

Deformations of Δ 'ed cat's with
(+Lowen, Genovese) E-shedine

Reminder abelian case

k ground field (+Lowen)

$S \twoheadrightarrow R$ commutative finite dim.
 k -algebras

Res:
$$\begin{array}{ccc} \text{Lin}(\text{cat}(R)) & \longrightarrow & \text{Lin}(\text{cat}(S)) \\ \parallel & & \downarrow \\ & & \mathcal{A} \longmapsto \mathcal{A}_S \end{array}$$

$\{ \text{cat's linear}/R \}$

\exists left adjoint $R \otimes_S -$

$$\text{Ob}(R \otimes_S b) = \text{Ob}(b)$$

$$(R \otimes_S B)(A, B) = R \otimes_S B(A, B)$$

B is S -flat if Hom-spaces
are S -flat

$$\Rightarrow R \otimes_S B \text{ } S\text{-flat}$$

Def ^{left} $\mathcal{A} \in \text{Lcn Cat flat } (R)$

An S -deformation of \mathcal{A} is a flat lift
of \mathcal{A} under $R \otimes_S -$

Def (\mathcal{A}, S) 2-category
of S -deps of \mathcal{A}

Fact If $(S, R) = (k[\epsilon], k)$
 $\epsilon^2 = 0$

$$\text{Def } (\mathcal{A}, S) \cong \cong \underline{\text{HH}}^2(\mathcal{A})$$

Hochschild cohomology

Variants $S \rightarrow R$

$$\text{AbCat}(R) \longrightarrow \text{AbCat}(S)$$

$$A \longmapsto A_S$$

\exists nice right adjoint

$$\text{Hom}_S(R, -)$$

$$B \in \text{AbCat}(S)$$

$$\text{Hom}_S(R, \mathcal{D}) \stackrel{\text{full}}{\subseteq} B$$

$$\text{Ob}(\text{Hom}_S(R, \mathcal{D})) =$$

$$\{ B \in \text{Ob}(B) \mid \exists B=0 \}$$

$$I = \ker(S \rightarrow R)$$

Need notion of flatness

Assume $A \in \text{Ab Cat}(R)$

has enough injectives

(general case: consider $\text{Ind } A$)

Def A is flat if
 $\text{Inj}(A)$ is flat.

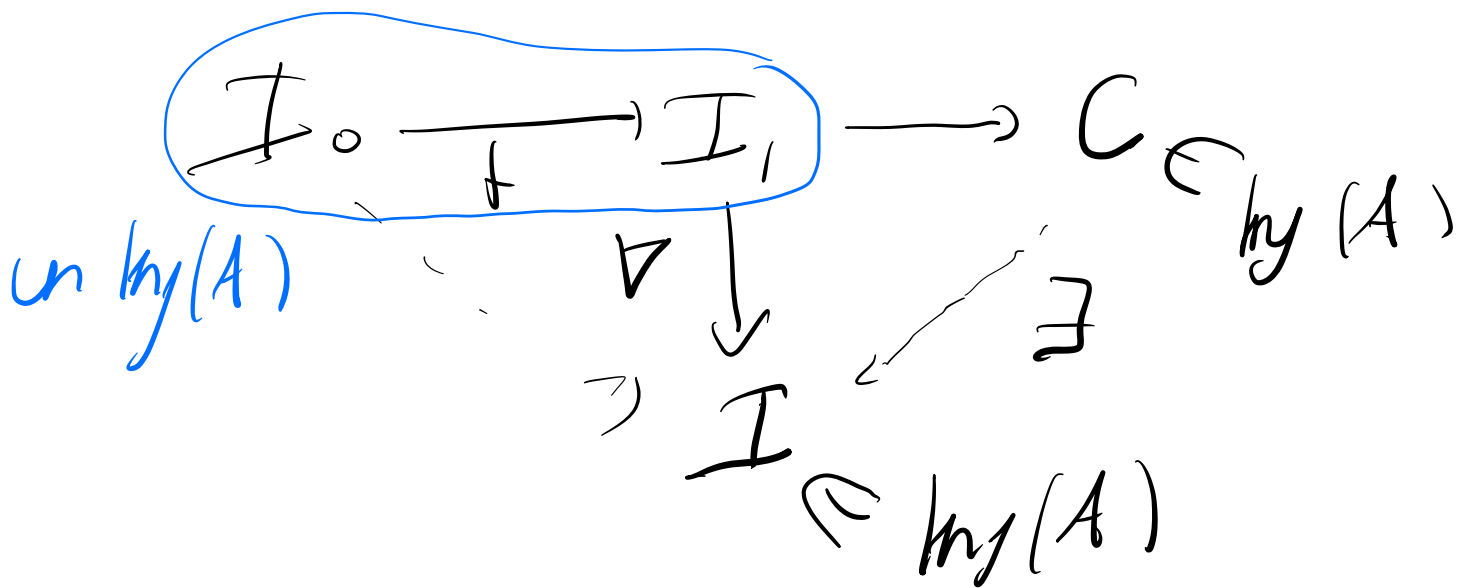
Def $A \in \text{Ab Cat}$ flat (R) .

