Homological Mirror Symmetry for chain type polynomials

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Invertible polynomials

Invertible polynomial $w \in \mathbb{C}[x_1, \ldots, x_n]$:

$$w = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} x_j^{a_{ij}},$$

where $c_i \neq 0$, $A = (a_{ij})$ is nondegenerate and w has an isolated critical point at the origin (can make $c_i = 1$).

Dual invertible polynomial $w^\vee$: replace A by the transposed matrix A^t.

Group of symmetries $G_w \subset (\mathbb{C}^*)^n$ consists of diagonal transformations g such that $w(gx) = w(x)$.

Invertible polynomials

Invertible polynomials have been classified by Kreuzer and Skarke. Atomic types:

- $w = x_1^{a_1}$ (Fermat)
- $w = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, where $a_i > 1$ (chain)
- $w = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, where $a_i > 1$ (loop)

For chain polynomial $p_a = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, where $a = (a_1, \ldots, a_n)$, the dual is $p_{a^\vee}$, where $a^\vee = (a_n, \ldots, a_1)$. The group of diagonal symmetries of p_a, G_a, consists of $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ such that

$$\lambda_1^{a_1}\lambda_2 = 1, \ \ldots \lambda_{n-1}^{a_{n-1}}\lambda_n = 1, \ \lambda_n^{a_n} = 1,$$

so it is cyclic of order $a_1 \ldots a_n$.

Mirror symmetry for invertible polynomials

LG Mirror Symmetry: $(\mathbb{C}^n/G_w, w)$ and $(\mathbb{C}^n, w^\vee)$ are mirror dual.

There are many levels of this duality. E.g., one can compare Frobenius algebras, Frobenius manifolds, Cohomological field theories. This involves Saito-Givental's theory on the B-side and Fan-Jarvis-Ruan-Witten theory on the A-side.

Today: will discuss the relevant equivalence of categories, i.e., homological mirror symmetry. Almost the entire talk will be about the B-side. For more details on the A-side, see Umut's talk available online.

B-side category

On the B-side to the pair $(\mathbb{C}^n/G_w, w)$ we associate the category of graded matrix factorizations.

Set $S = \mathbb{C}[x_1, \ldots, x_n]$. Recall that a matrix factorization of w is a $\mathbb{Z}_2$-graded free S-module $P_0 \oplus P_1$ equipped with odd S-linear operator δ such that $\delta^2 = w \cdot \mathrm{id}$.

There is a natural $\mathbb{Z}_2$-dg-category of matrix factorizations: $\mathrm{hom}((P, \delta_P), Q, \delta_P) = \mathrm{Hom}_0(P, Q) \oplus \mathrm{Hom}_1(P, Q)$ where the differential is given by $d(f) = \delta_Q f - (-1)^{|f|} f \delta_P$.

If G is a finite group of symmetries of w then there is a natural definition of the category of G-equivariant matrix factorizations.

B-side category

To define a $\mathbb{Z}$-graded category of matrix factorizations one needs to use a $\mathbb{C}^*$-action. Note that an invertible polynomial is quasi-homogeneous, i.e., there exists a grading $\deg(x_i) = d_i > 0$ such that w is homogeneous of degree d.

Consider the algebraic group $\Gamma_w \subset (\mathbb{C}^*)^n$ together with a natural homomorphism $\chi : \Gamma_w \to \mathbb{C}^*$ consisting of all $g \in (\mathbb{C}^*)^n$ such that $w(gx) = \chi(x)w(x)$. It is an extension of $\mathbb{C}^*$ by G_w. Then there is a natural $\mathbb{Z}$-graded dg-category of Γ_w-equivariant matrix factorizations of w.

Equivalently, we can consider the abelian group L_w dual to Γ_w, and consider the category of L_w-graded matrix factorizations of w. We denote this category by $\mathrm{MF}_{gr}(w)$.

A-side category and equivalence

On the A-side, consider Fukaya-Seidel category $DF(w)$ of the map $w : \mathbb{C}^n \to \mathbb{C}$.
Objects are Lagrangians, such that outside of compact set they project to radial lines (the ray $\mathbb{R}_{<0}$ is not allowed).

