Semiorthogonal indecomposability of irregular surfaces

Shinnosuke Okawa

Osaka University

Bondal 60
December 17, 2021

- We work over $\mathbf{k} = \overline{\mathbf{k}}$, $\operatorname{char} \mathbf{k} = 0$.

DK hypothesis

$X \dashrightarrow Y$: a step of minimal model program (MMP)
$\Rightarrow \mathbf{D}(X) = \langle \mathbf{D}(Y)^{\perp}, \mathbf{D}(Y) \rangle$: semiorthogonal decomposition (SOD)

Remark (Motivation)

$\Leftarrow$ is not true in general. SOD is finer than MMP.

Example

$H^i(X, \mathcal{O}_X) = 0$ for $i > 0 \Rightarrow \mathbf{D}(X) = \langle \mathcal{O}_X^{\perp}, \mathcal{O}_X \rangle$ ($\rightsquigarrow$ (quasi-)phantoms)

Problem (Main)

Which minimal X is SI?

This is a particular case of the more general

Problem

For which X do all SODs come from MMP?

Problem (Main)

Which minimal X is SI?

- There are already some results in this direction.
- They seem to suggest that "in most cases" all SODs come from MMP.
- The precise meaning of "most cases" is to be clarified even in dimension 2.
- Today we show the following

Theorem (Main)

X: minimal surface with $H^1(X, \mathcal{O}_X) \neq 0 \Rightarrow X$ is SI

Combined with earlier results, this implies

Corollary

- *X: minimal surface of $\mathrm{kod}(X) = 0$ or 1. Then*
  X is not SI $\iff H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$.
- *X: minimal surface of $\mathrm{kod}(X) = 2$. Then*
  X is not SI $\Rightarrow H^1(\mathcal{O}_X) = 0$ and $\mathrm{Bs}(\omega_X)$ is not contractible to points.

Corollary

- X: *minimal surface of* $\mathrm{kod}(X) = 0$ *or* 1. *Then* X *is not SI* $\iff$ $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$.
- X: *minimal surface of* $\mathrm{kod}(X) = 2$. *Then* X *is not SI* $\Rightarrow H^1(\mathcal{O}_X) = 0$ *and* $\mathrm{Bs}(\omega_X)$ *is not contractible to points.*

It seems fair to state the following

Conjecture

minimal surface X *is almost always SI: i.e., it is not SI if and only if* $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ *(*$q = p_g = 0$*).*

- Main point: usefulness of the relative point of view.
- $f\colon X \to Y$: projective morphism. Then
  minimal model theory of X *over* $Y \leftrightarrow$ f*-linear SOD of* $\mathbf{D}(X)$
- Even for the study of X itself, the relative point of view is indispensable. The same is true for $\mathbf{D}(X)$.

Definition

- L: invertible sheaf on X. The base locus of L is defined by

$$\operatorname{Bs} L \coloneqq \bigcap_{\sigma \in H^0(X,L)} Z(\sigma) \subseteq X$$

- $\operatorname{Bs} \omega_X$: canonical base locus of X

Theorem (Kawatani and O 2015)

- *X: smooth projective variety*
- *$\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: SOD*

$\Rightarrow$ *either $\operatorname{Supp} s \subseteq \operatorname{Bs} \omega_X$ holds for all objects $s \in \mathcal{S}$, or the same holds for the objects of $\mathcal{L}$.*

- By applying mutation, we always assume that the former holds.
- $\mathcal{S}$: small component $\quad \mathcal{L}$: large component
- X is SI $\iff$ $\mathcal{S} = 0$ for any $\mathcal{S}$.

Theorem (Kawatani and O 2015)

- X: *smooth projective variety*
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: *SOD*

$\Rightarrow$ *either* $\operatorname{Supp} s \subseteq \operatorname{Bs} \omega_X$ *holds for all objects* $s \in \mathcal{S}$, *or the same holds for the objects of* $\mathcal{L}$.

This theorem gives a sufficient condition for semiorthogonal indecomposability.