An S -def of A is a
flat lift under $\text{Hom}_S(R, -)$

Def (A, S) : 2-cat of S -defs
of A

How to study?

Fact $\text{Inj}(A)$ has weak cokernels



$C = \text{Inj hull}(\text{coker } f)$

Classical fact (from weak cokernels)

$\text{mod}(\text{Inj}(A)^{\circ})$ is abelian

finitely presented
right $\text{Inj}(A)$ modules

We have moreover

$$A \cong \text{mod} (\text{trg}(A)^{\circ})$$

$$A \longmapsto A(A, -)$$

Fact

$$\text{Dex}^{\text{right}}(A, S) \cong$$

$$\text{Dex}^{\text{left}}(\text{trg}(A), S)$$

Leads to

$$\text{HH}_{\text{Ab}}^*(A) := \text{HH}^*(\text{trg}(A))$$

Note: ~~X~~

$$\text{HH}_{\text{LinCat}}^*(A)$$

Fact $\mathcal{H}(S, R) = (k[\epsilon], k)$

then $\text{Def}(A, k[\epsilon]) / \cong \cong$

$$\text{HH}_{Ab}^2(A)$$

Δ' ed cat's w. t-structure

Motivation 1

The "curvature problem"

$$\mathcal{L} \left\{ \begin{array}{l} A_\infty \\ dg \end{array} \right\} - \text{cat}/k$$

$\text{HH}^2(\mathcal{L})$ corresponds

to (left) $k[\epsilon]$ deformations
of \mathcal{L} as A_∞ category

$$d^2 = [c_1, -1]$$

"curvature" $|c_1| = 2$

Not convenient for homological algebra.

Rk If E lives in degrees ≤ 0

(i.e. $\forall A, B \in E, \forall i \geq 0$

$$H^i(E(A, B)) = 0)$$

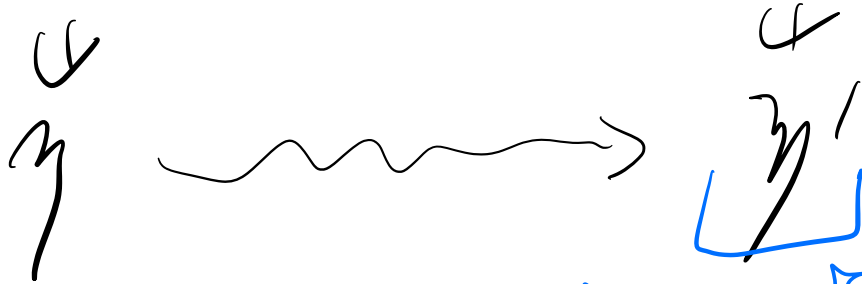
then no curvature problem!

Partial solution (Keller-Lavren)

("Morita deformations")

If $E \xrightarrow{\text{Morita}} E'$ then
(Keller)

$$HH^2(t) \cong HH^2(t')$$



may have representative without curvature

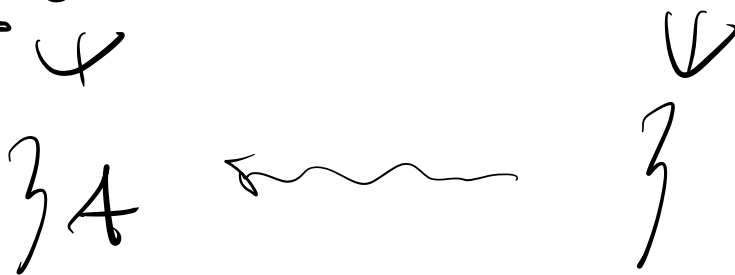
Use this to deform t'

Not clear to which γ this applies.

