Semiorthogonal indecomposability of irregular surfaces

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2 Recap of a result by Kawatani-O

3 Relative version

4 Result of Pirozhkov and applications

• We work over $\mathbf{k} = \overline{\mathbf{k}}$, char $\mathbf{k} = 0$.

DK hypothesis

 $X \dashrightarrow Y$: a step of minimal model program (MMP) $\Rightarrow \mathbf{D}(X) = \langle \mathbf{D}(Y)^{\perp}, \mathbf{D}(Y) \rangle$: semiorthogonal decomposition (SOD)

Remark (Motivation)

 \Leftarrow is not true in general. SOD is finer than MMP.

Example

$$H^i(X, \mathcal{O}_X) = 0$$
 for $i > 0 \Rightarrow \mathbf{D}(X) = \langle \mathcal{O}_X^{\perp}, \mathcal{O}_X \rangle$ (\rightsquigarrow (quasi-)phantoms)

Problem (Main)

Which minimal X is SI?

This is a particular case of the more general

Problem

For which X do all SODs come from MMP?

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Problem (Main)

Which minimal X is SI?

- There are already some results in this direction.
- They seem to suggest that "in most cases" all SODs come from MMP.
- The precise meaning of "most cases" is to be clarified even in dimension 2.
- Today we show the following

Theorem (Main)

X: minimal surface with $H^1(X, \mathcal{O}_X) \neq 0 \Rightarrow X$ is SI

Combined with earlier results, this implies

Corollary

- X: minimal surface of kod(X) = 0 or 1. Then X is not SI $\iff H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0.$
- X: minimal surface of kod(X) = 2. Then X is not $SI \Rightarrow H^1(\mathcal{O}_X) = 0$ and $Bs(\omega_X)$ is not contractible to points.

Corollary

- X: minimal surface of kod(X) = 0 or 1. Then X is not SI $\iff H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0.$
- X: minimal surface of kod(X) = 2. Then X is not SI ⇒ H¹(O_X) = 0 and Bs(ω_X) is not contractible to points.

It seems fair to state the following

Conjecture

minimal surface X is almost always SI: i.e., it is not SI if and only if $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ ($q = p_g = 0$).

- Main point: usefulness of the relative point of view.
- $f: X \to Y$: projective morphism. Then minimal model theory of X over $Y \leftrightarrow f$ -linear SOD of $\mathbf{D}(X)$
- Even for the study of X itself, the relative point of view is indispensable. The same is true for D(X).



3 Relative version

4 Result of Pirozhkov and applications

Definition

• L: invertible sheaf on X. The base locus of L is defined by

$$\operatorname{Bs} L \coloneqq \bigcap_{\sigma \in H^0(X,L)} Z(\sigma) \subseteq X$$

• Bs ω_X : canonical base locus of X

Theorem (Kawatani and O 2015)

- X: smooth projective variety
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: SOD

 \Rightarrow either $\operatorname{Supp} s \subseteq \operatorname{Bs} \omega_X$ holds for all objects $s \in S$, or the same holds for the objects of \mathcal{L} .

- By applying mutation, we always assume that the former holds.
- S: small component \mathcal{L} : large component
- X is SI $\iff S = 0$ for any S.

Theorem (Kawatani and O 2015)

- X: smooth projective variety
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: SOD

 \Rightarrow either $\operatorname{Supp} s \subseteq \operatorname{Bs} \omega_X$ holds for all objects $s \in S$, or the same holds for the objects of \mathcal{L} .

This theorem gives a sufficient condition for semiorthogonal indecomposability.

Lemma

 $\mathcal{S}=0 \iff \mathcal{S}\otimes\omega_X=\mathcal{S}.$

Corollary (A sufficient condition for the semiorthogonal indecomposability)

 $\forall Z \subset \operatorname{Bs} \omega_X$: connected component, ω_X is trivial on a Zariski open neighborhood of Z (e.g. $\operatorname{Bs} \omega_X$ is a finite set) $\Rightarrow S = 0$. Namely, X is SI.

Corollary (of Corollary)

If $Bs \omega_X = \emptyset$, i.e. if ω_X is globally generated, then X is SI.

Theorem (Kawatani and O 2015)

- X: smooth projective variety
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: SOD

 \Rightarrow either $\operatorname{Supp} s \subseteq \operatorname{Bs} \omega_X$ holds for all objects $s \in S$, or the same holds for the objects of \mathcal{L} .

There are 2 proofs.