Lemma

$\mathcal{S} = 0 \iff \mathcal{S} \otimes \omega_X = \mathcal{S}$.

Corollary (A sufficient condition for the semiorthogonal indecomposability)

$\forall Z \subset \operatorname{Bs} \omega_X$: *connected component,* ω_X *is trivial on a Zariski open neighborhood of* Z *(e.g.* $\operatorname{Bs} \omega_X$ *is a finite set)* $\Rightarrow \mathcal{S} = 0$. *Namely,* X *is SI.*

Corollary (of Corollary)

If $\operatorname{Bs} \omega_X = \emptyset$, *i.e. if* ω_X *is globally generated, then* X *is SI.*

Theorem (Kawatani and O 2015)

- X: *smooth projective variety*
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: *SOD*

$\Rightarrow$ *either* $\operatorname{Supp} s \subseteq \operatorname{Bs} \omega_X$ *holds for all objects* $s \in \mathcal{S}$, *or the same holds for the objects of* $\mathcal{L}$.

There are 2 proofs.

- (elementary and geometric) For each closed point $x \notin \operatorname{Bs} \omega_X$ we prove $\mathbf{k}(x) \in \mathcal{L}$. Use a section $\sigma \in H^0(X, \omega_X)$ such that $\sigma(x) \neq 0$. Here the canonical sheaf ω_X appears due to the fact that $- \otimes \omega_X[\dim X]$ is the Serre functor.
- (more informative) Proof by means of the Hochschild homology and its additivity with respect to SODs (see Pirozhkov 2020a).

$f\colon X \to Y$: projective morphism, X: nonsingular

Definition

- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$ is f-linear if $\mathcal{S}$ is stable under the action of $\mathrm{perf}(Y)$; i.e.,

  $$\mathcal{S} \otimes f^* E \subseteq \mathcal{S}$$

  holds for all $E \in \mathrm{perf}(Y)$.
- We say X is SI over Y or f is SI if X admits no non-trivial f-linear SOD.

Remark

f-linear SOD of X should be compared to MMP relative to f (MMP over Y).

Example (Rational elliptic surface)

Consider $f\colon X \to \mathbb{P}^1$: relatively minimal rational elliptic surface.

- X is minimal over $\mathbb{P}^1$, but not over $\mathrm{Spec}\,\mathbf{k}$ ($\mathrm{kod}(X) = -\infty$).
- There is no f-linear SOD of $\mathbf{D}(X)$, but $\mathbf{D}(X)$ admits (many) SODs.

$f\colon X \to Y$: projective morphism, X: nonsingular

Definition (Relative base loci)

- L: invertible sheaf on X. The relative base locus $\mathrm{Bs}_f(L) \subseteq X$ of L over Y is defined as follows.

$$f^* f_* L \otimes L^{-1} \to \mathcal{O}_X \to \mathcal{O}_{\mathrm{Bs}_f(L)} \to 0$$

- $\mathrm{Bs}_f(\omega_X)$: the relative canonical base locus of X over Y

Remark

- $\mathrm{Bs}_{X \to \mathrm{Spec}\,\mathbf{k}}(L) = \mathrm{Bs}(L)$.
- If Y is projective, we have the following alternative descriptions of $\mathrm{Bs}_f(L)$:

$$\mathrm{Bs}_f(L) = \bigcap_{M \in \mathrm{Pic}(Y)} \mathrm{Bs}(L \otimes f^* M) = \mathrm{Bs}(L \otimes f^* H)$$

  $H \in \mathrm{Pic}(Y)$: sufficiently ample
- $\mathrm{Bs}_f(L) = \mathrm{Bs}_f(L \otimes f^* M)$ for any $M \in \mathrm{Pic}(Y)$.

Theorem (Relative version of Kawatani-O)

- $f\colon X \to Y$: *projective morphism,* X: *nonsingular*
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: *$f$-linear SOD*

$\Rightarrow \operatorname{Supp} s \subseteq \operatorname{Bs}_f \omega_X$ *holds for all* $s \in \mathcal{S}$, *possibly after mutation.*

Proof.