Conjecture. For every invertible polynomial w, there is an equivalence $DF(w) \simeq \mathrm{MF}_{gr}(w^\vee)$.

Our result: proof for chain polynomials, i.e.,
$DF(p_a) \simeq \mathrm{MF}_{gr}(p_{a^\vee})$, where $a = (a_1, \ldots, a_n)$, $a^\vee = (a_n, \ldots, a_1)$, under some (standard) assumptions on the A-side.

Exceptional collections

An object E in a triangulated category $\mathcal{D}$ is called exceptional if $\mathrm{Ext}^{>0}(E,E) = 0$, $\mathrm{Hom}(E,E) = k$, the ground field.
A collection $(E_1, \ldots, E_n)$ of exceptional objects is exceptional if $\mathrm{Ext}^*(E_i, E_j) = 0$ for $i > j$ and all $\mathrm{Ext}^*(E_i, E_j)$ are finite dimensional.

An exceptional collection is full if it generates $\mathcal{D}$, i.e., $\mathrm{Hom}(E_i, X) = 0$ for all i implies that $X = 0$.

Standard example: $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ in $D^b(\mathrm{Coh}\,\mathbb{P}^n)$.

If $\mathrm{Ext}^{>0}(E_i, E_j) = 0$ for all i, j the collection is called strong. The category generated by a full strong exceptional collection is equivalent to $D^b(A - \mathrm{mod})$, where A is the Hom-algebra of the collection.

Aramaki-Takahashi collection

For a regular sequence $a_1, \ldots, a_m \in S$ such that $w = a_1 b_1 + \ldots + a_m b_m$, can define a Koszul matrix factorization

$$\mathrm{stab}(a_1, \ldots, a_m) = (\bigwedge^{\bullet}(S^m), \sum_i a_i \iota_{e_i^*} + \sum_i b_i e_i \wedge ?)$$

It depends only on the ideal $(a_1, \ldots, a_m)$ and corresponds to the module $S/(a_1, \ldots, a_n)$ under the equivalence of $\mathrm{MF}(w)$ with the singularity category of $S/(w)$.

In our case $w = p_a = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \ldots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, and we consider the matrix factorization

$$E = \begin{cases} \mathrm{stab}(x_2, x_4, \ldots, x_n), & n \text{ even}, \\ \mathrm{stab}(x_1, x_3, \ldots, x_n), & n \text{ odd}. \end{cases}$$

Aramaki-Takahashi collection

Recall that we consider L-graded matrix factorizations where the grading group L is generated by $\overline{x}_1, \ldots, \overline{x}_n$ with the relations

$$a_1\overline{x}_1 + \overline{x}_2 = a_2\overline{x}_2 + \overline{x}_3 = \ldots = a_{n-1}\overline{x}_{n-1} + \overline{x}_n = a_n\overline{x}_n = \overline{p}.$$

Set $\tau = (-1)^n\overline{x}_1$ and consider the twist operation $M(i) := M(i\tau)$ on graded matrix factorizations.

Theorem ([Aramaki-Takahashi]). $(E, E(1), \ldots, E(\mu^\vee - 1))$ is a full exceptional collection in $MF_{gr}(p_a)$, where $\mu^\vee = \mu(a^\vee) = a_1 \ldots a_n - a_1 \ldots a_{n-1} + a_1 \ldots a_{n-2} - \ldots$ (thiis is the Milnor number of the dual singularity $p_{a^\vee} = 0$).

We refer to $(E, \ldots, E(\mu^\vee - 1))$ as AT collection.

Ext-algebra

The shift functor T on $\mathrm{MF}_{gr}(p_a)$ satisfies $2T = \overline{p}$. So we can define a bigger abelian group $\widetilde{L}$ generated by L and T, and the Ext-algebra of the collection will be the $\widetilde{L}$-graded algebra

$$\mathcal{B}_a := \bigoplus_{\ell \in \widetilde{L}} \mathrm{Hom}^0(E, E(\ell)).$$

This algebra was computed by [Aramaki-Takahashi]. Assume for simplicity that $a_1 > 2$. Then

$$\mathcal{B}_a \simeq \begin{cases} k[x_1, x_3, \dots, x_{n-1}]/(x_1^{a_1}, \dots, x_{n-1}^{a_{n-1}}), & n \text{ even}, \\ k[x_0, x_2, \dots, x_{n-1}]/(x_0^2, x_2^{a_2}, \dots, x_{n-1}^{a_{n-1}}), & n \text{ odd}, \end{cases}$$

where $\deg(x_0) = \tau + T$.

