

On Isotropic and Numerical equivalence of cycles

$DM(k)$ - Voevodsky Δ -ed motivic category

$DM(k/k, \mathbb{F}_p)$ - Isotropic motivic category

= localiz. of $DM(k; \mathbb{F}_p)$ by the subcat. gen. by the motives of p -anisotropic varieties

X - anis., if degrees of all closed points are divisible by p .

Such categories should be much simpler than $DM(k)$, if k is "flexible": $k = k_0(t_1, t_2, \dots) = k_0(\mathbb{P}^\infty)$
 k_0 - some other field

In such a situation, Hom's between compact objects should be finite groups.

E/k - fin. gen. extension. Have a partial order:

$E/k \geq F/k$, if X - smooth, $k(X) = E$
 Y - proj., $k(Y) = F$

\exists a corresp. $X \xrightarrow{d} Y$ of degree

prime to p . $Ch = CH/p$
 $\Leftrightarrow Ch_*(X \times Y) \xrightarrow{\pi_*} Ch_*(X)$ is surj.

Get "isotropic realizations"

$$\Psi_{E,p}: DM(k) \rightarrow DM(\tilde{E}/E, \mathbb{F}_p)$$

$\tilde{E} = E(P^\infty)$, E - runs over the equivalence classes of fin. gen. ext. of k under the above order.

$DM(\tilde{E}/E, \mathbb{F}_p)$ should be much simpler than $DM(k)$

$H_{\mathcal{M}}^{*,*'}(\tilde{E}/E, \mathbb{F}_p)$ - is rigid (doesn't depend on the choice of a flexible field)

\uparrow
 isotropic motivic coh. of a point
 $\underline{p=2}$

and $\simeq \bigwedge_{\mathbb{F}_2} (z_i | i=0,1,\dots)$

\uparrow "dual" to Q_i - Milnor's op-h

$\deg(Q_i) = (2^i - 1) \lfloor \frac{2^{i+1} - 1}{2} \rfloor$ $Q_i(z_i) = 1$

$\underline{p=2}$

F. Tanania: Computation of isotropic stable homotopy groups of

spheres = E_2 -term of the classical Adams spectral sequence.

Pure part

$\text{Chow}(k/k, \mathbb{F}_p)$ — isotropic Chow motives

Hom's — isotropic Chow groups.

$$\text{Ch}_{k/k}^* = \underset{\substack{\uparrow \\ \text{an oriented} \\ \text{coh. theory with localiz.}}}{\text{Ch}}_{\substack{\text{CH}_p \\ \text{an oriented} \\ \text{coh. theory with localiz.}}} / (\text{anisotropic classes})$$

$\text{im}(\text{Ch}_*(Y) \xrightarrow{f_*} \text{Ch}_*(X))$
 Y — anis.

an oriented
coh. theory with localiz.

X — smooth proj.

$$\text{Ch}_{k/k}^*(X) \times \text{Ch}_{k/k}^*(X) \longrightarrow \mathbb{F}_p$$

We get a surj.: $\text{Ch}_{k/k}^* \twoheadrightarrow \text{Ch}_{\text{Num}(p)}^*$

Conjecture k — flexible, then

$$\text{Ch}_{k/k}^* = \text{Ch}_{\text{Num}}^*$$

It implies :

- $\text{Chow}(k/k, \mathbb{F}_p)$ is equiv. to $\text{Chow}(k, \mathbb{F}_p)$
 N_{num}
 - Isotropic Chow groups should be finite groups.
 - $\Psi_{E,p}$ - provide points of the Balmer's spectrum $\text{Spc}(\text{DM}^c(k))$
 \rightarrow get many new points
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The conj. was known for :

- 1) Ch^1
- 2) $\dim(X) \leq 5$
- 3) $\text{Ch}_z, z \leq 2$

Can expand this ...

Theorem] The conj. is true for:

- 1) $\dim(X) \leq 2p$
- 2) $\text{Ch}_z, z < p$
- 3) $\text{Ch}^z, z \leq p, \dim X < p^2 - p + z$

In particular, for a given X , the conj. is true for suff. large p .

"Sketch of the proof":

Need to show: u -num. \Rightarrow u -anis

Easy for divisors: $|D|$ -very ample linear system,

D_2 -generic repres. Then $Ch^*(X) \rightarrow Ch^*(D_2)$ is surjective \Rightarrow if D is num. triv., then D_2 -anis.

The same arguments give: any num. trivial complete intersection is anis.

In general

The Main case: $u \in Ch^z(X)$, $z \leq p$, $\dim X < zp$

Have $V \in K_0(X)$, s.t. $c_z(V) = u$ and $c_i(V) = 0$, $\forall i < z$.

$Z \subset X$, $\text{codim} = z$, $c_z(\mathcal{O}_Z) = (z-1)! [Z]$.

Then we apply the linear comb. of Adams op-s to get all Chern classes of V num. triv ($\dim(X) < zp$).

Proposition $V \in K_0(X)$, s.t. $c_*(V)$ is num. triv, then $c_*(V)$ is anis.

"Proof" Can assume: $V = \bigoplus_{i=1}^N \mathcal{O}(a_i)$.
 A_i - repres. of a_i very ample

$\bigcup_i A_i$ - divisor with strict normal crossings

$$\bar{N} = \{1, 2, \dots, N\}, \quad I \subset \bar{N}, \quad A_I = \bigcap_{i \in I} A_i$$

The Main Construction:

$\pi: \tilde{X} \rightarrow X$ - blow-up all A_I s.

$\tilde{\sigma}_I$ = the class of the pre-image of A_I

The $\pi^*(c_*(V)) = c_*(\tilde{V})$, where

$$\tilde{V} = \bigoplus_{j=1}^M \mathcal{O}(b_j), \quad b_j = \sum_{|I| \geq j} \tilde{\sigma}_I$$

B_j - repr. of b_j .

Note: $M \leq \dim X$

Moreover, the top Chern class of \tilde{V} is a complete inters.

$$B_M = \bigcap_j B_j$$

If we substitute B_j by generic repres.

of the resp. linear systems, then $B_{\overline{M}}$ will be anis. (as a num. tri. complete int.)

Apply the Main Constr. again:

$$\widetilde{V} = \bigoplus_{\ell=1}^M O(C_{\ell}), \quad C_M = \text{class of the pre-image of } B_{\overline{M}}$$

$\Rightarrow C_M$ is anis. \Rightarrow can remove $O(C_M)$ from \widetilde{V} and decrease the $\dim(V)$.

\Rightarrow done by the induction on $\dim(V)$. \square