- The elementary and geometric argument works equally well.
- Proof by means of the relative Hochschild homology should also be possible (need to be confirmed).

□

Theorem (Relative version of Kawatani-O)

- $f: X \to Y$: *projective morphism*, X: *nonsingular*
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: *$f$-linear SOD*

$\Rightarrow \operatorname{Supp} s \subseteq \operatorname{Bs}_f \omega_X$ *holds for all* $s \in \mathcal{S}$*, possibly after mutation.*

As in the case where $Y = \operatorname{Spec} \mathbf{k}$, we have

Corollary (A sufficient condition for the relative semiorthogonal indecomposability)

$\forall Z \subset \operatorname{Bs}_f \omega_X$: *connected component,* ω_X *is trivial over* Y *on a Zariski open neighborhood of* Z *(e.g.* $\operatorname{Bs}_f \omega_X$ *is a finite set)* $\Rightarrow \mathcal{S} = 0$*. Namely,* X *is SI over* Y*.*

In some cases we have $\operatorname{Bs}_f \omega_X \subsetneq \operatorname{Bs} \omega_X$, so we can say something stronger about f-linear SODs.

Definition (Pirozhkov 2020b)

A variety Y is stably semiorthogonally indecomposable (SSI) if for any projective morphism $f\colon X \to Y$ with X nonsingular, any SOD of $\mathbf{D}(X)$ is f-linear.

Example

$Y = \operatorname{Spec} R$: affine variety is SSI, as R classically generates $\operatorname{perf} R$.

Theorem (Pirozhkov 2020b)

If Y admits a finite morphism to an abelian variety, then Y is SSI.

Proof is based on the facts that Pic^0 is a spanning class for abelian varieties and that SODs of $\mathbf{D}(X)$ are invariant under the action of $\operatorname{Pic}^0(X)$.

Remark (Comparable fact of MMP)

If Y is as in the theorem, then any MMP of X is relative to $f\colon X \to Y$.

Remark (Trivial but crucial for us)

$f\colon X \to Y$: morphism of smooth projective varieties such that Y is SSI (e.g., a curve of positive genus). Then X is SI $\iff$ X is SI over Y.

Remark

$f\colon X \to Y$: morphism of smooth projective varieties such that Y is SSI (e.g., a curve of positive genus). Then X is SI $\iff$ X is SI over Y.

Example

$\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$: albanese morphism of X. Then X is SI if $\mathrm{Bs}_{\mathrm{alb}_X}\,\omega_X$ is a finite set.

Remark (relations to previous research)

$$\mathrm{Bs}_{\mathrm{alb}_X}\,\omega_X = \bigcap_{M \in \mathrm{Pic}(\mathrm{Alb}(X))} \mathrm{Bs}(\omega_X \otimes \mathrm{alb}_X^* M)$$

$$\overset{*}{\subseteq} \bigcap_{M' \in \mathrm{Pic}^0(X)} \mathrm{Bs}(\omega_X \otimes M') =: \mathrm{PBs}(\omega_X).$$

Hence if $\mathrm{PBs}(\omega_X)$ is a finite set, then X is SI. This is due to [Lin 2021], where $\mathrm{PBs}(\omega_X)$ is called the *paracanonical base locus* of X.

Remark (relations to previous research)

$$\mathrm{Bs}_{\mathrm{alb}_X}\,\omega_X = \bigcap_{M\in\mathrm{Pic}(\mathrm{Alb}(X))} \mathrm{Bs}(\omega_X\otimes \mathrm{alb}_X^* M) \overset{*}{\subseteq} \bigcap_{M'\in\mathrm{Pic}^0(X)} \mathrm{Bs}(\omega_X\otimes M') =: \mathrm{PBs}(\omega_X).$$

Hence if $\mathrm{PBs}(\omega_X)$ is a finite set, then X is SI. This is due to [Lin 2021], where $\mathrm{PBs}(\omega_X)$ is called the *paracanonical base locus* of X.

As an application, Xun Lin proved the following

Theorem (Lin 2021)

For a smooth projective curve C of genus g, $\mathrm{Sym}^k C$ is SI for $1\le k\le g-1$.