Examples: $n \leq 3$

For $n = 1$, $p = x_1^{a_1}$, $E = \mathrm{stab}(x_1)$.
The only nontrivial Ext in the AT-collection $(E, \ldots, E(a_1 - 2))$ are given by $x_0 \in \mathrm{Ext}^1(E(i), E(i+1))$, where $x_0^2 = 0$.

The element x_0 comes from the extension of S-modules

$$0 \to S/(x_1)(-\overline{x}_1) \xrightarrow{x_1} S/(x_1^2) \to S/(x_1) \to 0$$

which gives rise to an exact triangle in the category of matrix factorizations with $S/(x_1)$ going to E, and $S/(x_1)(-\overline{x}_1)$ going to $E(1)$.

For $n = 2$, $p = x_1^{a_1} x_2 + x_2^{a_2}$, $E = \mathrm{stab}(x_2)$.
The AT-collection $(E, E(1), \ldots, E(\mu^\vee - 1))$, where $\mu^\vee = a_1 a_2 - a_1 + 1$, is strong (no higher Ext's), and the Hom-algebra is generated by $x_1 : E(i) \to E(i+1)$, with the relation $x_1^{a_1} = 0$.

Examples: $n \leq 3$

For $n = 3$, $p = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$, $E = \mathrm{stab}(x_1, x_3)$.

The AT-collection $(E, E(1), \ldots, E(\mu^\vee - 1))$ is no longer strong. We have generators $x_0 \in \mathrm{Ext}^1(E(i), E(i+1))$ and $x_2 \in \mathrm{Ext}^2(E(i), E(i+a_1))$. (Note that $\overline{x}_2 = \overline{p} - a_1\overline{x}_1 = 2T + a_1\tau$.)

The corresponding A_∞-algebra is homotopically nontrivial. In fact, one can calculate that

$$m_{a_1}(x_0, \ldots, x_0) = x_2.$$

Socle of the Ext-algebra: glimpse of a recursion

The algebra $\mathcal{B}_a$ is Gorenstein, with 1-dimensional socle generated by $x_1^{a_1-1}x_3^{a_3-1}\dots x_{n-1}^{a_{n-1}-1}$ if n is even (resp., $x_0x_2^{a_2-1}\dots x_{n-1}^{a_{n-1}-1}$ if n is odd).

In the grading group L we have relations

$$\overline{x}_i \equiv (-1)^i a_1 a_2 \dots a_{i-1} \cdot \overline{x_1} \mod 2T \cdot \mathbb{Z}.$$

So say for even n, the socle lives in degree $N\tau$, where

$$N = (a_1-1)+a_1a_2(a_3-1)+\dots+a_1a_2\dots a_{n-2}(a_{n-1}-1) = \mu^\vee(a-)$$

modulo $2T \cdot \mathbb{Z}$, where $a- = (a_1, \dots, a_{n-1})$.

In other words, this gives a basis vector in $\mathrm{Ext}^*(E, E(\mu^\vee(a-)))$ and $\mathrm{Ext}^*(E, E(i)) = 0$ for $i > \mu^\vee(a-)$ in the AT-collection.

Recursion. I: VGIT embedding

Plan: 1) Identify the subcategory $\langle E, E(1), \dots, E(\mu^\vee(a-) - 1)\rangle$, generated by the initial segment of the AT-collection, with the previous category $\mathrm{MF}_{gr}(p_{a-})$, where $a- = (a_1, \dots, a_{n-1})$.
2) Recover the entire AT-collection from this initial segment.

