ext}(\mathcal{M}, \mathcal{N})) = \chi(\mathrm{Ext}(\mathcal{N}, \mathcal{M}))$$

Similar to D-modules $\supset$ "vector bundles"

$\boxed{q \text{ is fixed}}$

$q \in \mathbb{C}^\times$

$\underline{|q|<1}$

$\mathcal{E}$ alg. vector bundle

$\downarrow$

$\mathbb{G}_m = \mathbb{C}^\times$ $\quad X =$ coordinate on $\mathbb{C}^\times$

+ iso: $\mathcal{E} \rightsquigarrow$ pullback of $\mathcal{E}$ by $X \to qX$

$q^{\mathbb{Z}}$ $\quad \rightsquigarrow$ bundle on $E = \mathbb{C}^\times / q^{\mathbb{Z}}$

Classification by Ramis – Sauloy – Zhang:
2009

Subproblem: germ of punctured disk $0<|x|\ll 1$

bundle on it
with meron. structure at 0
$\Leftrightarrow$ vector space over
field $\mathbb{C}\{x\}[x^{-1}]$

q-equiv. bundles.

For  $/q^{\mathbb{Z}}$: $\rightsquigarrow$ holom. bundle on $E = \mathbb{C}^\times/q^{\mathbb{Z}}$

filtration by rational #s

$\varphi_j \in \text{fiber}|_{x_0} \rightsquigarrow \underbrace{\varphi_1 \in \text{fiber}|_{x_1 = qx_0}}_{\text{const.}\, j^2} \rightsquigarrow \varphi_2 : \cdots$

$|\varphi_j| \le e^{\dots}$

$$x \to x^{\frac{1}{2}\frac{a}{b}\frac{\log x}{\log q}} = e^{\frac{a}{b}\frac{(\log x)^2}{(\log q)}}$$

Filtration of hol. bundle $\mathcal{F}$ / $E_q = \mathbb{C}^\times / q^{\mathbb{Z}}$

$\mathcal{F}_{\le a/b}$ $\quad \mathrm{gr}_{a/b}\, \mathcal{F}_{a/b}$ is semistable with slope a/b

Note: this is not Harder-Narasimhan filtration (opposite order on $\mathbb{Q}$)

it is "anti HN"

$\Longrightarrow$ RH corr. over $\mathbb{G}_{m,\mathbb{C}}$

q-diff vector bundles $\overset{\sim}{\longleftrightarrow}$ $\mathcal{F} \in \mathrm{Vect}(E_q)$ with two anti HN filtration include singularities

$\rightsquigarrow$ A_q-Hol modules $\sim$ non-abelian A_∞-cat

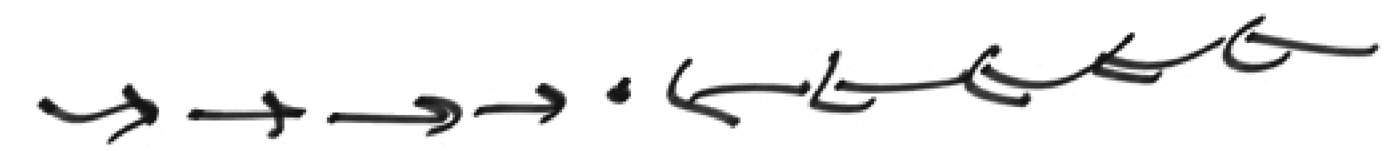
Coh. sheaves on E_q with the ext. rel

allow torsion sheaves

$\leftarrow \infty \cup \mathbb{Q}$

$$M = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}[x, x^{-1}] / (x - q^n x_0)$$