- (elementary and geometric) For each closed point $x \notin \operatorname{Bs} \omega_X$ we prove $\mathbf{k}(x) \in \mathcal{L}$. Use a section $\sigma \in H^0(X, \omega_X)$ such that $\sigma(x) \neq 0$. Here the canonical sheaf ω_X appears due to the fact that $\otimes \omega_X[\dim X]$ is the Serre functor.
- (more informative) Proof by means of the Hochschild homology and its additivity with respect to SODs (see Pirozhkov 2020a).





4 Result of Pirozhkov and applications

Relative version

 $f\colon X\to Y\colon$ projective morphism, $X\colon$ nonsingular

Definition

• $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$ is *f*-linear if \mathcal{S} is stable under the action of perf(Y); i.e.,

 $\mathcal{S}\otimes f^*E\subseteq \mathcal{S}$

holds for all $E \in perf(Y)$.

• We say X is SI over Y or f is SI if X admits no non-trivial f-linear SOD.

Remark

f-linear SOD of X should be compared to MMP relative to f (MMP over Y).

Example (Rational elliptic surface)

Consider $f: X \to \mathbb{P}^1$: relatively minimal rational elliptic surface.

- X is minimal over \mathbb{P}^1 , but not over $\operatorname{Spec} \mathbf{k} \ (\operatorname{kod}(X) = -\infty)$.
- There is no f-linear SOD of $\mathbf{D}(X)$, but $\mathbf{D}(X)$ admits (many) SODs.

Relative version

 $f\colon X\to Y\colon$ projective morphism, $X\colon$ nonsingular

Definition (Relative base loci)

L: invertible sheaf on X. The relative base locus Bs_f(L) ⊆ X of L over Y is defined as follows.

$$f^*f_*L \otimes L^{-1} \to \mathcal{O}_X \to \mathcal{O}_{\mathrm{Bs}_f(L)} \to 0$$

• $\operatorname{Bs}_f(\omega_X)$: the relative canonical base locus of X over Y

Remark

- $\operatorname{Bs}_{X \to \operatorname{Spec} \mathbf{k}}(L) = \operatorname{Bs}(L).$
- If Y is projective, we have the following alternative descriptions of $Bs_f(L)$:

$$\operatorname{Bs}_f(L) = \bigcap_{M \in \operatorname{Pic}(Y)} \operatorname{Bs}(L \otimes f^*M) = \operatorname{Bs}(L \otimes f^*H)$$

 $H \in \operatorname{Pic}(Y)$: sufficiently ample • $\operatorname{Bs}_f(L) = \operatorname{Bs}_f(L \otimes f^*M)$ for any $M \in \operatorname{Pic}(Y)$.

Theorem (Relative version of Kawatani-O)

- $f: X \to Y$: projective morphism, X: nonsingular
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: *f*-linear SOD

 \Rightarrow Supp $s \subseteq Bs_f \omega_X$ holds for all $s \in S$, possibly after mutation.

Proof.

- The elementary and geometric argument works equally well.
- Proof by means of the relative Hochschild homology should also be possible (need to be confirmed).

Theorem (Relative version of Kawatani-O)

- $f: X \to Y$: projective morphism, X: nonsingular
- $\mathbf{D}(X) = \langle \mathcal{S}, \mathcal{L} \rangle$: *f*-linear SOD

 \Rightarrow Supp $s \subseteq Bs_f \omega_X$ holds for all $s \in S$, possibly after mutation.

As in the case where $Y = \operatorname{Spec} \mathbf{k}$, we have

Corollary (A sufficient condition for the relative semiorthogonal indecomposability)

 $\forall Z \subset Bs_f \, \omega_X$: connected component, ω_X is trivial over Y on a Zariski open neighborhood of Z (e.g. $Bs_f \, \omega_X$ is a finite set) $\Rightarrow S = 0$. Namely, X is SI over Y.

In some cases we have $Bs_f \omega_X \subsetneq Bs \omega_X$, so we can say something stronger about *f*-linear SODs.

2 Recap of a result by Kawatani-O

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Definition (Pirozhkov 2020b)

A variety Y is stably semiorthogonally indecomposable (SSI) if for any projective morphism $f: X \to Y$ with X nonsingular, any SOD of $\mathbf{D}(X)$ is f-linear.

Example

 $Y = \operatorname{Spec} R$: affine variety is SSI, as R classically generates $\operatorname{perf} R$.

Theorem (Pirozhkov 2020b)

If Y admits a finite morphism to an abelian variety, then Y is SSI.

