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DG quotients of DG categories

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Abstract

Keller introduced a notion of quotient of a differential graded category modulo a full differential graded subcategory which agrees with Verdier's notion of quotient of a triangulated category modulo a triangulated subcategory. This work is an attempt to further develop his theory.

Keywords: DG category; Triangulated category; Derived category; Localization

Conventions. We fix a commutative ring k and write $\otimes$ instead of $\otimes_k$ and "DG category" instead of "differential graded k-category." If $\mathcal{A}$ is a DG category, we write "DG module over $\mathcal{A}$" instead of "DG functor from $\mathcal{A}$ to the DG category of complexes of k-modules" (more details on the DG module terminology can be found in Appendix C). Unless stated otherwise, all categories are assumed to be small. Triangulated categories are systematically viewed as $\mathbb{Z}$-graded categories (see A.1). A triangulated subcategory $\mathcal{C}'$ of a triangulated subcategory $\mathcal{C}$ is required to be full, but we do not require it to be strictly full (i.e., to contain all objects of $\mathcal{C}$ isomorphic to an object of $\mathcal{C}'$). In the definition of quotient of a triangulated category we do not require the subcategory to be thick (see A.2, A.3).

1. Introduction

1.1. It has been clear to the experts since the 1960s that Verdier's notions of derived category and triangulated category [56,57] are not quite satisfactory: when you pass to the

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homotopy category you forget too much. This is why Grothendieck developed his derivator theory [17,40].

A different approach was suggested by Bondal and Kapranov [4]. According to [4], one should work with pretriangulated DG categories rather than with triangulated categories in Verdier's sense (e.g., with the DG category of bounded above complexes of projective modules rather than the bounded above derived category of modules). Hopefully, the part of homological algebra most relevant for algebraic geometry will be rewritten using DG categories or rather the more flexible notion of A_∞-category due to Fukaya and Kontsevich (see [14,15,24,25,30,31,33,36,37]), which goes back to Stasheff's notion of A_∞-algebra [51,52].

One of the basic tools developed by Verdier [56,57] is the notion of quotient of a triangulated category by a triangulated subcategory. Keller [23] has started to develop a theory of quotients in the DG setting. This work is an attempt to further develop his theory. I tried to make this article essentially self-contained, in particular, it can be read independently of [23].

The notion of quotient in the setting of A_∞-categories is being developed by Kontsevich and Soibelman [33] and Lyubashenko and Ovsienko [38].

1.2. The basic notions related to that of DG category are recalled in Section 2. Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. Let $\mathcal{A}^{\mathrm{tr}}$ denote the triangulated category associated to $\mathcal{A}$ (we recall its definition in 2.4). A *DG quotient* (or simply a *quotient*) of $\mathcal{A}$ modulo $\mathcal{B}$ is a diagram of DG categories and DG functors

$$\mathcal{A} \xleftarrow{\approx} \widetilde{\mathcal{A}} \xrightarrow{\xi} \mathcal{C} \tag{1.1}$$

such that the DG functor $\widetilde{\mathcal{A}} \to \mathcal{A}$ is a quasi-equivalence (see 2.3 for the definition), the functor $\mathrm{Ho}(\widetilde{\mathcal{A}}) \to \mathrm{Ho}(\mathcal{C})$ is essentially surjective, and the functor $\widetilde{\mathcal{A}}^{\mathrm{tr}} \to \mathcal{C}^{\mathrm{tr}}$ induces an equivalence $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to \mathcal{C}^{\mathrm{tr}}$. Keller [23] proved that a DG quotient always exists (recall that our DG categories are assumed to be small, otherwise even the existence of $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}$ is not clear). We recall his construction of the DG quotient in Section 4, and give a new construction in Section 3.

The new construction is reminiscent of but easier than Dwyer–Kan localization [11–13]. It is very simple under a certain flatness assumption (which is satisfied automatically if one works over a field): one just kills the objects of $\mathcal{B}$ (see 3.1). Without this assumption one has to first replace $\mathcal{A}$ by a suitable resolution (see 3.5).

The idea of Keller's original construction of the DG quotient (see Section 4) is to take the orthogonal complement of $\mathcal{B}$ as a DG quotient, but as the orthogonal complement of $\mathcal{B}$ in $\mathcal{A}$ is not necessarily big enough he takes the complement not in $\mathcal{A}$ but in its ind-version $\underrightarrow{\mathcal{A}}$ studied by him in [22]. The reason why it is natural to consider the orthogonal complement in $\underrightarrow{\mathcal{A}}$ is explained in 1.5. Of course, instead of $\underrightarrow{\mathcal{A}}$ one can use the pro-version $\underleftarrow{\mathcal{A}}$.

Keller's construction using $\underrightarrow{\mathcal{A}}$ (respectively $\underleftarrow{\mathcal{A}}$) is convenient for considering right (respectively left) derived DG functors (see Section 5).

In 6.1 we show that the DG quotient of $\mathcal{A}$ modulo $\mathcal{B}$ is "as unique as possible," so one can speak of *thhe* DG quotient of $\mathcal{A}$ modulo $\mathcal{B}$ ("thhe" is the homotopy version of

"the"). In 1.6.2 and 1.7 we give another explanation of uniqueness. Unfortunately, both explanations are somewhat clumsy.

1.3. Hom complexes of the DG quotient

We are going to describe them first as objects of the derived category of k-modules (see 1.3.1), then in a stronger sense (see 1.3.2). We will do it by successive approximation starting with less precise and less technical statements.

1.3.1. Each construction of the DG quotient shows that if $X, Y \in \operatorname{Ob}\mathcal{A}$, $\widetilde{X}, \widetilde{Y} \in \operatorname{Ob}\widetilde{\mathcal{A}}$, $\widetilde{X} \mapsto X$, $\widetilde{Y} \mapsto Y$ then the complex

$$\operatorname{Hom}_{\mathcal{C}}\big(\xi\big(\widetilde{X}\big), \xi\big(\widetilde{Y}\big)\big) \tag{1.2}$$

viewed as an object of the derived category of complexes of k-modules is canonically isomorphic to

$$\operatorname{Cone}\big(h_Y \overset{L}{\otimes}_{\mathcal{B}} \tilde{h}_X \to \operatorname{Hom}(X, Y)\big), \tag{1.3}$$

where h_Y is the right DG $\mathcal{B}$-module defined by $h_Y(Z) := \operatorname{Hom}(Z, Y)$, $Z \in \mathcal{B}$, and $\tilde{h}_X$ is the left DG $\mathcal{B}$-module defined by $\tilde{h}_X(Z) := \operatorname{Hom}(X, Z)$, $Z \in \mathcal{B}$. One can compute $h_Y \overset{L}{\otimes}_{\mathcal{B}} \tilde{h}_X$ using a semi-free resolution of h_Y or h_X (see C.8 for the definition of "semi-free"), and this corresponds to Keller's construction of the DG quotient. If h_Y or $\tilde{h}_X$ is homotopically flat over k (see 3.3 for the definition of "homotopically flat") then one can compute $h_Y \overset{L}{\otimes}_{\mathcal{B}} \tilde{h}_X$ using the bar resolution, and this corresponds to the new construction of the DG quotient (see Remarks 3.6(i)).

1.3.2. Let $D(\mathcal{A})$ denote the derived category of *right* DG modules over $\mathcal{A}$. By 2.7 the functor $D(\mathcal{A}) \to D(\widetilde{\mathcal{A}})$ is an equivalence, so for fixed $\widetilde{Y} \in \operatorname{Ob}\widetilde{\mathcal{A}}$ the complex (1.2) defines an object of $D(\mathcal{A})$. This object is canonically isomorphic to (1.3). Quite similarly, for fixed $\widetilde{X} \in \operatorname{Ob}\widetilde{\mathcal{A}}$ the complex (1.2) viewed as an object of $D(\widetilde{\mathcal{A}}^{\circ})$ is canonically isomorphic to (1.3). If $\widetilde{\mathcal{A}}$ is homotopically flat over k (see 3.3) then (1.2) and (1.3) are canonically isomorphic in $D(\widetilde{\mathcal{A}} \otimes_k \widetilde{\mathcal{A}}^{\circ})$ (see Remarks 3.6(i)). (Without the homotopical flatness assumption they are canonically isomorphic as objects of the category $D(\mathcal{A} \overset{L}{\otimes} \mathcal{A}^{\circ})$ defined in E.5.)

1.3.3. Let $(1.2)_Y$ denote (1.2) viewed as an object of $D(\mathcal{A})$. The morphism $(1.3)_Y \to (1.2)_Y$ mentioned in 1.3.1 and 1.3.2 is uniquely characterized by the following property: the composition $h_Y := \operatorname{Hom}(?, Y) \to (1.3)_Y \to (1.2)_Y$ equals the obvious morphism $\operatorname{Hom}(?, Y) \to (1.2)_Y$. To prove the existence and uniqueness of such a morphism, we may assume that $\widetilde{\mathcal{A}} = \mathcal{A}$ and the DG functor $\widetilde{\mathcal{A}} \to \mathcal{A}$ equals $\mathrm{id}_{\mathcal{A}}$. Rewrite the DG $\mathcal{A}^{\circ}$-module $X \mapsto h_Y \overset{L}{\otimes}_{\mathcal{B}} \tilde{h}_X$ as $L\operatorname{Ind}\operatorname{Res} h_Y$ (here $\operatorname{Res}: D(\mathcal{A}) \to D(\mathcal{B})$ is the restriction functor and $L\operatorname{Ind}$ is its left adjoint, i.e., the derived induction functor) and notice that $\operatorname{Hom}(L\operatorname{Ind}\operatorname{Res} h_Y, M) = 0$ for every DG $\mathcal{A}^{\circ}$-module M with $\operatorname{Res} M = 0$, in particular

for $M = (1.2)_Y$. As $\mathrm{Res}\,(1.2)_Y = 0$ and $\mathrm{Res}\cdot L\,\mathrm{Ind} \simeq \mathrm{id}$, the fact that our morphism $(1.3)_Y \to (1.2)_Y$ is an isomorphism is equivalent to the implication (i) $\Rightarrow$ (ii) in the following proposition.