finite

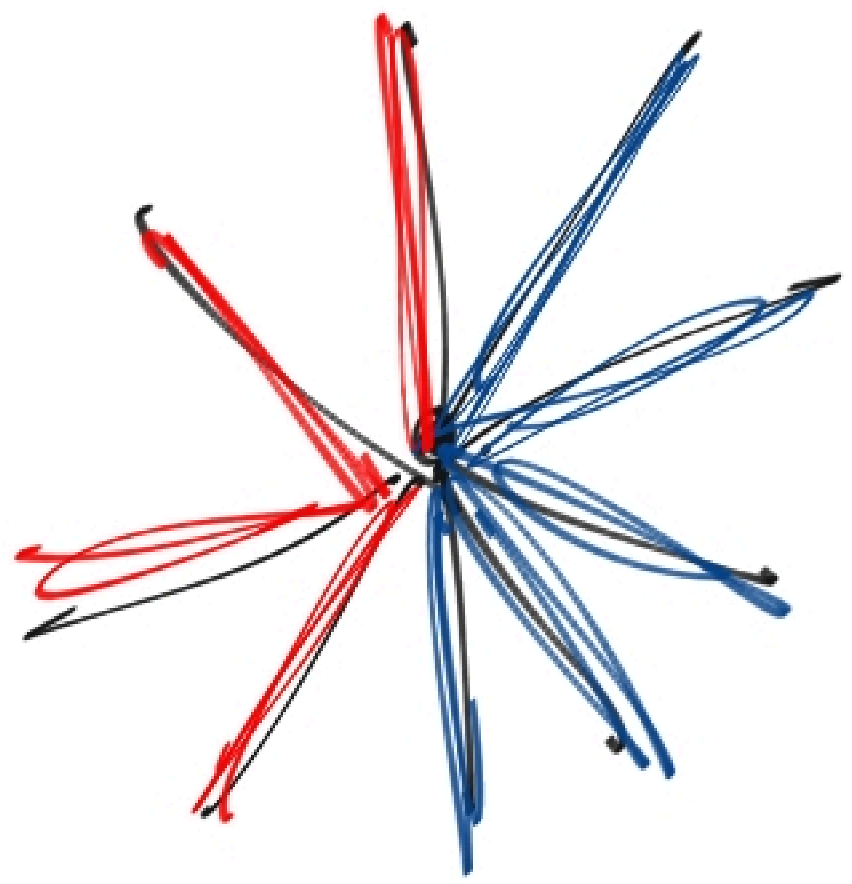


two filtrations

Fukaya (

) $\subset \mathbb{R}^2$

Fix a finite $\overset{N}{\text{coll.}}$ of $\mathbb{Q}$-rays in $\mathbb{R}^2$



$D^b(A_{N-1}\text{-mod})$

$\mathrm{Fukaya}(\;) \longrightarrow \oplus N$ copies $D^b(\mathbb{C}\text{-mod})$

Full subcat. of $\mathrm{Fuk}(\;) \otimes D^b(\mathrm{Coh}(E_q))$

Condition: Apply i-th functor $\mathrm{Fu}(\;) \to D^b(\mathbb{C}\text{-m})$ $i = 1 \ldots n$ we land in $[\ldots]$ of semistable ~~bundles~~ coh. sheaves of given slope

Generalization to $\forall$ dim

(hypothetical) $A_n^{(q)} = \left(A_1^{(q)}\right)^{\otimes n}$ $\qquad 0 < |q| < 1$

Hol. D-modules in n-dim $\qquad \mathcal{M}_{\leq 0} \subset \mathcal{M}_{\leq 1} \ldots$