To do 1) we use the VGIT construction of [BFK], as explained in [Favero-Kaplan-Kelly].
Consider the polynomial
$W = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_{n+1}^{a_n}$, invariant with respect to the $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ with the weights

$$c_1 = -1, c_2 = a_1, c_3 = -a_1 a_2, \dots, c_n = \pm a_1 \dots a_{n-1}, c_{n+1} = -c_n.$$

Main idea: on the open subset $U_+ = (x_{n+1} \neq 0)$ the $\mathbb{G}_m$-action allows to reduce to $x_{n+1} = 1$ which gives $W_+ = p_a(x_1, \dots, x_n)$, while on $U_- = (x_n \neq 0)$ the $\mathbb{G}_m$-action allows to reduce to $x_n = 1$ which gives $W_- = p_{a-}(x_1, \dots, x_{n-1}) + x_{n+1}^{a_n}$.

VGIT embedding

For every interval $I \subset \mathbb{Z}$, one defines the window subcategory $\mathcal{W}_I \subset \mathrm{MF}_\Gamma(W)$ consisting of matrix factorizations F such that $F|_0$ has weights in I with respect to the above $\mathbb{G}_m \subset \Gamma$.

Set $\alpha_n = a_1 \dots a_n + a_1 \dots a_{n-2} + \dots$, and consider the intervals

$$I^- = [0, \alpha_{n-1} - 1] \subset I^+ = [0, a_1 \dots a_{n-1} + \alpha_{n-2} - 1].$$

Theorem([FKK]). The restriction to U_+ induces an equivalence $\mathcal{W}_{I^+} \xrightarrow{\sim} \mathrm{MF}_{\Gamma_+}(W_+) = \mathrm{MF}_{gr}(p_a)$, while the restriction to U_- gives $\mathcal{W}_{I^-} \xrightarrow{\sim} \mathrm{MF}_{\Gamma_-}(W_-) = \mathrm{MF}_{gr}(p_{a-} + x_{n+1}^{a_n})$.

We also have a fully faithful embedding $\mathrm{MF}_{gr}(p(a-)) \to \mathrm{MF}_{gr}(p(a-) + x_{n+1}^{a_n})$ sending F to $F \boxtimes \mathrm{stab}(x_{n+1})$.

VGIT embedding

So we get a fully faithful embedding

$$\Phi : \mathrm{MF}_{gr}(p_{a-}) \to \mathrm{MF}_{gr}(W_-) \simeq \mathcal{W}_{I^-} \subset \mathcal{W}_{I^+} \xrightarrow{\sim} \mathrm{MF}_{gr}(p_a).$$

Naive hope: the image of the AT-collection in $\mathrm{MF}_{gr}(p(a-))$ will give a segment of the AT-collection in $\mathrm{MF}_{gr}(p_a)$.

This is false. For example, for $n = 2$, the AT-collection in $\mathrm{MF}_{gr}(p_{a_1,a_2})$ consists of $E(i)$, where $E = \mathtt{stab}(x_2)$. But Φ(AT-collection) will consist of $F(i)$, where $F = \mathtt{stab}(x_1, x_2)$.

Mutations

For $\mathcal{C} \subset \mathcal{D}$ an admissible subcategory (e.g., generated by an exceptional collection) there exist left and right adjoints $\lambda_{\mathcal{C}}, \rho_{\mathcal{C}} : \mathcal{D} \to \mathcal{C}$ to the inclusion of $\mathcal{C}$. Consider the subcategories ${}^{\perp}\mathcal{C}, \mathcal{C}^{\perp} \subset \mathcal{D}$, where $X \in {}^{\perp}\mathcal{C}$ (resp., $X \in \mathcal{C}^{\perp}$) if $\mathrm{Hom}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Hom}(\mathcal{C}, X) = 0$).

For $X \in {}^{\perp}\mathcal{C}$, we have an exact triangle $C \to X \to L_{\mathcal{C}}(X) \to \ldots$, where $C = \rho_{\mathcal{C}}(X)$ and $L_{\mathcal{C}}(X) = \lambda_{\mathcal{C}^{\perp}}(X)$. This triangle implies that $L_{\mathcal{C}}(X) \in \mathcal{C}^{\perp}$. The functor

$$L_{\mathcal{C}} : {}^{\perp}\mathcal{C} \to \mathcal{C}^{\perp}$$

is an equivalence, called left mutation through $\mathcal{C}$.

