Deformations of Δ'ed cat's with t-structure

(+ Lowen, Genovese)

Reminder abelian case

(+ Lowen)

k ground field

$S \twoheadrightarrow R$ commutative finite dim. k-algebras

Res: $\mathrm{LinCat}(R) \longrightarrow \mathrm{LinCat}(S)$

$\qquad \| \qquad\qquad\qquad \mathcal{A} \longmapsto \mathcal{A}_S$

$\{$ cat's linear$/R\}$

$\exists$ left adjoint $R \otimes_S -$

$\mathrm{Ob}(R \otimes_S \mathcal{B}) = \mathrm{Ob}(\mathcal{B})$

$(R \otimes_S \mathfrak{a})(A,B) = R \otimes_S \mathfrak{a}(A,B)$

$\mathfrak{a}$ is S-flat if Hom-spaces are S-flat

$\Rightarrow R \otimes_S \mathfrak{a}$ S-flat

Def $\mathfrak{a} \in \mathrm{LinCat}_{flat}(R)$

An "left" S-deformation of $\mathfrak{a}$ is a flat lift of $\mathfrak{a}$ under $R \otimes_S -$

Def $(\mathfrak{a}, S)$ 2-category of S-defs of $\mathfrak{a}$

Fact If $(S,R) = (k[\varepsilon], k)$ $\varepsilon^2 = 0$

$\mathrm{Def}(\mathfrak{a},S)/_{\cong} \cong \underbrace{HH^2(\mathfrak{a})}$

Hochschild cohomology

<u>Variant</u> $S \twoheadrightarrow R$

$$\mathrm{AbCat}(R) \longrightarrow \mathrm{AbCat}(S)$$
$$\mathcal{A} \longmapsto \mathcal{A}_S$$

$\exists$ nice <u>right adjoint</u>

$$\mathrm{Hom}_S(R, -)$$

$\mathcal{B} \in \mathrm{AbCat}(S)$

$$\mathrm{Hom}_S(R, \mathcal{B}) \overset{\text{full}}{\subseteq} \mathcal{B}$$

$$\mathrm{Ob}(\mathrm{Hom}_S(R, \mathcal{B})) =$$

$$\{ B \in \mathrm{Ob}(\mathcal{B}) \mid IB = 0 \}$$

$I = \ker(S \to R)$

<u>Need notion of flatness</u>

Assume $A \in AbCat(R)$

has enough injectives

(general case: consider Ind A)

<u>Def</u> A is flat if

Inj(A) is flat.

<u>Def</u> $A \in AbCat_{flat}(R)$.

An S-def of A is a

flat lift under $Hom_S(R, -)$ (right)

<u>Def</u> (A, S): 2-cat of S-defs of A

How to study?

Fact $\mathrm{Inj}(A)$ has weak cokernels

$$\begin{tikzcd}
I_0 \arrow[r, "f"'] & I_1 \arrow[r] \arrow[d, "\forall"'] & C \in \mathrm{Inj}(A) \arrow[dl, dashed, "\exists"] \\
 & I \in \mathrm{Inj}(A) &
\end{tikzcd}$$

in $\mathrm{Inj}(A)$

$C = \mathrm{Inj}\ \mathrm{hull}\ (\mathrm{coker}\, f)$

Classical fact (from weak cokernels)

$\underbrace{\mathrm{mod}(\mathrm{Inj}(A)^{\circ})}_{\text{finitel presented right } \mathrm{Inj}(A) \text{ modules}}$ is abelian

We have moreover

$$\mathcal{A} \simeq \mathrm{mod}(\mathrm{Inj}(\mathcal{A})^{\circ})$$

$$A \longmapsto \mathcal{A}(A,-)$$

<u>Fact</u>

$$\mathrm{Def}^{\text{right}}(\mathcal{A},S) \simeq$$

$$\mathrm{Def}^{\text{left}}(\mathrm{Inj}(\mathcal{A}),S)$$

<u>Leads to</u>

$$HH^*_{Ab}(\mathcal{A}) := HH^*(\mathrm{Inj}(\mathcal{A}))$$

Note: $\neq$

$$HH^*_{LinCat}(\mathcal{A})$$

<u>Fact</u> $\mathcal{H}(S,R) = (k[\epsilon],k)$

then $\mathrm{Def}(A, k[\varepsilon])/\cong \;\cong\; HH^2_{Ab}(A)$

Δ'ed Cat's w. t-structure

Motivation I

The "curvature problem"

$\mathcal{C}$ $\left\{\begin{matrix} A_\infty \\ dg \end{matrix}\right\}$-cat$/k$

$HH^2(\mathcal{C})$ correspond to (left) $k[\varepsilon]$ deformations of $\mathcal{C}$ as c A_∞ category

$$d^2 = [c, -] \qquad \underbrace{c}_{\text{"curvature"}} = m_0$$

$|c| = 2$

Not convenient for homological algebra.