1.4. Proposition. *Let $\xi : \mathcal{A} \to \mathcal{C}$ be a DG functor and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory such that the objects of $\xi(\mathcal{B})$ are contractible and* $\mathrm{Ho}(\xi) : \mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{C})$ *is essentially surjective. Then the following properties are equivalent*:

(i) *$\xi : \mathcal{A} \to \mathcal{C}$ is a DG quotient of $\mathcal{A}$ modulo $\mathcal{B}$;*
(ii) *for every $Y \in \mathcal{A}$ the DG $\mathcal{A}^\circ$-module*

$$X \mapsto \mathrm{Cone}\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}\big(\xi(X), \xi(Y)\big)\big) \tag{1.4}$$

is in the essential image of the derived induction functor $L\,\mathrm{Ind} : D(\mathcal{B}) \to D(\mathcal{A})$;
(ii°) *for every $X \in \mathcal{A}$ the DG $\mathcal{A}$-module*

$$Y \mapsto \mathrm{Cone}\big(\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}\big(\xi(X), \xi(Y)\big)\big)$$

is in the essential image of $L\,\mathrm{Ind} : D(\mathcal{B}^\circ) \to D(\mathcal{A}^\circ)$.

The proof is contained in 9.3.

Remark. A DG $\mathcal{A}^\circ$-module M belongs to the essential image of the derived induction functor $L\,\mathrm{Ind} : D(\mathcal{B}) \to D(\mathcal{A})$ if and only if the morphism $L\,\mathrm{Ind}\,\mathrm{Res}\,M \to M$ is a quasi-isomorphism.

1.5. On Keller's construction of the DG quotient

As explained in 10.2, the next proposition follows directly from Proposition 1.4. The symbol $\mathrm{Ho}^\cdot$ below denotes the graded homotopy category (see 2.3).

1.5.1. Proposition. *Let $\xi : \mathcal{A} \to \mathcal{C}$ be a DG quotient of $\mathcal{A}$ modulo $\mathcal{B}$ and let $\xi^* : D(\mathcal{C}) \to D(\mathcal{A})$ be the corresponding restriction functor. Then*

(a) *the composition* $\mathrm{Ho}^\cdot(\mathcal{C}) \hookrightarrow D(\mathcal{C}) \to D(\mathcal{A})$ *is fully faithful*;
(b) *an object of $D(\mathcal{A})$ belongs to its essential image if and only if it is isomorphic to* $\mathrm{Cone}(L\,\mathrm{Ind}\,\mathrm{Res}\,a \to a)$ *for some $a \in \mathrm{Ho}^\cdot(\mathcal{A}) \subset D(\mathcal{A})$, where* $L\,\mathrm{Ind}$ (*respectively* Res) *is the derived induction* (*respectively restriction*) *functor corresponding to $\mathcal{B} \hookrightarrow \mathcal{A}$.*

Remark. In fact, the whole functor $D(\mathcal{C}) \to D(\mathcal{A})$ is fully faithful (see Theorem 1.6.2(ii) or Proposition 4.6(ii)).

1.5.2. So if $\xi : \mathcal{A} \to \mathcal{C}$ is a DG quotient then $\mathrm{Ho}^\cdot(\mathcal{C})$ identifies with a full subcategory of $D(\mathcal{A})$. But $D(\mathcal{A}) = \mathrm{Ho}^\cdot(\underset{\to}{\mathcal{A}})$, where $\underset{\to}{\mathcal{A}}$ is the DG category of semi-free DG $\mathcal{A}^\circ$-modules (see C.8). Thus, $\mathrm{Ho}^\cdot(\mathcal{C})$ identifies with the graded homotopy category of a certain DG subcategory of $\underset{\to}{\mathcal{A}}$. This is the DG quotient $\mathcal{A} \nearrow \mathcal{B}$ from Section 4.

1.6. Universal property of the DG quotient

1.6.1. 2-category of DG categories

There is a reasonable way to organize all (small) DG categories into a 2-category *DGcat*, i.e., to associate to each two DG categories $\mathcal{A}_1, \mathcal{A}_2$ a *category of quasi-functors* $T(\mathcal{A}_1, \mathcal{A}_2)$ and to define weakly associative composition functors $T(\mathcal{A}_1, \mathcal{A}_2) \times T(\mathcal{A}_2, \mathcal{A}_3) \to T(\mathcal{A}_1, \mathcal{A}_3)$ so that for every DG category $\mathcal{A}$ there is a weak unit object in $T(\mathcal{A}, \mathcal{A})$. Besides, each $T(\mathcal{A}_1, \mathcal{A}_2)$ is equipped with a graded k-category structure, and if $\mathcal{A}_2$ is pretriangulated in the sense of 2.4 then $T(\mathcal{A}_1, \mathcal{A}_2)$ is equipped with a triangulated structure. We need *DGcat* to formulate the universal property 1.6.2 of the DG quotient. The definition of *DGcat* will be recalled in Appendix E. Here are two key examples.

Examples.

(i) Let $\mathcal{K}$ be a DG model of the derived category of complexes of k-modules (e.g., $\mathcal{K} =$ the DG category of semi-free DG k-modules). Then $T(\mathcal{A}, \mathcal{K})$ is the derived category of DG $\mathcal{A}$-modules. (If $\mathcal{K}$ is not small then $T(\mathcal{A}, \mathcal{K})$ is defined to be the direct limit of $T(\mathcal{A}, \mathcal{K}')$ for all small full DG subcategories $\mathcal{K}' \subset \mathcal{K}$.)

(ii) If $\mathcal{A}_0$ is the DG category with one object whose endomorphism DG algebra equals k then $T(\mathcal{A}_0, \mathcal{A})$ is the graded homotopy category $\mathrm{Ho}^{\cdot}(\mathcal{A})$.

It is clear from the definition of $T(\mathcal{A}_1, \mathcal{A}_2)$ (see Appendix E) or from example (ii) above that $\Phi \in T(\mathcal{A}_1, \mathcal{A}_2)$ induces a graded functor $\mathrm{Ho}^{\cdot}(\mathcal{A}_1) \to \mathrm{Ho}^{\cdot}(\mathcal{A}_2)$ and thus $\mathrm{Ho}^{\cdot}$ becomes a (non-strict) 2-functor from *DGcat* to that of graded categories. It is also clear from Appendix E that one has a bigger 2-functor $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{tr}}$ from *DGcat* to the 2-category of triangulated categories (with triangulated functors as 1-morphisms).

A DG functor $F: \mathcal{A}_1 \to \mathcal{A}_2$ defines an object $\Phi_F \in T(\mathcal{A}_1, \mathcal{A}_2)$ (see E.7.1). Thus, one gets a 2-functor $DGcat_{\mathrm{naive}} \to DGcat$, where $DGcat_{\mathrm{naive}}$ is the 2-category with DG categories as objects, DG functors as 1-morphisms, and degree zero morphisms of DG functors as 2-morphisms. If F is a quasi-equivalence then Φ_F is invertible. So a diagram $\mathcal{A}_1 \xleftarrow{\approx} \widetilde{\mathcal{A}}_1 \xrightarrow{F} \mathcal{A}_2$ still defines an object of $T(\mathcal{A}_1, \mathcal{A}_2)$. All isomorphism classes of objects of $T(\mathcal{A}_1, \mathcal{A}_2)$ come from such diagrams (see E.7.2 and B.5).

1.6.2. Main Theorem. *Let $\mathcal{B}$ be a full DG subcategory of a DG category $\mathcal{A}$. For all pairs $(\mathcal{C}, \xi)$, where $\mathcal{C}$ is a DG category and $\xi \in T(\mathcal{A}, \mathcal{C})$, the following properties are equivalent*:

(i) *the functor* $\mathrm{Ho}(\mathcal{A}) \to \mathrm{Ho}(\mathcal{C})$ *corresponding to ξ is essentially surjective, and the functor $\mathcal{A}^{\mathrm{tr}} \to \mathcal{C}^{\mathrm{tr}}$ corresponding to ξ induces an equivalence $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to \mathcal{C}^{\mathrm{tr}}$;*

(ii) *for every DG category $\mathcal{K}$ the functor $T(\mathcal{C}, \mathcal{K}) \to T(\mathcal{A}, \mathcal{K})$ corresponding to ξ is fully faithful and $\Phi \in T(\mathcal{A}, \mathcal{K})$ belongs to its essential image if and only if the image of Φ in $T(\mathcal{B}, \mathcal{K})$ is zero.*

A pair $(\mathcal{C}, \xi)$ *satisfying* (i), (ii) *exists and is unique in the sense of DGcat.*

A weaker version of the universal property was proved by Keller, who worked not with the 2-category *DGcat* but with the category whose morphisms are 2-isomorphism classes of 1-morphisms of *DGcat* (see [23, Theorem 4.6, Proposition 4.1, and Lemma 4.2]). Theorem 1.6.2 will be proved in 11.2 using the following statement, which easily follows (see 11.1) from Proposition 1.4.