$\dim \mathcal{M}_{\leq k} \leq \text{const} \cdot k^n$

$\rightsquigarrow$ sing. support: $\mathbb{Q}$-Lagr. polyhedral cone

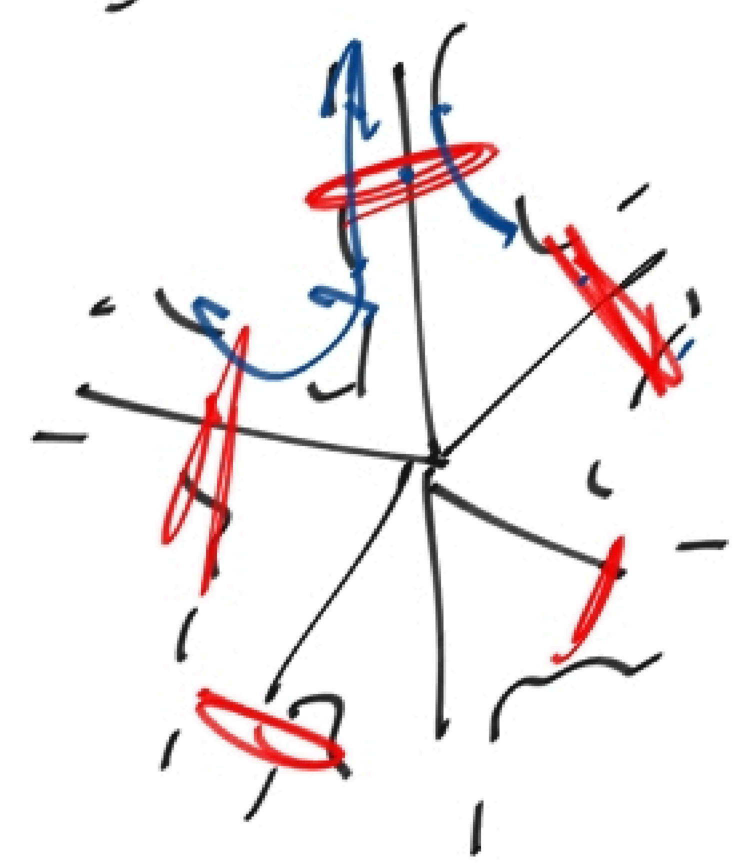
$L \subset \mathbb{R}^{2n}$

$L = \coprod_{\text{fin}} L_\alpha \qquad L_\alpha \sim \mathbb{R}^n_{\geq 0}$

$\mathbb{Q}$-Lagrangian.

Fix L; Consider only Hol qu $\mathcal{D}$-modules $\boxed{\text{with Sing.Supp} \subset L.}$

1): $\mathcal{F}(L) :=$ A_∞ cat. defined $/\mathbb{Z}$



U = tubular neighborhood

transversal lagr. disc D_α to gener. point of L_α octants

WF = wrapped Fukaya

$\mathcal{A}_L := \mathrm{End}_{WF}(\oplus D_\alpha)$ finite-dim modules of $\mathcal{A}_L$

$\forall$ L_α functor. $\phi_{\alpha}(L) \to D^b(\text{vect.-spaces})$

Ambiguity by shifts: $\mathbb{Z}$

$$\beta \quad L_\alpha \rightsquigarrow pt \in LGr(\mathbb{R}^{2n})$$

$$\begin{tikzcd}
pt \in LGr(\mathbb{R}^{2n}) \\
\text{choice } \widetilde{pt} \in \widetilde{LGr}(\mathbb{R}^{2n}) \arrow[u, "\mathbb{Z}\text{-covering}"']
\end{tikzcd}$$

2) $D^b(Coh((E_q)^n))$

$\widetilde{Sp(2n,\mathbb{Z})}$ equivariant

$\mathbb{R}^n$ vertical $\rightsquigarrow$

for each $\mathbb{Q}$-$\widetilde{pt} \in \widetilde{LGr}(\mathbb{R}^{2n})$

$\downarrow$

full subcategory

$Coh_{\text{zero-dim supp.}}((E_q)^n)$

$SL(2,\mathbb{Z})$, $\otimes$ line bundles -invariant hypothetical

Formulation:

Hol. Dim, simple CL $\overset{\sim}{\longleftrightarrow}$ full subcategory of $\mathcal{F}(L) \otimes D^b(\mathrm{Coh}(E_q)^n)$

consisting of those object

s.t $\forall$ $\vdash$ / L_α octants $\mathcal{F}(L) \xrightarrow{\Phi_\alpha} D^b(\mathrm{Vect})$,

the image lies in the corresponding full subcategory of $D^b(\mathrm{Coh}(E_q)^n)$

$|q|=1$?

RH corresp. for $|q|<1$ $(n=1)$

$q = e^{2\pi i \tau}$ $\quad \mathrm{Im}\,\tau > 0$

$\tau' = -\frac{1}{\tau}$

(more generally: $\tau' = \frac{a\tau+b}{c\tau+d}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$)

$q' = e^{2\pi i \tau'}$ $\quad E_q \sim E_{q'}$

Hol. A_q-mod $\simeq$ Hol $A_{q'}$-mod

(analytic equivalence)

"Betti" $\boxed{\cancel{\ast}\ \cancel{\ast} \otimes D^b(E_q)}$

∞ many alg. str. on moduli stack $\underline{\text{analytic}}$

∞ many ell. Khov. versions.

Claim: for $|q|=1$ there is no Betti - alg. str.

(?)

still ∞ many de Rham alg. st

$PSL(2,\mathbb{Z})$ acts on $\mathbb{RP}^1 \ni \tau$.

$|q|=1$

$q \notin \mu_\infty$.

$\tau = \sqrt{2}$!