<u>Rk</u> If $\mathcal{E}$ lives in degrees ≤ 0

(i.e. $\forall A, B \in \mathcal{E}$, $\forall i > 0$
$H^i(\mathcal{E}(A,B)) = 0$)

then no curvature problem!

<u>Partial solution</u> (Keller-Lowen)

("Morita deformations")

If $\mathcal{E} \overset{\text{Morita}}{\rightsquigarrow} \mathcal{E}'$ then

(Keller)

$$HH^2(t) \cong HH^2(t')$$

$$\eta \in HH^2(t) \rightsquigarrow \boxed{\eta'} \in HH^2(t')$$

may have representative without curvature

Use this to deform t'

Not clear to which η this applies.

<u>Note</u> $\quad A \in AbCat(k)$

$$t = D^+_{dg}(A)$$

$$HH^*_{Ab}(A) \cong HH^*(t)$$

$$\eta_A \in HH^*_{Ab}(A) \leftsquigarrow \eta \in HH^*(t)$$

$$D^{+}_{dg}\big(\underbrace{A_{\eta_A}}\big)$$

deformation of A corresponding to η_A

← realises η in some "natural way"

"Right deformation."

Lift for

$$\underbrace{R\mathrm{Hom}_{k[\varepsilon]}(k, -)}$$

variant of Toën's Internal Hom.

NO CURVATURE PROBLEM!

Motivation 2

"more general dg-base-rings"

A Ab Cat / k

As above

$\mathrm{Def}^{\text{right}}(A, k[\varepsilon])/\cong \; \cong HH^2_{Ab}(A)$

$HH^0_{Ab}(A)$
$\parallel$
Nat iso's $\mathrm{Id} \to \mathrm{Id}$ of defs.

$HH^1_{Ab}(A)$
$\parallel$
Aut eq's of defs.

What about $HH^n_{Ab}(\mathcal{A})$

$n \geq 3$?

<u>Idea</u> $\mathcal{A} \subseteq \underbrace{D^+_{dg}(\mathcal{A})}$

Δ'ed with t-structure

$$HH^1(\mathcal{A}) \underset{\text{correspond to}}{\overset{\text{should}}{\longleftrightarrow}} k[\varepsilon] \text{ defs}$$

$|\varepsilon| = 2-1$

of $D^+(\mathcal{A})$ as Δ'ed category with t-structure

<u>Derived injectives</u> (Rizzardo VdB, Shaul, Krause)

$\mathcal{T}$ (pre) Δ'ed cat with t-structure with heart $\mathcal{T}^\heartsuit$

$I \in \mathrm{Inj}(\mathcal{T}^\heartsuit)$

$L(I)$ (if existing) is defined by

cohom functor

$$\mathcal{T}(-, \underbrace{L(I)}_{\text{"derived injective"}}) = \mathcal{T}^\heartsuit(H^0(-), I)$$

Properties

(1) $L(I) \in \mathrm{Ob}(\mathcal{T}_{\geq 0})$

$H^0(L(I)) = I$

(2) $(L(I))_{I \in \mathrm{Inj}(A)} \subseteq \mathcal{T}$

full subcat.

dg-cat living in degree ≤ 0

$H^0(\mathrm{Der\,Tny}(\mathcal{T})) \cong \mathrm{Tny}(\mathcal{T}^\heartsuit)$

H^0 of Hom's:

Rk $\quad H^{-i}(-)$ $i > 0$ are

interesting bimodules over

$H^0(-) = \mathrm{Tny}(\mathcal{T}^\heartsuit)$

(detects non-trivial t-structures)

(3) If $\mathcal{T} = D^+_{dg}(\mathcal{A})$

$L(I) = I$

$\mathrm{Der\,Tny}(\mathcal{T}) \cong \mathrm{Tny}(\mathcal{T}^\heartsuit)$

<u>Def</u> $\mathcal{T}$ has <u>enough derived injectives</u> if:

(1) $\mathcal{T}^{\heartsuit}$ has enough injectives.

(2) $L(I)$ exists $\forall I \in \mathrm{Inj}(\mathcal{T}^{\heartsuit})$

<u>Thm</u> If (0) $\mathcal{T}$ has left bounded t-structure;

(1) $\exists$ enough derived injectives;

(2) $\bigoplus_{i \in \mathbb{N}} A_i$ exists if

$$A_i \in \mathrm{Ob}(\mathcal{T}_{\geq n_i})$$

$$n_i \longrightarrow \infty$$

$$\Longrightarrow \quad \mathcal{T} \cong \mathrm{Tw}^{+}\left(\mathrm{Der}_{fry}\left(\mathcal{T}^{\heartsuit}\right)^{\circ}\right)^{\circ}$$

↳ 1-sided twisted complexes.

<u>Fact</u>

$$\mathrm{Def}^{\mathrm{right}}\left(\mathcal{T},\ k[\varepsilon]\right) \qquad |\varepsilon| = 2-n$$

$$\cong \quad \underbrace{\mathrm{HH}^{n}}_{\text{ordinary Hochschild cohomology}}(\mathcal{T})$$

HENCE NO CURVATURE PROBLEM.