1.6.3. Proposition. *Let $\xi : \mathcal{A} \to \mathcal{C}$ be a quotient of a DG category $\mathcal{A}$ modulo a full DG subcategory $\mathcal{B}$. If a DG category $\mathcal{K}$ is homotopically flat over k then $\xi \otimes \mathrm{id}_{\mathcal{K}} : \mathcal{A} \otimes \mathcal{K} \to \mathcal{C} \otimes \mathcal{K}$ is a quotient of the DG category $\mathcal{A} \otimes \mathcal{K}$ modulo $\mathcal{B} \otimes \mathcal{K}$.*

1.7. More on uniqueness

Let $(\mathcal{C}_1, \xi_1)$ and $(\mathcal{C}_2, \xi_2)$, $\xi_i \in T(\mathcal{A}, \mathcal{C}_i)$, be DG quotients of $\mathcal{A}$ modulo $\mathcal{B}$. Then one has an object $\Phi \in T(\mathcal{C}_1, \mathcal{C}_2)$ defined up to unique isomorphism. In fact, the graded category $T(\mathcal{C}_1, \mathcal{C}_2)$ comes from a certain DG category (three choices of which are mentioned in E.8) and one would like to lift Φ to a homotopically canonical object of this DG category. The following argument shows that this is possible under reasonable assumptions. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are homotopically flat over k in the sense of 3.3, these assumptions hold for the Keller model (see E.8, in particular (E.4)).

Suppose that $T(\mathcal{A}, \mathcal{C}_i)$ (respectively $T(\mathcal{C}_1, \mathcal{C}_2)$) is realized as the graded homotopy category of a DG category $\mathbb{DG}(\mathcal{A}, \mathcal{C}_i)$ (respectively $\mathbb{DG}(\mathcal{C}_1, \mathcal{C}_2)$) and suppose that the graded functor

$$T(\mathcal{A}, \mathcal{C}_1) \times T(\mathcal{C}_1, \mathcal{C}_2) \times T(\mathcal{A}, \mathcal{C}_2)^\circ \to \{\text{Graded } k\text{-modules}\}$$

defined by $(F_1, G, F_2) \mapsto \bigoplus_n \mathrm{Ext}^n(F_2, GF_1)$ is lifted to a DG functor

$$\Psi : \mathbb{DG}(\mathcal{A}, \mathcal{C}_1) \times \mathbb{DG}(\mathcal{C}_1, \mathcal{C}_2) \times \mathbb{DG}(\mathcal{A}, \mathcal{C}_2)^\circ \to k\text{-DGmod}, \tag{1.5}$$

where k-DGmod is the DG category of complexes of k-modules. We claim that once ξ_i, $i \in \{1, 2\}$, is lifted to an object of $\mathbb{DG}(\mathcal{A}, \mathcal{C}_i)$ one can lift $\Phi \in T(\mathcal{C}_1, \mathcal{C}_2)$ to an object of $\mathbb{DG}(\mathcal{C}_1, \mathcal{C}_2)$ in a homotopically canonical way. Indeed, once ξ_i is lifted to an object of $\mathbb{DG}(\mathcal{A}, \mathcal{C}_i)$ the DG functor (1.5) yields a DG functor $\psi : \mathbb{DG}(\mathcal{C}_1, \mathcal{C}_2) \to k$-DGmod such that the corresponding graded functor $T(\mathcal{C}_1, \mathcal{C}_2) \to \{\text{Graded } k\text{-modules}\}$ is corepresentable (it is corepresentable by Φ). Such a functor defines a homotopically canonical object of $\mathbb{DG}(\mathcal{C}_1, \mathcal{C}_2)$ (see Lemma C.16.2, C.16.3).

1.8. What do DG categories form?

To formulate uniqueness of the DG quotient in a more elegant and precise way than in 1.7, one probably has to spell out the relevant structure on the class of all DG categories (which is finer than the structure of 2-category). I hope that this will be done by the experts. Kontsevich and Soibelman are working on this subject. They introduce in [33,34] a notion of homotopy n-category so that a homotopy 1-category is the same as an A_∞-category (the notion of homotopy category is defined in [34] with respect to some category of

"spaces," and in this description of the results of [34] we assume that "space" = "complex of k-modules"). They show that homotopy 1-categories form a homotopy 2-category and they hope that homotopy n-categories form a homotopy $(n+1)$-category. They also show that the notion of homotopy n-category is closely related to the little n-cubes operad. E.g., they prove in [32,34] that endomorphisms of the identity 1-morphism of an object of a homotopy 2-category form an algebra over the chain complex of the little squares operad (Deligne's conjecture). As DG categories are A_∞-categories, we will hopefully understand what DG categories form as soon as Kontsevich and Soibelman publish their results.

In the available texts they assume that the ground ring k is a field. Possibly the case of an arbitrary ground ring k is not much harder for experts, but a non-expert like myself becomes depressed when he comes to the conclusion that DG models of the triangulated category $T(\mathcal{A},\mathcal{K})$ are available only if you first replace $\mathcal{A}$ or $\mathcal{K}$ by a resolution which is homotopically flat over k (see E.8).

1.9. Structure of the article

In Section 2 we recall the basic notions related to DG categories. In Sections 3, 4 we give the two constructions of the quotient DG category. In Sections 5 and 7 we discuss the notion of derived DG functor. The approach of Section 5 is based on Keller's construction of the DG quotient, while the approach of Section 7 is based on any DG quotient satisfying a certain flatness condition, e.g., the DG quotient from Section 3. In Section 6 we give an explanation of the uniqueness of DG quotient. In Sections 8–11 we prove the theorems formulated in Sections 3–7.

Finally, there are Appendices A–E; hopefully they make this article essentially self-contained.

2. DG categories: recollections and notation

2.1. We fix a commutative ring k and write $\otimes$ instead of $\otimes_k$ and "DG category" instead of " differential graded k-category." So a DG category is a category $\mathcal{A}$ in which the sets $\mathrm{Hom}(X,Y)$, $X,Y\in\mathrm{Ob}\,\mathcal{A}$, are provided with the structure of a $\mathbb{Z}$-graded k-module and a differential $d:\mathrm{Hom}(X,Y)\to\mathrm{Hom}(X,Y)$ of degree 1 so that for every $X,Y,Z\in\mathrm{Ob}\,\mathcal{A}$ the composition map $\mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z)\to\mathrm{Hom}(X,Z)$ comes from a morphism of complexes $\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Z)\to\mathrm{Hom}(X,Z)$. Using the super commutativity isomorphism $A\otimes B\xrightarrow{\sim}B\otimes A$ in the category of DG k-modules one defines for every DG category $\mathcal{A}$ the dual DG category $\mathcal{A}^\circ$ with $\mathrm{Ob}\,\mathcal{A}^\circ=\mathrm{Ob}\,\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}^\circ}(X,Y)=\mathrm{Hom}_{\mathcal{A}}(Y,X)$ (details can be found in [22, §1.1]).

The *tensor product* of DG categories $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

(i) $\mathrm{Ob}(\mathcal{A}\otimes\mathcal{B}):=\mathrm{Ob}\,\mathcal{A}\times\mathrm{Ob}\,\mathcal{B}$; for $a\in\mathrm{Ob}\,\mathcal{A}$ and $b\in\mathrm{Ob}\,\mathcal{B}$ the corresponding object of $\mathcal{A}\otimes\mathcal{B}$ is denoted by $a\otimes b$;

(ii) $\mathrm{Hom}(a\otimes b,a'\otimes b'):=\mathrm{Hom}(a,a')\otimes\mathrm{Hom}(b,b')$ and the composition map is defined by $(f_1\otimes g_1)(f_2\otimes g_2):=(-1)^{pq}f_1f_2\otimes g_1g_2$, $p:=\deg g_1$, $q:=\deg f_2$.

2.2. Remark. Probably the notion of DG category was introduced around 1964 (G.M. Kelly [29] refers to it as a new notion used in [28] and in an unpublished work by Eilenberg and Moore).

2.3. Given a DG category $\mathcal{A}$ one defines a graded category $\mathrm{Ho}^{\cdot}(\mathcal{A})$ with $\mathrm{Ob}\,\mathrm{Ho}^{\cdot}(\mathcal{A}) = \mathrm{Ob}\,\mathcal{A}$ by replacing each Hom complex by the direct sum of its cohomology groups. We call $\mathrm{Ho}^{\cdot}(\mathcal{A})$ the *graded homotopy category* of $\mathcal{A}$. Restricting ourselves to the 0th cohomology of the Hom complexes, we get the *homotopy category* $\mathrm{Ho}(\mathcal{A})$.

A DG functor F is said to be a *quasi-equivalence* if $\mathrm{Ho}^{\cdot}(F) : \mathrm{Ho}^{\cdot}(\mathcal{A}) \to \mathrm{Ho}^{\cdot}(\mathcal{B})$ is fully faithful and $\mathrm{Ho}(F)$ is essentially surjective. We will often use the notation $\mathcal{A} \xrightarrow{\approx} \mathcal{B}$ for a quasi-equivalence from $\mathcal{A}$ to $\mathcal{B}$. The following two notions are less reasonable. $F : \mathcal{A} \to \mathcal{B}$ is said to be a *quasi-isomorphism* if $\mathrm{Ho}^{\cdot}(F)$ is an isomorphism. We say that $F : \mathcal{A} \to \mathcal{B}$ is a *DG equivalence* if it is fully faithful and for every object $X \in \mathcal{B}$ there is a closed isomorphism of degree 0 between X and an object of $F(\mathcal{A})$.