The inverse functor is provided by the functor of the right mutation

$$R_{\mathcal{C}} : \mathcal{C}^{\perp} \to {}^{\perp}\mathcal{C}.$$

Dual exceptional collection

If $E_0, \ldots, E_n$ is an exceptional collection then the left dual exceptional collection is $(F_{-n}, \ldots, F_0)$ given by

$$F_0 = E_0,\ F_{-1} = L_{E_0} E_1,\ F_{-2} = L_{E_0,E_1} E_2, \ldots,\ F_{-n} = L_{E_0,\ldots,E_{n-1}} E_n.$$

It has the property $\mathrm{Hom}^*(E_i, F_{-j}) = 0$ for $i \neq j$ while $\mathrm{Hom}^*(E_i, F_{-i})$ is 1-dimensional.

Example. The left dual collection to $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ on $\mathbb{P}^n$ is $(\Omega^n(n), \ldots, \Omega^1(1), \mathcal{O})$.

Dual of the inital segment of AT-collection

We want to relate $(E, \ldots, E(\mu^\vee(a-) - 1))$ in $\mathrm{MF}_{gr}(p_a)$ to the image of $\Phi : \mathrm{MF}_{gr}(p_{a-}) \to \mathrm{MF}_{gr}(p_a)$. This requires passing to a dual collection and an extra mutation.

Consider the subcategory $\mathcal{C} = \langle E(-\mu^\vee(a - -)), \ldots, E(-1)\rangle$. Then the image of the AT-collection in $\mathrm{MF}_{gr}(p_{a-})$ under Φ (suitably twisted) is left dual to the collection $L_{\mathcal{C}}(E, \ldots, E(\mu^\vee - 1))$.

This image (up to a twist) is given by $(F(\mu^\vee(a-) - 1), \ldots, F(1), F)$, where

$$F = \begin{cases} \mathrm{stab}(x_1, x_3, \ldots, x_{n-1}, x_n), & n \text{ even}, \\ \mathrm{stab}(x_2, x_4, \ldots, x_{n-1}, x_n), & n \text{odd} \end{cases}.$$

Helices

If $(E_1, \ldots, E_n)$ is an exceptional collection in $\mathcal{D}$ then it extends to a helix $(E_i)_{i \in \mathbb{Z}}$ in $\mathcal{D}$, where

$$E_{i-n} = L_{E_{i-n+1},\ldots,E_{i-1}} E_i, \quad E_{i+n} = R_{E_{i+1},\ldots,E_{i+n-1}} E_i.$$

For $(\mathcal{O}, \ldots, \mathcal{O}(n))$ on $\mathbb{P}^n$ this gives $(\mathcal{O}(i))_{i \in \mathbb{Z}}$ (up to a shift).

If $(E_1, \ldots, E_n)$ generates $\mathcal{D}$ then $\mathcal{S}_{\mathcal{D}}(E_i) \simeq E_{i-n}$, where $\mathcal{S}_{\mathcal{D}}$ is the Serre functor on $\mathcal{D}$, i.e., $\mathrm{Hom}(X, Y)^* \simeq \mathrm{Hom}(Y, \mathcal{S}_{\mathcal{D}}(X))$.

The Serre functor on the category $\mathrm{MF}_{gr}(p_a)$ has form $\mathcal{S}(F) = F(-\mu^\vee)[m]$ for some $m \in \mathbb{Z}$. Hence, the helix generated by the AT-collection is simply $(E(i))_{i \in \mathbb{Z}}$ (up to shift).

Helices and the AT-collections

Let $(E_0, \ldots, E_{M-1})$ denote the AT-collection in $\mathcal{D} = \mathrm{MF}_{gr}(p_a)$, where $M = \mu^\vee$. We replace it and its initial segment $(E_0, \ldots, E_{m-1})$, where $m = \mu^\vee(a-)$, by the left dual collections $(F_{-M+1}, \ldots, F_0)$ and $(F_{-m+1}, \ldots, F_0)$. Let $\mathcal{C} = \langle F_{-m+1}, \ldots, F_0 \rangle$. Consider the functor $\lambda : \mathcal{D} \to \mathcal{C}$ left adjoint to the inclusion.