Analogy: X alg (compact) $/\mathbb{C}$ $\qquad$ $\overline{X}$ alg $/\overline{\mathbb{C}}$

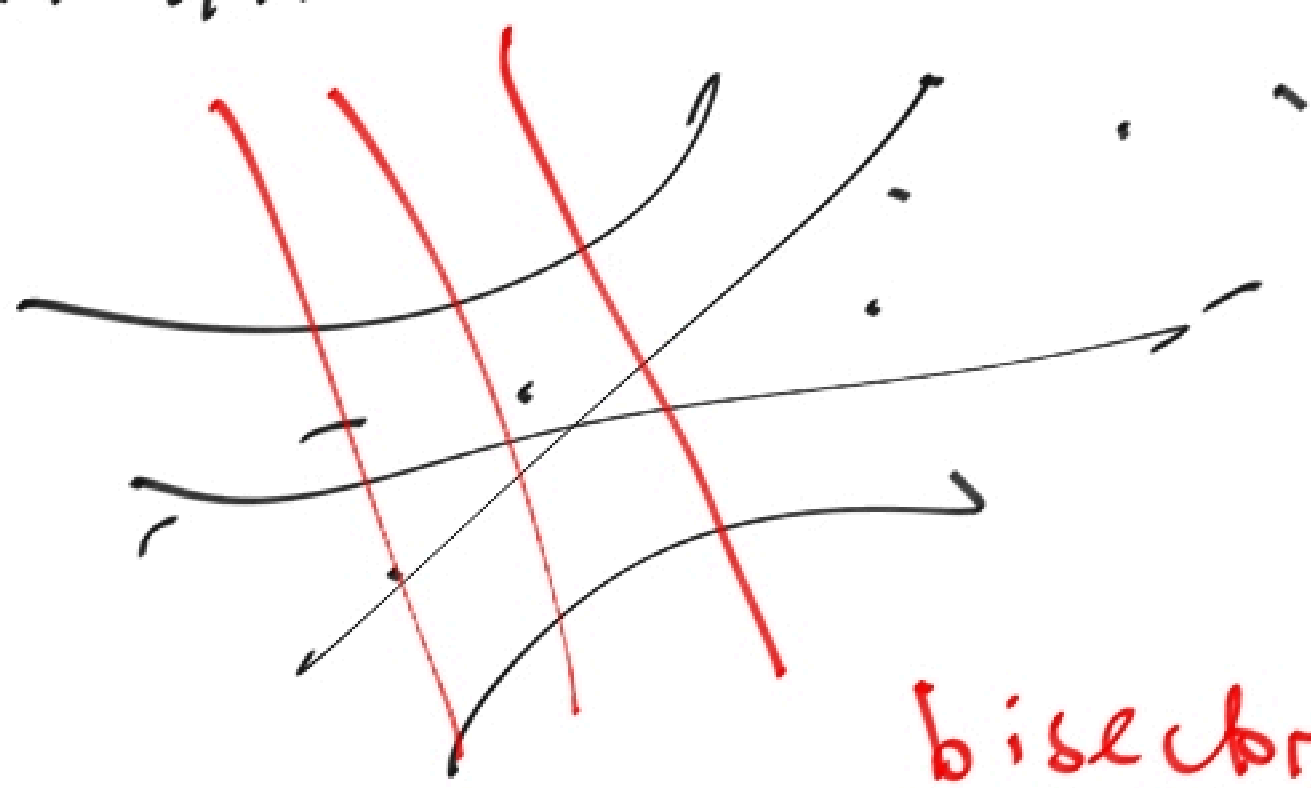
T. Mochizuki.
(Conjecture of Kashiwara)

hol $\mathcal{D}$-mod on X $\simeq$ Hol. $\mathcal{D}$-mod on $\overline{X}$

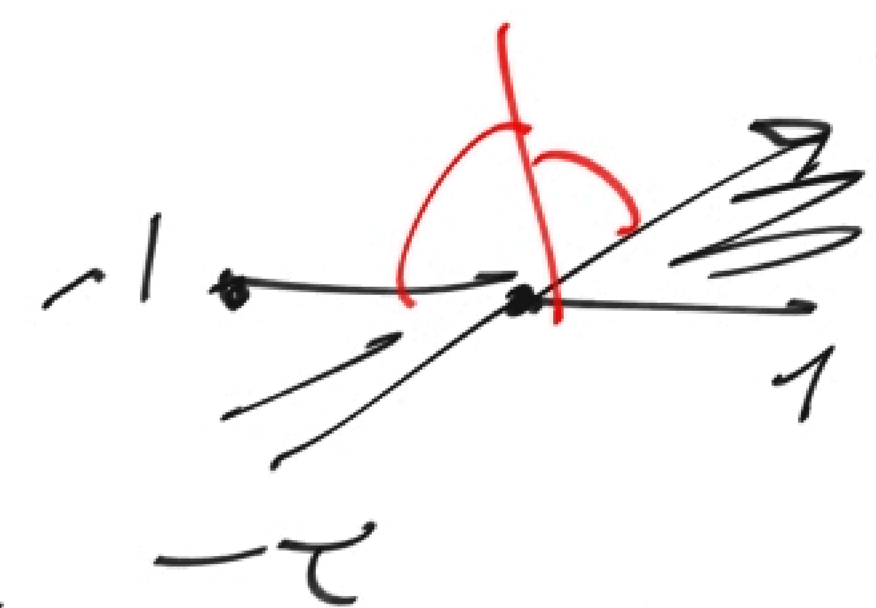
$\mathcal{E} \longmapsto \underline{\mathrm{Hom}}(\mathcal{E}, \text{distribution on } X)$