2.4. To a DG category $\mathcal{A}$ Bondal and Kapranov associate a triangulated category $\mathcal{A}^{\mathrm{tr}}$ (or $\mathrm{Tr}^{+}(\mathcal{A})$ in the notation of [4]). It is defined as the homotopy category of a certain DG category $\mathcal{A}^{\text{pre-tr}}$. The idea of the definition of $\mathcal{A}^{\text{pre-tr}}$ is to formally add to $\mathcal{A}$ all cones, cones of morphisms between cones, etc. Here is the precise definition from [4]. The objects of $\mathcal{A}^{\text{pre-tr}}$ are "one-sided twisted complexes," i.e., formal expressions $(\bigoplus_{i=1}^{n} C_i[r_i], q)$, where $C_i \in \mathcal{A}$, $r_i \in \mathbb{Z}$, $n \geqslant 0$, $q = (q_{ij})$, $q_{ij} \in \mathrm{Hom}(C_j, C_i)[r_i - r_j]$ is homogeneous of degree 1, $q_{ij} = 0$ for $i \geqslant j$, $dq + q^2 = 0$. If $C, C' \in \mathrm{Ob}\,\mathcal{A}^{\text{pre-tr}}$, $C = (\bigoplus_{j=1}^{n} C_j[r_j], q)$, $C' = (\bigoplus_{i=1}^{m} C'_i[r'_i], q')$ then the $\mathbb{Z}$-graded k-module $\mathrm{Hom}(C, C')$ is the space of matrices $f = (f_{ij})$, $f_{ij} \in \mathrm{Hom}(C_j, C'_i)[r'_i - r_j]$, and the composition map $\mathrm{Hom}(C, C') \otimes \mathrm{Hom}(C', C'') \to \mathrm{Hom}(C, C'')$ is matrix multiplication. The differential $\mathrm{d} : \mathrm{Hom}(C, C') \to \mathrm{Hom}(C, C')$ is defined by $\mathrm{d}f := \mathrm{d}_{\mathrm{naive}} f + q'f - (-1)^l fq$ if $\deg f_{ij} = l$, where $\mathrm{d}_{\mathrm{naive}} f := (\mathrm{d}f_{ij})$.

$\mathcal{A}^{\text{pre-tr}}$ contains $\mathcal{A}$ as a full DG subcategory. If $X, Y \in \mathcal{A}$ and $f : X \to Y$ is a closed morphism of degree 0 one defines $\mathrm{Cone}(f)$ to be the object $(Y \oplus X[1], q) \in \mathcal{A}^{\text{pre-tr}}$, where $q_{12} \in \mathrm{Hom}(X, Y)[1]$ equals f and $q_{11} = q_{21} = q_{22} = 0$.

Remark. As explained in [4], one has a canonical fully faithful DG functor (the Yoneda embedding) $\mathcal{A}^{\text{pre-tr}} \to \mathcal{A}^{\circ}$-DGmod, where $\mathcal{A}^{\circ}$-DGmod is the DG category of DG $\mathcal{A}^{\circ}$-modules; a DG $\mathcal{A}^{\circ}$-module is DG-isomorphic to an object of $\mathcal{A}^{\text{pre-tr}}$ if and only if it is finitely generated and semi-free in the sense of C.8. Quite similarly one can identify $\mathcal{A}^{\text{pre-tr}}$ with the DG category dual to that of finitely generated semi-free DG $\mathcal{A}$-modules.

A non-empty DG category $\mathcal{A}$ is said to be *pretriangulated* if for every $X \in \mathcal{A}$, $k \in \mathbb{Z}$ the object $X[k] \in \mathcal{A}^{\text{pre-tr}}$ is homotopy equivalent to an object of $\mathcal{A}$ and for every closed morphism f in $\mathcal{A}$ of degree 0 the object $\mathrm{Cone}(f) \in \mathcal{A}^{\text{pre-tr}}$ is homotopy equivalent to an object of $\mathcal{A}$. We say that $\mathcal{A}$ is *strongly pretriangulated* ($+$-pretriangulated in the terminology of [4]) if the same is true with "homotopy equivalent" replaced by "DG-isomorphic" (a DG-isomorphism is an invertible closed morphism of degree 0).

If $\mathcal{A}$ is pretriangulated then every closed degree 0 morphism $f : X \to Y$ in $\mathcal{A}$ gives rise to the usual triangle $X \to Y \to \mathrm{Cone}(f) \to X[1]$ in $\mathrm{Ho}(\mathcal{A})$. Triangles of this type

and those isomorphic to them are called distinguished. Thus, if $\mathcal{A}$ is pretriangulated then $\mathrm{Ho}^{\cdot}(\mathcal{A})$ becomes a triangulated category (in fact, the Yoneda embedding identifies $\mathrm{Ho}^{\cdot}(\mathcal{A})$ with a triangulated subcategory of $\mathrm{Ho}^{\cdot}(\mathcal{A}^{\circ}\text{-DGmod})$).

If $\mathcal{A}$ is pretriangulated (respectively strongly pretriangulated) then every object of $\mathcal{A}^{\text{pre-tr}}$ is homotopy equivalent (respectively DG-isomorphic) to an object of $\mathcal{A}$. As explained in [4], the DG category $\mathcal{A}^{\text{pre-tr}}$ is always strongly pretriangulated, so $\mathcal{A}^{\text{tr}} := \mathrm{Ho}^{\cdot}(\mathcal{A}^{\text{pre-tr}})$ is a triangulated category.

2.5. Proposition. *If a DG functor $F : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence then the same is true for the corresponding DG functor $F^{\text{pre-tr}} : \mathcal{A}^{\text{pre-tr}} \to \mathcal{B}^{\text{pre-tr}}$.*

The proof is standard.

2.6. Remark. Skipping the condition "$q_{ij} = 0$ for $i \geqslant j$" in the definition of $\mathcal{A}^{\text{pre-tr}}$ one gets the definition of the DG category Pre-Tr($\mathcal{A}$) considered by Bondal and Kapranov [4]. In Proposition 2.5 one *cannot* replace $\mathcal{A}^{\text{pre-tr}}$ and $\mathcal{B}^{\text{pre-tr}}$ by Pre-Tr($\mathcal{A}$) and Pre-Tr($\mathcal{B}$). E.g., suppose that $\mathcal{A}$ and $\mathcal{B}$ are DG algebras (i.e., DG categories with one object), namely $\mathcal{A}$ is the de Rham algebra of a C^{∞} manifold M with trivial real cohomology and nontrivial π_1, $\mathcal{B} = \mathbb{R}$, and $F : \mathcal{A} \to \mathcal{B}$ is the evaluation morphism corresponding to a point of M. Then Pre-Tr(F) : Pre-Tr($\mathcal{A}$) $\to$ Pre-Tr($\mathcal{B}$) is not a quasi-equivalence. To show this notice that $K_0(M) \otimes \mathbb{Q} = \mathbb{Q}$, so there exists a vector bundle ξ on M with an integrable connection ∇ such that ξ is trivial but (ξ, ∇) is not. ξ-valued differential forms form a DG $\mathcal{A}$-module M which is free as a graded $\mathcal{A}$-module. Considering M as an object of Pre-Tr($\mathcal{A}$), we see that Pre-Tr(F) is not a quasi-equivalence.

2.7. Derived category of DG modules

Let $\mathcal{A}$ be a DG category. Following [22], we denote by $D(\mathcal{A})$ the derived category of DG $\mathcal{A}^{\circ}$-modules, i.e., the Verdier quotient of the homotopy category of DG $\mathcal{A}^{\circ}$-modules by the triangulated subcategory of acyclic DG $\mathcal{A}^{\circ}$-modules. According to [6, Theorem 10.12.5.1] (or [22, Example 7.2]) if a DG functor $\mathcal{A} \to \mathcal{B}$ is a quasi-equivalence then the restriction functor $D(\mathcal{B}) \to D(\mathcal{A})$ and its left adjoint functor (the derived induction functor) are equivalences. This also follows from Remark 4.3 because $D(\mathcal{A})$ can be identified with the homotopy category of semi-free DG $\mathcal{A}^{\circ}$-modules (see C.8).

2.8. Given DG functors $\mathcal{A}' \to \mathcal{A} \leftarrow \mathcal{A}''$ one defines $\mathcal{A}' \times_{\mathcal{A}} \mathcal{A}''$ to be the fiber product in the category of DG categories. This is the most naive definition (one takes the fiber product both at the level of objects and at the level of morphisms). More reasonable versions are discussed in Appendix D.

2.9. To a DG category $\mathcal{A}$ we associate a new DG category $\mathcal{M}or\,\mathcal{A}$, which is equipped with a DG functor Cone : $\mathcal{M}or\,\mathcal{A} \to \mathcal{A}^{\text{pre-tr}}$. The objects of $\mathcal{M}or\,\mathcal{A}$ are triples (X, Y, f), where $X, Y \in \mathrm{Ob}\,\mathcal{A}$ and f is a closed morphism $X \to Y$ of degree 0. At the level of

objects $\mathrm{Cone}(X, Y, f)$ is the cone of f. We define $\mathrm{Hom}((X, Y, f), (X', Y', f'))$ to be the subcomplex

$$\big\{u \in \mathrm{Hom}\big(\mathrm{Cone}(f), \mathrm{Cone}\big(f'\big)\big) \bigm| \pi' u i = 0\big\},$$

where $i : Y \to \mathrm{Cone}(f)$ and $\pi' : \mathrm{Cone}(f') \to X'[1]$ are the natural morphisms. At the level of morphisms, $\mathrm{Cone} : \mathrm{Hom}((X, Y, f), (X', Y', f')) \to \mathrm{Hom}(\mathrm{Cone}(f), \mathrm{Cone}(f'))$ is defined to be the natural embedding. Composition of the morphisms of $\mathcal{M}or\,\mathcal{A}$ is defined so that $\mathrm{Cone} : \mathcal{M}or\,\mathcal{A} \to \mathcal{A}^{\text{pre-tr}}$ becomes a DG functor. There is an obvious DG functor $\mathcal{M}or\,\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ such that $(X, Y, f) \mapsto (X, Y)$.