Note $A \in \text{AbCat}(k)$

$$L = D_{dg}^+(A)$$

$$HH_{Ab}^*(A) \cong HH^*(L)$$



$$D_{dy}^+ (A \eta_A)$$

deformation of A
corresponding to η_A

← realises η
in some

“natural way”

lift for

“Right deformation”

$$R\text{Hom}_{k[a]}(k, -)$$

variant of Toën's
Internal Hom.

NO CURVATURE PROBLEM!

Motivation 2

↳ more general dg-base-rings

$A \quad A \text{ cat} / k$

As above

$$\text{Dey}^{\text{right}}(A, k[c]) \cong \cong \text{HH}_{Ab}^2(A)$$

$$\text{HH}_{Ab}^0(A)$$

\cong

Nat iso's
 $\text{Id} \rightarrow \text{Id}$
of defs.

$$\text{HH}_{Ab}^1(A)$$

\cong

Aut eq's
of defs.

What about $\mathrm{HH}_{Ab}^n(A)$

$n \geq 3$?

Idea $A \subseteq \underline{D_{\mathrm{alg}}^+(A)}$
 Δ' 'ed with t -structure

$\mathrm{HH}^1(A)$ $\xleftrightarrow[\text{correspond to}]{\text{should}}$ $k[\epsilon]$ defs
 $|\mathcal{E}| = 2 - f$

of $D^+(A)$ as Δ' 'ed
category with t -structure

Derived injectives (Rizzardo VdB
Shaul
Krause)

\mathcal{Z} (pre) Δ' 'ed cat with
 t -structure with heart \mathcal{Z}^\heartsuit

$$I \in \text{Mod}(Z^\heartsuit)$$

$L(I)$ (if existing) is

defined by

$$\tau(-, L(I)) = \tau^\heartsuit(H^0(-), I)$$

cohom functor
↓

“derived injective”

Properties

$$(1) \quad L(I) \in \text{Ob}(\tau_{\geq 0})$$

$$H^0(L(I)) = I$$

$$(2) \quad (L(I))_{I \in \text{Mod}(A)} \subseteq \tau$$

full subcat.

dg-cat living in degree ≤ 0

$$H^0(\text{Der } \text{trg}(\mathcal{Z})) \cong \text{trg}(\mathcal{Z}^\heartsuit)$$

H^0 of Hom's: $i > 0$
Rk $H^{-i}(-)$ are

interesting bundles over

$$H^0(-) = \text{trg}(\mathcal{Z}^\heartsuit)$$

(detects non-trivial t-structures)

$$(3) \text{ If } \mathcal{Z} = D_{\text{dg}}^+(A)$$

$$L(\mathbb{I}) = \mathbb{I}$$

$$\text{Der } \text{trg}(\mathcal{Z}) \cong \text{trg}(\mathcal{Z}^\heartsuit)$$

Def \mathcal{T} has enough
derived injectives if:

(1) \mathcal{T}^\heartsuit has enough
injectives.

(2) $L(I)$ exists $\forall I \in \text{Inj}(\mathcal{T}^\heartsuit)$

Thm If (0) \mathcal{T} has left bounded t -
structure;

(1) Enough derived injectives;

(2) $\bigoplus_{i \in \mathbb{N}} A_i$ exists if

$A_i \in \text{Ob}(\mathcal{T}_{\geq n_i})$

$n_i \rightarrow \infty$

$\Rightarrow \mathcal{Z} \equiv Tw^+ (\text{Density } (\mathcal{Z}^\vee)^0)^0$
 \hookrightarrow 1-sided twisted complexes.

Fact

Def^{right} $(\mathcal{Z}, k[\mathcal{E}]) \quad |\mathcal{E}| = 2-n$

$\cong \underbrace{HH^n(\mathcal{Z})}$

ordinary Hochschild
cohomology

HENCE NO CURVATURE
PROBLEM.