Proof.

Check $\mathrm{PBs}(\omega)=\emptyset$. □

On the other hand, $\mathrm{Bs}(\omega)\neq\emptyset$ for k close to $g-1$.

Remark (ctnd.)

[Caucci 2021] showed that the inclusion $\overset{*}{\subseteq}$ is in fact an equality.

Theorem (Main, bis)

X: minimal surface with $H^1(\mathcal{O}_X) \neq 0 \Rightarrow X$ is SI.

<u>Proof</u> Consider alb_X. There are 2 cases.

1. alb_X is generically finite.
   - Let $f\colon X \to Y$ be the Stein factorization of alb_X.
   - $\mathrm{Bs}_f\,\omega_X = \mathrm{Bs}_{\mathrm{alb}_X}\,\omega_X$, since $Y \to \mathrm{Alb}(X)$ is finite.

Theorem (Konno 2008)

There is a factorization of f

$$X \xrightarrow{\ g\ } X' \longrightarrow Y, \qquad X \overset{f}{\curvearrowright} Y$$

s.t. X' has only rational singularities and $\mathrm{Bs}_f\,\omega_X \subseteq \mathrm{Exc}(g) \cup$ (a finite set).

- Konno 2008 $\Rightarrow \mathcal{S}$ is supported in $\mathrm{Exc}(g) \cup$ (a finite set)$\Rightarrow \mathcal{S}$ is g-linear.

Theorem (Konno 2008)

$\mathrm{Bs}_g(\omega_X) = \emptyset$.

Proof (ctnd.)

2. $\mathrm{alb}_X(X)$ is a curve.
   - $\alpha\colon X \to C$: the Stein factorization of alb_X
   - We may assume that the generic fiber is of genus ≥ 2.

Theorem (Konno 2010)

There is a factorization of α

$$X \xrightarrow{f} Y \longrightarrow C, \quad \alpha\colon X \to C$$

such that $f\colon X \to Y$ is a birational contraction to a proper normal algebraic space and $\mathrm{Bs}_\alpha\,\omega_X \subseteq \mathrm{Exc}(f) \cup$ (a finite set).

- Konno 2010 $\Rightarrow$ $\mathcal{S}$ is supported in $\mathrm{Exc}(f) \cup$ (a finite set)$\Rightarrow$ $\mathcal{S}$ is f-linear.
- Apply to f the arguments of Case 1. $\Rightarrow \mathcal{S} = 0$.

Caucci, Federico (Oct. 2021). "Paracanonical base locus, Albanese morphism, and semi-orthogonal indecomposability of derived categories". arXiv: 2110.06795 [math.AG] (cit. on p. 19).

Kawatani, Kotaro and O (2015). "Nonexistence of semiorthogonal decompositions and sections of the canonical bundle". arXiv: 1508.00682 [math.AG] (cit. on pp. 8–10).

Konno, Kazuhiro (2008). "On the fixed loci of the canonical systems over normal surface singularities". In: *Asian J. Math.* 12.4, pp. 449–464. ISSN: 1093-6106. DOI: 10.4310/AJM.2008.v12.n4.a3 (cit. on p. 21).

— (2010). "Canonical fixed parts of fibred algebraic surfaces". In: *Tohoku Math. J. (2)* 62.1, pp. 117–136. ISSN: 0040-8735. DOI: 10.2748/tmj/1270041029 (cit. on p. 22).

Lin, Xun (July 2021). "On nonexistence of semi-orthogonal decompositions in algebraic geometry". arXiv: 2107.09564 [math.AG] (cit. on pp. 18, 19).

Pirozhkov, Dmitrii (June 2020a). "Admissible subcategories of del Pezzo surfaces". In: *arXiv e-prints*, arXiv:2006.07643, arXiv:2006.07643. arXiv: 2006.07643 [math.AG] (cit. on p. 10).

— (Nov. 2020b). "Stably semiorthogonally indecomposable varieties". arXiv: 2011.12743 [math.AG] (cit. on p. 17).