2.10. Given a DG category $\mathcal{A}$ one has the "stupid" DG category $\mathcal{M}or_{\mathrm{stup}}\,\mathcal{A}$ equipped with a DG functor $F : \mathcal{M}or_{\mathrm{stup}}\,\mathcal{A} \to \mathcal{A} \times \mathcal{A}$: it has the same objects as $\mathcal{M}or\,\mathcal{A}$ (see 2.9), $\mathrm{Hom}((X, Y, f), (X', Y', f'))$ is the subcomplex

$$\big\{(u, v) \in \mathrm{Hom}\big(X, X'\big) \times \mathrm{Hom}\big(Y, Y'\big) \bigm| f' u = v f\big\},$$

$F(X, Y, f) := (X, Y)$, $F(u, v) = (u, v)$, and composition of the morphisms of $\mathcal{M}or_{\mathrm{stup}}\,\mathcal{A}$ is defined so that $F : \mathcal{M}or_{\mathrm{stup}}\,\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ becomes a DG functor. There are canonical DG functors $\Phi : \mathcal{M}or_{\mathrm{stup}}\,\mathcal{A} \to \mathcal{M}or\,\mathcal{A}$ and $\Psi : \mathcal{M}or\,\mathcal{A} \to \mathcal{M}or_{\mathrm{stup}}\,\mathcal{A}$ such that $\Phi(X, Y, f) := (X, Y, f)$, $\Psi(X, Y, f) := (Y, \mathrm{Cone}(f), i)$, where $i : Y \to \mathrm{Cone}(f)$ is the natural morphism. So one gets the DG functor

$$\Phi\Psi : \mathcal{M}or\,\mathcal{A} \to \mathcal{M}or\,\mathcal{A}. \tag{2.1}$$

3. A new construction of the DG quotient

3.1. Construction

Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. We denote by $\mathcal{A}/\mathcal{B}$ the DG category obtained from $\mathcal{A}$ by adding for every object $U \in \mathcal{B}$ a morphism $\varepsilon_U : U \to U$ of degree -1 such that $\mathrm{d}(\varepsilon_U) = \mathrm{id}_U$ (we add neither new objects nor new relations between the morphisms).

So for $X, Y \in \mathcal{A}$ we have an isomorphism of graded k-modules (but not an isomorphism of complexes)

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y), \tag{3.1}$$

where $\mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X, Y)$ is the direct sum of tensor products $\mathrm{Hom}_{\mathcal{A}}(U_n, U_{n+1}) \otimes k[1] \otimes \mathrm{Hom}_{\mathcal{A}}(U_{n-1}, U_n) \otimes k[1] \otimes \cdots \otimes k[1] \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(U_0, U_1)$, $U_0 := X$, $U_{n+1} := Y$, $U_i \in \mathcal{B}$ for $1 \leqslant i \leqslant n$ (in particular, $\mathrm{Hom}^0_{\mathcal{A}/\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$); the morphism (3.1) maps $f_n \otimes \varepsilon \otimes f_{n-1} \otimes \cdots \otimes \varepsilon \otimes f_0$ to $f_n \varepsilon_{U_n} f_{n-1} \cdots \varepsilon_{U_1} f_0$, where ε is the canonical generator of $k[1]$. Using the formula $\mathrm{d}(\varepsilon_U) = \mathrm{id}_U$ one can easily find the differential on the l.h.s.

of (3.1) corresponding to the one on the r.h.s. The image of $\bigoplus_{n=0}^{N} \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X,Y)$ is a subcomplex of $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$, so we get a filtration on $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$. The map (3.1) induces an isomorphism of complexes

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X,Y) \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y). \tag{3.2}$$

3.2. Example. If $\mathcal{A}$ has a single object U with $\mathrm{End}_{\mathcal{A}} U = R$ then $\mathcal{A}/\mathcal{A}$ has a single object U with $\mathrm{End}_{\mathcal{A}/\mathcal{A}} U = \widetilde{R}$, where the DG algebra $\widetilde{R}$ is obtained from the DG algebra R by adding a new generator ε of degree -1 with $\mathrm{d}\varepsilon = 1$. As a DG R-bimodule, $\widetilde{R}$ equals $\mathrm{Cone}(\mathrm{Bar}(R) \to R)$, where $\mathrm{Bar}(R)$ is the bar resolution of the DG R-bimodule R. Both descriptions of $\widetilde{R}$ show that it has zero cohomology.

A more interesting example can be found in 3.7.

3.3. The triangulated functor $\mathcal{A}^{\mathrm{tr}} \to (\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$ maps $\mathcal{B}^{\mathrm{tr}}$ to zero and therefore induces a triangulated functor $\Phi : \mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to (\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$. Here $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}$ denotes Verdier's quotient (see Appendix A). We will prove that if k is a field then Φ is an equivalence. For a general ring k this is true under an additional assumption. E.g., it is enough to assume that $\mathcal{A}$ is homotopically flat over k (we prefer to use the name "homotopically flat" instead of Spaltenstein's name "K-flat" which is probably due to the notation $K(\mathcal{C})$ for the homotopy category of complexes in an additive category $\mathcal{C}$). A DG category $\mathcal{A}$ is said to be *homotopically flat* over k if for every $X, Y \in \mathcal{A}$ the complex $\mathrm{Hom}(X,Y)$ is homotopically flat over k in Spaltenstein's sense [50], i.e., for every acyclic complex C of k-modules $C \otimes_k \mathrm{Hom}(X,Y)$ is acyclic. In fact, homotopical flatness of $\mathcal{A}$ can be replaced by one of the following weaker assumptions:

$$\mathrm{Hom}(X,U) \text{ is homotopically flat over } k \text{ for all } X \in \mathcal{A},\ U \in \mathcal{B}; \tag{3.3}$$

$$\mathrm{Hom}(U,X) \text{ is homotopically flat over } k \text{ for all } X \in \mathcal{A},\ U \in \mathcal{B}. \tag{3.4}$$

Here is our first main result.

3.4. Theorem. *Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. If either* (3.3) *or* (3.4) *holds then $\Phi : \mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to (\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$ is an equivalence.*

The proof is contained in Section 8.

3.5. If (3.3) and (3.4) are not satisfied one can construct a diagram (1.1) by choosing a homotopically flat resolution $\widetilde{\mathcal{A}} \xrightarrow{\approx} \mathcal{A}$ and putting $\mathcal{C} := \widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}$, where $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}$ is the full subcategory of objects whose image in $\mathcal{A}$ is homotopy equivalent to an object of $\mathcal{B}$. Here "homotopically flat resolution" means that $\widetilde{\mathcal{A}}$ is homotopically flat and the DG functor $\widetilde{\mathcal{A}} \to \mathcal{A}$ is a quasi-equivalence (see 2.3). The existence of homotopically flat resolutions of $\mathcal{A}$ follows from Lemma B.5.

3.6. Remarks.

(i) If (3.3) or (3.4) holds then one can compute (1.3) using the bar resolution of the DG $\mathcal{B}$-module $\tilde{h}_X$ or the DG $\mathcal{B}^\circ$-module h_Y. The corresponding complex representing the object (1.3) of the derived category is precisely $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$.

(ii) Let $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ be as in 3.5 and suppose that (3.3) or (3.4) holds for both $\mathcal{B}\subset\mathcal{A}$ and $\widetilde{\mathcal{B}}\subset\widetilde{\mathcal{A}}$. Then the DG functor $\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}\to\mathcal{A}/\mathcal{B}$ is a quasi-equivalence, i.e., it induces an equivalence of the corresponding homotopy categories. This follows from Theorem 3.4. One can also directly show that if $X,Y\in\mathrm{Ob}(\mathcal{A}/\mathcal{B})=\mathrm{Ob}\,\mathcal{A}$ are the images of $\widetilde{X},\widetilde{Y}\in\mathrm{Ob}(\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}})=\mathrm{Ob}\,\widetilde{\mathcal{A}}$ then the morphism $\mathrm{Hom}_{\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}}(\widetilde{X},\widetilde{Y})\to\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$ is a quasi-isomorphism (use (3.2) and notice that the morphism $\mathrm{Hom}^n_{\widetilde{\mathcal{A}}/\widetilde{\mathcal{B}}}(\widetilde{X},\widetilde{Y})\to\mathrm{Hom}^n_{\mathcal{A}/\mathcal{B}}(X,Y)$ is a quasi-isomorphism for every n; this follows directly from the definition of Hom^n and the fact that (3.3) or (3.4) holds for $\mathcal{B}\subset\mathcal{A}$ and $\widetilde{\mathcal{B}}\subset\widetilde{\mathcal{A}}$).

(iii) Usually the DG category $\mathcal{A}/\mathcal{B}$ is huge. E.g., if $\mathcal{A}$ is the DG category of all complexes from some universe U and $\mathcal{B}\subset\mathcal{A}$ is the subcategory of acyclic complexes then the complexes $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y)$, $X,Y\in\mathcal{A}$, are not U-small for obvious reasons (see [18, §1.0] for the terminology) even though $(\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$ is a U-category. But it follows from Theorem 3.4 that whenever $(\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$ is a U-category there exists an A_∞-category $\mathcal{C}$ with U-small Hom complexes equipped with an A_∞-functor $\mathcal{C}\to\mathcal{A}/\mathcal{B}$ which is a quasi-equivalence (so one can work with $\mathcal{C}$ instead of $\mathcal{A}/\mathcal{B}$).

(iv) The DG category $\mathcal{A}/\mathcal{B}$ defined in 3.1 depends on the ground ring k, so the full notation should be $(\mathcal{A}/\mathcal{B})_k$. Given a morphism $k_0\to k$, we have a canonical functor $F:(\mathcal{A}/\mathcal{B})_{k_0}\to(\mathcal{A}/\mathcal{B})_k$. If (3.3) or (3.4) holds for both k_0 and k then the functor $(\mathcal{A}/\mathcal{B})_{k_0}\to(\mathcal{A}/\mathcal{B})_k$ is a quasi-isomorphism by Theorem 3.4.