$\tau \to \tau' = -\frac{1}{\tau}$ $\qquad$ $\forall\, \tau \in \mathbb{C} - \mathbb{R}_{\leq 0}$

"Distribution" $\qquad$ Meromorphic function on $\mathbb{C}_z$

1) 

poles only in neigh. of bisectors



$(\mathbb{R}_+\tau \oplus \mathbb{R}_+) \sqcup (\mathbb{R}_-\tau \oplus \mathbb{R}_-)$

2) on $V \setminus$ $\leq$ exp. growth

On {"distributions"} action of

$\mathcal{A}^{(q)}$ $\mathcal{A}^{(q')}$

$$f(z) \longrightarrow \left\{ \left(\begin{array}{l} f(z+1) \\ e^{2\pi i \tau z} f(z) \end{array} \right) \right. \qquad \mathcal{A}_q$$

$$\otimes$$

$$f(z) \longrightarrow \left\{ \begin{array}{l} f(z \cancel{\;} \frac{1}{\tau}) \\ e^{2\pi i z} f(z) \end{array} \right. \qquad \mathcal{A}_{q'}$$

holon. A_q - mod $\rightsquigarrow$ holon. $A_{q'}$ - mod.

it is ~~not~~ what we expect!

for $|q|<1$ $\quad$ $\operatorname{Im}\tau > 0$

Differ by: Strange involution

"Betti side" $|q|<1$: Coh. Sheaves on E_q + two Anti HN fr.

$\boxed{\text{antipodal involution on } E_q}$

$t \to -t$ in group law on E_q

Main conjecture:

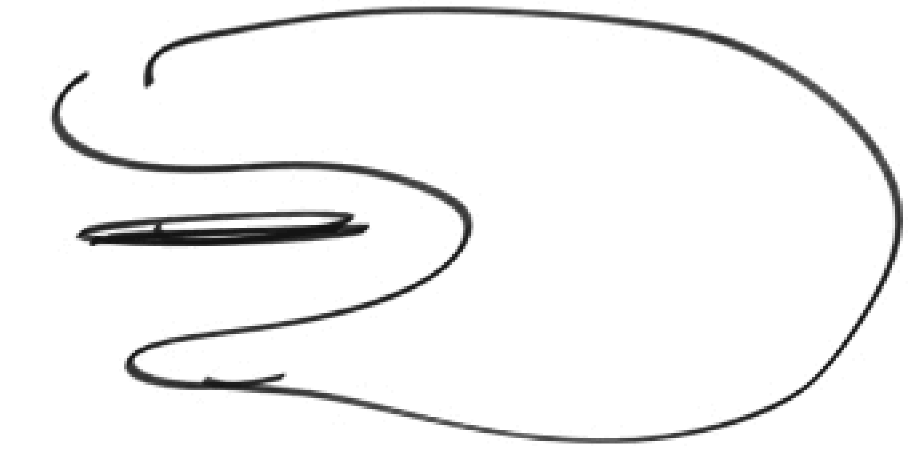
analytic antipodal involution on $\mathrm{Hol}^{-}(A^{(q)}\text{-mod})$

defined for $|q| \neq 1$

extends analytically to $|q| = 1$

Reasoning: Merom function ~ "distribution"

texts $\tau \in \mathbb{R}_{>0}$



$\rightarrow$ Analytic Moduli stack of A_∞-categories

over $\mathbb{CP}^1 \cup \mathbb{CP}^1$

$\mathbb{CP}^1 - \mathbb{RP}^1$

$/ \; PGL(2, \mathbb{Z})$

Elliptic difference equations

1. $(\mathbb{CP}^2 - \mathbb{RP}^2) / GL(3, \mathbb{Z})$

no difference between Betti & de Rham.