Key result: $(\lambda(F_{-M+1}), \ldots, \lambda(F_0))$ is a segment of the helix generated by $(F_{-m+1}, \ldots, F_0)$ (note that $\lambda(F_{-i}) = F_{-i}$ for $0 \le i \le m-1$), and the functor λ induces isomorphisms

$$\mathrm{Ext}^*(F_{-j}, F_{-i}) \xrightarrow{\sim} \mathrm{Ext}^*(\lambda(F_{-j}), \lambda(F_{-i}))$$

for $i \le j$ (from left to right).

Crucial fact for the proof: $\mathrm{Ext}^*(E_0, E_m)$ is 1-dimensional and $\mathrm{Ext}^*(E_0, E_i) \otimes \mathrm{Ext}^*(E_i, E_m) \to \mathrm{Ext}^*(E_0, E_m)$ is a perfect pairing.

Example: M=m+1

Assume $\text{Ext}^*(E_0, E_m)$ is 1-dimensional and $\text{Ext}^*(E_0, E_i) \otimes \text{Ext}^*(E_i, E_m) \to \text{Ext}^*(E_0, E_m)$ is a perfect pairing. Consider $\mathcal{C} = \langle E_0, \ldots, E_{m-1} \rangle$, and let $\lambda, \rho : \mathcal{D} \to \mathcal{C}$ denote the left and right adjoints to the inclusion.

Claim: $\lambda(L_{\mathcal{C}} E_m) \simeq \rho(E_m) \simeq \mathcal{S}_{\mathcal{C}}(E_0)$ and λ induces isomorphisms $\text{Ext}^*(L_{\mathcal{C}} E_m, C) \to \text{Ext}^*(\mathcal{S}_{\mathcal{C}}(E_0), C)$ for $C \in \mathcal{C}$.

Proof: $\text{Hom}(C, \rho(E_m)) \simeq \text{Hom}(C, E_m) \simeq \text{Hom}(E_0, C)^\vee \simeq \text{Hom}(C, \mathcal{S}_{\mathcal{C}}(E_0))$.

Now, if $F_{-m}, \ldots, F_0$ is the dual collection to $E_0, \ldots, E_m$, then $\mathcal{C} = \langle F_{-m+1}, \ldots, F_0 \rangle$ and $F_{-m} = L_{\mathcal{C}}(E_m)$.

Recursion: conclusion

Starting from the AT-collection $(E_0, \ldots, E_{m-1})$ in $\mathrm{MF}_{gr}(p_{a-})$ for $a- = (a_1, \ldots, a_{n-1})$, we have a recipe for constructing a category with an exceptional collection, which will be equivalent to $\mathrm{MF}_{gr}(p_a)$ with its AT-collection.

Step 1. Extend $(E_0, \ldots, E_{m-1})$ to a helix $(E_i)_{i\in\mathbb{Z}}$, and take any segment of length M, say, $(E_0, \ldots, E_{M-1})$.
Step 2. Leave only Ext's from left to right (but remember the A_∞-structure).
Step 3. Pass to the right dual exceptional collection.

A-side

On the A-side we consider the perturbation $x_1 + p_a(x_1, \ldots, x_n) : \mathbb{C}^n \to \mathbb{C}$ that has only nondegenerate critical points. There are $\mu = \mu(a) = a_1 \ldots a_n - a_2 \ldots a_n + a_3 \ldots a_n - \ldots$ critical points, and they form one orbit with respect to the group of rotations through multiples of $2\pi/\mu$.

We consider a diffeomorphism φ given by the $2\pi/\mu$ rotation inside some circle containing critical values and is identity outside a bigger circle, together with its lift as a symplectomorphism Φ of $\mathbb{C}^n$.

A-side

We take a path γ from a large regular value to a critical value, and consider the set of paths

$$\gamma, \varphi(\gamma), \ldots, \varphi^{\mu-1}(\gamma).$$

Corresponding Lefshetz thimbles form an exceptional collection in the Fukaya-Seidel category $DF(a)$, which is an analog of the AT-collection.

Using Lefschetz bifibration method one can show the recursion that gives $DF(a_1, \ldots, a_n)$ from $DF(a_2, \ldots, a_n)$.