3.7. Example

3.7.1. Let $\mathcal{A}_0$ be the DG category with two objects X_1, X_2 freely generated by a morphism $f:X_1\to X_2$ of degree 0 with $\mathrm{d}f=0$ (so $\mathrm{Hom}(X_i,X_i)=k$, $\mathrm{Hom}(X_1,X_2)$ is the free module kf and $\mathrm{Hom}(X_2,X_1)=0$). Put $\mathcal{A}:=\mathcal{A}_0^{\text{pre-tr}}$. Let $\mathcal{B}\subset\mathcal{A}$ be the full DG subcategory with a single object $\mathrm{Cone}(f)$. Instead of describing the whole DG quotient $\mathcal{A}/\mathcal{B}$, we will describe only the full DG subcategory $(\mathcal{A}/\mathcal{B})_0\subset\mathcal{A}/\mathcal{B}$ with objects X_1 and X_2 (the DG functor $(\mathcal{A}/\mathcal{B})_0^{\text{pre-tr}}\to(\mathcal{A}/\mathcal{B})^{\text{pre-tr}}$ is a DG equivalence in the sense of 2.3, so $\mathcal{A}/\mathcal{B}$ can be considered as a full DG subcategory of $(\mathcal{A}/\mathcal{B})_0^{\text{pre-tr}}$). Directly using the definition of $\mathcal{A}/\mathcal{B}$ (see 3.1), one shows that $(\mathcal{A}/\mathcal{B})_0$ equals the DG category $\mathcal{K}$ freely generated by our original $f:X_1\to X_2$ and also a morphism $g:X_2\to X_1$ of degree 0, morphisms $\alpha_i:X_i\to X_i$ of degree -1, and a morphism $u:X_1\to X_2$ of degree -2 with the differential given by $\mathrm{d}f=\mathrm{d}g=0$, $\mathrm{d}\alpha_1=gf-1$, $\mathrm{d}\alpha_2=fg-1$, $\mathrm{d}u=f\alpha_1-\alpha_2 f$. On the other hand, one has the following description of $\mathrm{Ho}^\cdot((\mathcal{A}/\mathcal{B})_0)$.

3.7.2. Lemma. $\mathrm{Ext}^n_{\mathcal{A}/\mathcal{B}}(X_i,X_j)=0$ *for* $n\neq 0$, $\mathrm{Ext}^0_{\mathcal{A}/\mathcal{B}}(X_i,X_i)=k$, *and* $\mathrm{Ext}^0_{\mathcal{A}/\mathcal{B}}(X_1,X_2)$, $\mathrm{Ext}^0_{\mathcal{A}/\mathcal{B}}(X_2,X_1)$ *are free* k*-modules generated by* f *and* f^{-1}.

As $(\mathcal{A}/\mathcal{B})_0 = \mathcal{K}$, one gets the following corollary.

3.7.3. Corollary. *$\mathcal{K}$ is a resolution of the k-category $\mathbf{I}_2$ generated by the category $\mathbf{J}_2$ with 2 objects and precisely one morphism with any given source and target.*

Clearly, $\mathcal{K}$ is semi-free in the sense of B.4.

3.7.4. Proof of Lemma 3.7.2. By Theorem 3.4, $\mathrm{Ho}^{\cdot}(\mathcal{A}/\mathcal{B}) = \mathcal{A}_0^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}$. As $X_2 \in (\mathcal{B}^{\mathrm{tr}})^{\perp}$, the map $\mathrm{Ext}^n_{\mathcal{A}}(X_i, X_2) \to \mathrm{Ext}^n_{\mathcal{A}_0^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}}(X_i, X_2)$, $i = 1, 2$, is an isomorphism by A.4. Therefore, $\mathrm{Ext}^n_{\mathcal{A}/\mathcal{B}}(X_i, X_2)$ is as stated in the lemma. But $f : X_1 \to X_2$ becomes an isomorphism in $\mathrm{Ho}(\mathcal{A}/\mathcal{B})$, so $\mathrm{Ext}^n_{\mathcal{A}/\mathcal{B}}(X_i, X_1)$ is also as stated. □

3.7.5. Modification of the proof

In the above proof we used Theorem 3.4 and A.4 to show that $\varphi : \mathrm{Ext}^n_{\mathcal{A}}(X_i, X_2) \to \mathrm{Ext}^n_{\mathcal{A}/\mathcal{B}}(X_i, X_2)$ is an isomorphism. In fact, this follows directly from (3.2), which is an immediate consequence of the definition of $\mathcal{A}/\mathcal{B}$. Indeed, φ is induced by the canonical morphism $\alpha : \mathrm{Hom}_{\mathcal{A}}(X_i, X_2) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X_i, X_2)$. By (3.2) α is injective and $L := \mathrm{Coker}\,\alpha_i$ is the union of an increasing sequence of subcomplexes $0 = L_0 \subset L_1 \subset \cdots$ such that $L_n/L_{n-1} = \mathrm{Hom}^n_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X_i, X_2)$ for $n \geqslant 1$. Finally, $\mathrm{Hom}^n_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X_i, X_2)$ is acyclic for all $n \geqslant 1$ because the complex $\mathrm{Hom}_{\mathcal{A}}(U, X_2)$, $U := \mathrm{Cone}(f : X_1 \to X_2)$ is contractible.

3.7.6. Remarks.

(i) The DG category $\mathcal{K}$ from 3.7.1 and the fact that it is a resolution of $\mathbf{I}_2$ were known to Kontsevich [31]. One can come to the definition of $\mathcal{K}$ as follows. The naive guess is that already the DG category $\mathcal{K}'$ freely generated by f, g, α_1, α_2 as above is a resolution of $\mathbf{I}_2$, but one discovers a nontrivial element $\nu \in \mathrm{Ext}^{-1}(X_1, X_2)$ by representing $fgf - f$ as a coboundary in two different ways (notice that $f(gf - 1) = fgf - f = (fg - 1)f$). Killing ν one gets the DG category $\mathcal{K}$, which already turns out to be a resolution of $\mathbf{J}_2$.

(ii) The DG category $\mathcal{K}$ from 3.7.1 has a topological analog $\mathcal{K}_{\mathrm{top}}$. This is a topological category with two objects X_1, X_2 freely generated by morphisms $f \in \mathrm{Mor}(X_1, X_2)$, $g \in \mathrm{Mor}(X_2, X_1)$, continuous maps $\alpha_i : [0, 1] \to \mathrm{Mor}(X_i, X_i)$, and a continuous map $u : [0, 1] \times [0, 1] \to \mathrm{Mor}(X_1, X_2)$ with defining relations $\alpha_i(0) = \mathrm{id}_{X_i}$, $\alpha_1(1) = gf$, $\alpha_2(1) = fg$, $u(t, 0) = f\alpha_1(t)$, $u(t, 1) = \alpha_2(t)f$, $u(0, \tau) = f$, $u(1, \tau) = fgf$. It was considered by Vogt [58], who was inspired by an article of R. Lashof. The spaces $\mathrm{Mor}_{\mathcal{K}_{\mathrm{top}}}(X_i, X_j)$ are contractible. This can be easily deduced from Corollary 3.7.3 using a cellular decomposition of $\mathrm{Mor}_{\mathcal{K}_{\mathrm{top}}}(X_i, X_j)$ such that the composition maps

$$\mathrm{Mor}_{\mathcal{K}_{\mathrm{top}}}(X_i, X_j) \times \mathrm{Mor}_{\mathcal{K}_{\mathrm{top}}}(X_j, X_k) \to \mathrm{Mor}_{\mathcal{K}_{\mathrm{top}}}(X_i, X_k)$$

are cellular and the DG category that one gets by replacing the topological spaces $\mathrm{Mor}_{\mathcal{K}_{\mathrm{top}}}(X_i, X_j)$ by their cellular chain complexes equals $\mathcal{K}$.

4. The DG categories $\underrightarrow{\mathcal{A}}$ and $\underleftarrow{\mathcal{A}}$. Keller's construction of the DG quotient

The DG category $\mathcal{A}/\mathcal{B}$ from Section 3 depends on the ground ring k (see Remarks 3.6(iv)). Here we describe Keller's construction of a quotient DG category, which does not depend at all on k (if you like, assume $k = \mathbb{Z}$). The construction makes use of the DG category $\underrightarrow{\mathcal{A}}$ studied by him in [22], which may be considered as a DG version of the category of ind-objects. There is also a dual construction based on $\underleftarrow{\mathcal{A}}$ (a DG version of the category of pro-objects).

4.1. If $\mathcal{A}$ is a DG category we denote by $\underrightarrow{\mathcal{A}}$ the DG category of semi-free DG $\mathcal{A}^\circ$-modules (see C.8 for the definition of "semi-free"). The notation $\underrightarrow{\mathcal{A}}$ has been chosen because one can think of objects of $\underrightarrow{\mathcal{A}}$ as a certain kind of direct limits of objects of $\mathcal{A}^{\text{pre-tr}}$ (see Remark 4.2). We put $\underleftarrow{\mathcal{A}} := (\underrightarrow{\mathcal{A}^\circ})^\circ$. Of course, the DG categories $\underrightarrow{\mathcal{A}}$ and $\underleftarrow{\mathcal{A}}$ are not small. They are strongly pretriangulated in the sense of 2.4, and $\mathrm{Ho}(\underrightarrow{\mathcal{A}}) = \underrightarrow{\mathcal{A}}^{\text{tr}}$ identifies with the derived category $D(\mathcal{A})$ of DG $\mathcal{A}^\circ$-modules (see C.8). We have the fully faithful DG functors $\underleftarrow{\mathcal{A}} \leftarrow \mathcal{A} \to \underrightarrow{\mathcal{A}}$. Given a DG functor $\mathcal{B} \to \mathcal{A}$, one has the induction DG functors $\underrightarrow{\mathcal{B}} \to \underrightarrow{\mathcal{A}}$ and $\underleftarrow{\mathcal{B}} \to \underleftarrow{\mathcal{A}}$ (see C.9). In particular, if $\mathcal{B} \subset \mathcal{A}$ is a full subcategory then $\underrightarrow{\mathcal{B}}$, $\underleftarrow{\mathcal{B}}$ are identified with full DG subcategories of $\underrightarrow{\mathcal{A}}$, $\underleftarrow{\mathcal{A}}$.