Proof is based on the facts that Pic^{0} is a spanning class for abelian varieties and that SODs of $\mathbf{D}(X)$ are invariant under the action of $\operatorname{Pic}^{0}(X)$.

Remark (Comparable fact of MMP)

If Y is as in the theorem, then any MMP of X is relative to $f: X \to Y$.

Remark (Trivial but crucial for us)

 $f: X \to Y$: morphism of smooth projective varieties such that Y is SSI (e.g., a curve of positive genus). Then X is SI \iff X is SI over Y.

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Remark

 $f\colon X\to Y\colon \text{morphism of smooth projective varieties such that }Y\text{ is SSI (e.g., a curve of positive genus). Then }X\text{ is SI }\Longleftrightarrow X\text{ is SI over }Y.$

Example

 $alb_X \colon X \to Alb(X)$: albanese morphism of X. Then X is SI if $Bs_{alb_X} \omega_X$ is a finite set.

Remark (relations to previous research)

$$\operatorname{Bs}_{\operatorname{alb}_X} \omega_X = \bigcap_{M \in \operatorname{Pic}(\operatorname{Alb}(X))} \operatorname{Bs}(\omega_X \otimes \operatorname{alb}_X^* M)$$
$$\stackrel{*}{\subseteq} \bigcap_{M' \in \operatorname{Pic}^0(X)} \operatorname{Bs}(\omega_X \otimes M') \eqqcolon \operatorname{PBs}(\omega_X).$$

Hence if $PBs(\omega_X)$ is a finite set, then X is SI. This is due to [Lin 2021], where $PBs(\omega_X)$ is called the *paracanonical base locus* of X.

Remark (relations to previous research)

$$\operatorname{Bs}_{\operatorname{alb}_X} \omega_X = \bigcap_{M \in \operatorname{Pic}(\operatorname{Alb}(X))} \operatorname{Bs}(\omega_X \otimes \operatorname{alb}_X^* M) \stackrel{*}{\subseteq} \bigcap_{M' \in \operatorname{Pic}^0(X)} \operatorname{Bs}(\omega_X \otimes M') \eqqcolon \operatorname{PBs}(\omega_X).$$

Hence if $PBs(\omega_X)$ is a finite set, then X is SI. This is due to [Lin 2021], where $PBs(\omega_X)$ is called the *paracanonical base locus* of X.

As an application, Xun Lin proved the following

Theorem (Lin 2021)

For a smooth projective curve C of genus g, $\operatorname{Sym}^k C$ is SI for $1 \le k \le g-1$.

Proof.

Check $PBs(\omega) = \emptyset$.

On the other hand, $Bs(\omega) \neq \emptyset$ for k close to g-1.

Remark (ctnd.)

[Caucci 2021] showed that the inclusion \subseteq is in fact an equality.

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The case of surfaces

Theorem (Main, bis)

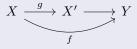
X: minimal surface with $H^1(\mathcal{O}_X) \neq 0 \Rightarrow X$ is SI.

<u>Proof</u> Consider alb_X . There are 2 cases.

- alb_X is generically finite.
 - Let $f: X \to Y$ be the Stein factorization of alb_X .
 - $\operatorname{Bs}_f \omega_X = \operatorname{Bs}_{\operatorname{alb}_X} \omega_X$, since $Y \to \operatorname{Alb}(X)$ is finite.

Theorem (Konno 2008)

There is a factorization of f



s.t. X' has only rational singularities and $Bs_f \omega_X \subseteq Exc(g) \cup (a \text{ finite set}).$

• Konno 2008 $\Rightarrow S$ is supported in $Exc(g) \cup (a \text{ finite set}) \Rightarrow S$ is g-linear.

Theorem (Konno 2008)

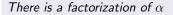
 $\operatorname{Bs}_g(\omega_X) = \emptyset.$

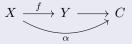
Proof (ctnd.)

(a) alb_X(X) is a curve.

- $\alpha: X \to C$: the Stein factorization of alb_X
- We may assume that the generic fiber is of genus ≥ 2 .

Theorem (Konno 2010)





such that $f: X \to Y$ is a birational contraction to a proper normal algebraic space and $Bs_{\alpha} \omega_X \subseteq Exc(f) \cup$ (a finite set).

- Konno 2010 $\Rightarrow S$ is supported in $Exc(f) \cup (a \text{ finite set}) \Rightarrow S$ is f-linear.
- Apply to f the arguments of Case 1. $\Rightarrow S = 0$.

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