4.2. Remark. Here is a small version of $\underrightarrow{\mathcal{A}}$. Fix an infinite set I and consider the following DG category $\mathcal{A}_I^{\text{pre-tr}\to}$ (which coincides with the DG category $\mathcal{A}^{\text{pre-tr}}$ from 2.4 if $I = \mathbb{N}$). To define an object of $\mathcal{A}_I^{\text{pre-tr}\to}$, make the following changes in the definition of an object of $\mathcal{A}^{\text{pre-tr}}$. First, replace $\bigoplus_{i=1}^n C_i[r_i]$ by $\bigoplus_{i\in I} C_i[r_i]$ and require the cardinality of $\{i \in I \mid C_i \neq 0\}$ to be strictly less than that of I. Second, replace the triangularity condition on q by the existence of an ordering of I such that $q_{ij} \neq 0$ only for $i < j$ and $\{i \in I \mid i < j\}$ is finite for every $j \in I$ (in other words, for $j \in I$ let $I_{<j}$ denote the set of $i \in I$ for which there is a finite sequence $i_0, \dots, i_n \in I$ with $n > 0$, $i_0 = j$, $i_n = i$ such that $q_{i_{k+1}i_k} \neq 0$, then for every $j \in I$ the set $I_{<j}$ should be finite and should not contain j). Morphisms of $\underrightarrow{\mathcal{A}}$ are defined to be matrices (f_{ij}) as in 2.4 such that $\{i \in I \mid f_{ij} \neq 0\}$ is finite for every $j \in I$. The DG functor $\mathcal{A} \to \underrightarrow{\mathcal{A}}$ extends in the obvious way to a fully faithful DG functor $\mathcal{A}_I^{\text{pre-tr}\to} \to \underrightarrow{\mathcal{A}}$.

One also has the DG category $\mathcal{A}_I^{\text{pre-tr}\leftarrow} := ((\mathcal{A}^\circ)_I^{\text{pre-tr}\to})^\circ$ and the fully faithful DG functor $\mathcal{A}_I^{\text{pre-tr}\to} \to \underleftarrow{\mathcal{A}}$.

4.3. Remark. A quasi-equivalence $F : \mathcal{A} \xrightarrow{\approx} \mathcal{B}$ induces quasi-equivalences

$$\underrightarrow{\mathcal{A}} \xrightarrow{\approx} \underrightarrow{\mathcal{B}}, \qquad \underleftarrow{\mathcal{A}} \xrightarrow{\approx} \underleftarrow{\mathcal{B}}, \qquad \mathcal{A}_I^{\text{pre-tr}\to} \xrightarrow{\approx} \mathcal{B}_I^{\text{pre-tr}\to}, \qquad \mathcal{A}_I^{\text{pre-tr}\leftarrow} \xrightarrow{\approx} \mathcal{B}_I^{\text{pre-tr}\leftarrow}$$

(the fact that $\underrightarrow{\mathcal{A}} \to \underrightarrow{\mathcal{B}}$ is a quasi-equivalence was mentioned in 2.7). This is a consequence of the following lemma.

4.4. Lemma. *A triangulated subcategory of* $\mathrm{Ho}(\underrightarrow{\mathcal{A}})$ *containing* $\mathrm{Ho}(\mathcal{A})$ *and closed under* (*infinite*) *direct sums coincides with* $\mathrm{Ho}(\underrightarrow{\mathcal{A}})$. *A triangulated subcategory of* $\mathrm{Ho}(\mathcal{A}_I^{\text{pre-tr}\to})$

containing $\mathrm{Ho}(\mathcal{A})$ *and closed under direct sums indexed by sets* J *such that* $\mathrm{Card}\, J < \mathrm{Card}\, I$ *coincides with* $\mathrm{Ho}(\mathcal{A}_I^{\text{pre-tr}\rightarrow})$.

This was proved by Keller [22, p. 69]. Key idea: if one has a sequence of DG $\mathcal{A}^\circ$-modules M_i and morphisms $f_i : M_i \to M_{i+1}$ then one has an exact sequence $0 \to M \xrightarrow{1-f} M \to \varinjlim M_i \to 0$, where $M := \bigoplus_i M_i$ and $f : M \to M$ is induced by the f_i's.

4.5. Now let $\mathcal{B} \subset \mathcal{A}$ be a full DG subcategory. Let $\mathcal{B}^\perp$ (respectively ${}^\perp\mathcal{B}$) denote the full DG subcategory of $\underrightarrow{\mathcal{A}}$ (respectively of $\underleftarrow{\mathcal{A}}$) that consists of objects X such that for every $b \in \mathcal{B}$ the complex $\mathrm{Hom}(b, X)$ (respectively $\mathrm{Hom}(X, b)$) is acyclic. Recall that $D(\mathcal{A}) = \mathrm{Ho}(\underrightarrow{\mathcal{A}}) = \underrightarrow{\mathcal{A}}^{\mathrm{tr}}$.

4.6. Proposition. *Let* $\xi : \mathcal{A} \to \mathcal{C}$ *be a quotient of a DG category* $\mathcal{A}$ *modulo* $\mathcal{B} \subset \mathcal{A}$. *Then*

(i) $\underrightarrow{\xi} : \underrightarrow{\mathcal{A}} \to \underrightarrow{\mathcal{C}}$ *is a quotient of* $\underrightarrow{\mathcal{A}}$ *modulo* $\underrightarrow{\mathcal{B}}$;
(i′) $\underleftarrow{\xi} : \underleftarrow{\mathcal{A}} \to \underleftarrow{\mathcal{C}}$ *is a quotient of* $\underleftarrow{\mathcal{A}}$ *modulo* $\underleftarrow{\mathcal{B}}$;
(ii) *the restriction functor* $D(\mathcal{C}) \to D(\mathcal{A})$ *is fully faithful, and its essential image consists precisely of objects of* $D(\mathcal{A})$ *annihilated by the restriction functor* $\rho : D(\mathcal{A}) \to D(\mathcal{B})$; *the functor* $D(\mathcal{A})/D(\mathcal{C}) \to D(\mathcal{B})$ *induced by* ρ *is an equivalence.*

See 10.3 for the proof.

4.7. Proposition.

(i) *The essential image of* $\underrightarrow{\mathcal{B}}^{\mathrm{tr}}$ *in* $\underrightarrow{\mathcal{A}}^{\mathrm{tr}}$ *is right-admissible in the sense of* A.6.
(ii) *The right orthogonal complement of* $\underrightarrow{\mathcal{B}}^{\mathrm{tr}}$ *in* $\underrightarrow{\mathcal{A}}^{\mathrm{tr}}$ *equals* $(\mathcal{B}^\perp)^{\mathrm{tr}}$.
(iii) *The functor* $(\mathcal{B}^\perp)^{\mathrm{tr}} \to \underrightarrow{\mathcal{A}}^{\mathrm{tr}}/\underrightarrow{\mathcal{B}}^{\mathrm{tr}}$ *is an equivalence.*
(iv) *The functor* $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to \underrightarrow{\mathcal{A}}^{\mathrm{tr}}/\underrightarrow{\mathcal{B}}^{\mathrm{tr}}$ *is fully faithful.*
(i°)–(iv°) *Statements* (i)–(iv) *remain true if one replaces* $\underrightarrow{\mathcal{A}}^{\mathrm{tr}}$ *and* $\underrightarrow{\mathcal{B}}^{\mathrm{tr}}$ *by* $\underleftarrow{\mathcal{A}}^{\mathrm{tr}}$ *and* $\underleftarrow{\mathcal{B}}^{\mathrm{tr}}$, *"right" by "left", and* $\mathcal{B}^\perp$ *by* ${}^\perp\mathcal{B}$.

The proof will be given in 10.1.

4.8. Remark. Keller [23] derives Proposition 4.6(i) from Neeman's theorem on compactly generated triangulated categories [47, Theorem 2.1]. Statements (i) and (iv) of Proposition 4.7 are particular cases of Lemmas 1.7 and 2.5 of Neeman's work [47].

4.9. Now let $\mathcal{A} \nearrow \mathcal{B} \subset \mathcal{B}^\perp$ be the full DG subcategory of objects $X \in \mathcal{B}^\perp$ such that for some $a \in \mathcal{A}$ and some closed morphism $f : a \to X$ of degree 0, the cone of f is homotopy equivalent to an object of $\underrightarrow{\mathcal{B}}$. Let $\mathcal{A} \swarrow \mathcal{B} \subset {}^\perp\mathcal{B}$ be the full DG subcategory of objects $X \in {}^\perp\mathcal{B}$ such that for some $a \in \mathcal{A}$ and some closed morphism $f : X \to a$ of degree 0, the cone of f is homotopy equivalent to an object of $\underleftarrow{\mathcal{B}}$. By Proposition 4.7 we have the fully faithful functor $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to \underrightarrow{\mathcal{A}}^{\mathrm{tr}}/\underrightarrow{\mathcal{B}}^{\mathrm{tr}} = (\mathcal{B}^\perp)^{\mathrm{tr}} = \mathrm{Ho}(\mathcal{B}^\perp)$, and its essential image equals $(\mathcal{A} \nearrow \mathcal{B})^{\mathrm{tr}}$. So we get an equivalence

$$\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \xrightarrow{\sim} (\mathcal{A} \nearrow \mathcal{B})^{\mathrm{tr}} \tag{4.1}$$

and a similar equivalence $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \xrightarrow{\sim} (\mathcal{A} \swarrow \mathcal{B})^{\mathrm{tr}}$.

4.10. Let us construct a diagram (1.1) with $\mathcal{C} = \mathcal{A} \nearrow \mathcal{B}$ such that the corresponding functor $\mathcal{A}^{\mathrm{tr}} \to (\mathcal{A} \nearrow \mathcal{B})^{\mathrm{tr}}$ induces (4.1) (so $\mathcal{A} \nearrow \mathcal{B}$ will become a DG quotient of $\mathcal{A}$ modulo $\mathcal{B}$). The DG category $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$ is defined as follows. First, consider the DG category $\mathcal{M}or \underrightarrow{\mathcal{A}}$ (see 2.9). Its objects are triples (a, Y, g), where $a, Y \in \underrightarrow{\mathcal{A}}$ and g is a closed morphism $a \to Y$ of degree 0. We define $\widetilde{\mathcal{A}} \subset \mathcal{M}or \underrightarrow{\mathcal{A}}$ to be the full DG subcategory of triples (a, Y, g) such that $a \in \mathcal{A}$, $Y \in \mathcal{A} \nearrow \mathcal{B} \subset \underrightarrow{\mathcal{A}}$, and $\mathrm{Cone}(a \xrightarrow{g} Y)$ is homotopy equivalent to an object of $\underrightarrow{\mathcal{B}}$. The DG functors $\mathcal{A} \leftarrow \widetilde{\mathcal{A}} \to \mathcal{A} \nearrow \mathcal{B}$ are defined by $(a, Y, g) \mapsto a$ and $(a, Y, g) \mapsto Y$.

4.11. Remarks.

(i) Let $\widetilde{\mathcal{A}}' \subset \mathcal{M}or \underrightarrow{\mathcal{A}}$ be the full DG subcategory of triples $(P, a, f) \in \mathcal{M}or \underrightarrow{\mathcal{A}}$ such that $P \in \underrightarrow{\mathcal{B}}$, $a \in \mathcal{A}$, and $\mathrm{Cone}(P \xrightarrow{f} a) \in \mathcal{B}^{\perp}$. The DG functor (2.1) (with $\mathcal{A}$ replaced by $\underrightarrow{\mathcal{A}}$) induces a quasi-equivalence $\widetilde{\mathcal{A}}' \xrightarrow{\approx} \widetilde{\mathcal{A}}$, so one can use $\widetilde{\mathcal{A}}'$ instead of $\widetilde{\mathcal{A}}$.

(ii) It follows from the definition of (2.1) that the image of the DG functor $\widetilde{\mathcal{A}}' \to \widetilde{\mathcal{A}}$ is contained in $\widetilde{\mathcal{A}}_{\mathrm{stup}} := \widetilde{\mathcal{A}} \cap \mathcal{M}or_{\mathrm{stup}} \underrightarrow{\mathcal{A}}$ (see 2.10 for the definition of $\mathcal{M}or_{\mathrm{stup}} \underrightarrow{\mathcal{A}} \subset \mathcal{M}or \underrightarrow{\mathcal{A}}$).

4.12. Dualizing the construction from 4.10 one gets the full DG subcategory $\widecheck{\mathcal{A}} \subset \mathcal{M}or \underleftarrow{\mathcal{A}}$ which consists of triples $(\overline{Y}, a, \overline{g})$ such that $\overline{Y} \in \mathcal{A} \swarrow \mathcal{B}$, $a \in \mathcal{A}$, and $\mathrm{Cone}(\overline{Y} \xrightarrow{\overline{g}} a)$ is homotopy equivalent to an object of $\underleftarrow{\mathcal{B}}$. Dualizing Remarks 4.11, one gets a DG category $\widecheck{\mathcal{A}}'$ equipped with a quasi-equivalence $\widecheck{\mathcal{A}}' \xrightarrow{\approx} \widecheck{\mathcal{A}}$; $\widecheck{\mathcal{A}}' \subset \mathcal{M}or \underleftarrow{\mathcal{A}}$ is the full DG subcategory of triples $(a, \overline{P}, \overline{f})$ such that $a \in \mathcal{A}$, $\overline{P} \in \underleftarrow{\mathcal{B}}$, and $\mathrm{Cone}(\overline{f}) \in {}^{\perp}\mathcal{B}$. The diagrams $\mathcal{A} \leftarrow \widecheck{\mathcal{A}} \to \mathcal{A} \swarrow \mathcal{B}$ and $\mathcal{A} \leftarrow \widecheck{\mathcal{A}}' \to \mathcal{A} \swarrow \mathcal{B}$ are also DG quotients of $\mathcal{A}$ modulo $\mathcal{B}$. The image of the DG functor $\widecheck{\mathcal{A}}' \to \widecheck{\mathcal{A}}$ is contained in $\widecheck{\widetilde{\mathcal{A}}}_{\mathrm{stup}} := \widecheck{\mathcal{A}} \cap \mathcal{M}or_{\mathrm{stup}} \underleftarrow{\mathcal{A}}$.

4.13. One can also include the diagrams constructed in 4.10 and 4.12 into a canonical commutative diagram of DG categories and DG functors

$$\begin{array}{ccccc}
\mathcal{A} & = & \mathcal{A} & = & \mathcal{A} \\
\uparrow & & \uparrow & & \uparrow \\
\widecheck{\mathcal{A}} & \xleftarrow{\approx} & \widecheck{\widetilde{\mathcal{A}}} & \xrightarrow{\approx} & \widetilde{\mathcal{A}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A} \swarrow \mathcal{B} & \xleftarrow{\approx} & \mathcal{A} \diagup\!\!\!\swarrow \mathcal{B} & \xrightarrow{\approx} & \mathcal{A} \nearrow \mathcal{B}
\end{array} \tag{4.2}$$

in which each column is a DG quotient of $\mathcal{A}$ modulo $\mathcal{B}$. The DG category $\overleftrightarrow{\mathcal{A}}$ is defined to be the fiber product $\overleftarrow{\mathcal{A}}' \times_{\mathcal{A}} \overrightarrow{\mathcal{A}}'$, where $\overleftarrow{\mathcal{A}}'$ and $\overrightarrow{\mathcal{A}}'$ were defined in Remarks 4.11, 4.12 (recall that "fiber product" is understood in the most naive sense, see 2.8). To define $\mathcal{A} \swarrow\!\!\!\nearrow \mathcal{B}$, we use the DG category $\underleftrightarrow{\mathcal{A}}$ such that $\mathrm{Ob}\,\underleftrightarrow{\mathcal{A}} := \mathrm{Ob}\,\underleftarrow{\mathcal{A}} \sqcup \mathrm{Ob}\,\underrightarrow{\mathcal{A}}$, $\underrightarrow{\mathcal{A}}$ and $\underleftarrow{\mathcal{A}}$ are full DG subcategories of $\underleftrightarrow{\mathcal{A}}$, and for $Y \in \mathrm{Ob}\,\underrightarrow{\mathcal{A}}$, $\overline{Y} \in \mathrm{Ob}\,\underleftarrow{\mathcal{A}}$ one has $\mathrm{Hom}(Y, \overline{Y}) := 0$, $\mathrm{Hom}(\overline{Y}, Y) := Y \otimes_{\mathcal{A}} \overline{Y}$ (recall that Y is a DG $\mathcal{A}^\circ$-module and $\overline{Y}$ is a DG $\mathcal{A}$-module, so $Y \otimes_{\mathcal{A}} \overline{Y}$ is well-defined, see C.3). For $a \in \mathcal{A}$ we denote by $\underleftarrow{a}$ (respectively $\underrightarrow{a}$) the image of a in $\underleftarrow{\mathcal{A}}$ (respectively $\underrightarrow{\mathcal{A}}$); we have the "identity" morphism $e = e_a : \underleftarrow{a} \to \underrightarrow{a}$. Now define $\mathcal{A} \swarrow\!\!\!\nearrow \mathcal{B} \subset \mathcal{M}or\,\underleftrightarrow{\mathcal{A}}$ to be the full DG subcategory of triples $(\overline{Y}, Y, f) \in \mathcal{M}or\,\underleftrightarrow{\mathcal{A}}$ such that $\overline{Y} \in \mathcal{A} \swarrow \mathcal{B} \subset \underleftarrow{\mathcal{A}}$, $Y \in \mathcal{A} \nearrow \mathcal{B} \subset \underrightarrow{\mathcal{A}}$, and $f : \overline{Y} \to Y$ can be represented as a composition $\overline{Y} \xrightarrow{g} \underleftarrow{a} \xrightarrow{e} \underrightarrow{a} \xrightarrow{h} Y$, $a \in \mathcal{A}$, so that $\mathrm{Cone}(g)$ is homotopy equivalent to an object of $\underleftarrow{\mathcal{B}}$ and $\mathrm{Cone}(h)$ is homotopy equivalent to an object of $\underrightarrow{\mathcal{B}}$ (g and h are closed morphisms of degree 0).

The DG functors $\mathcal{A} \swarrow\!\!\!\nearrow \mathcal{B} \to \mathcal{A} \nearrow \mathcal{B}$ and $\mathcal{A} \swarrow\!\!\!\nearrow \mathcal{B} \to \mathcal{A} \swarrow \mathcal{B}$ send $(\overline{Y}, Y, f) \in \mathcal{A} \swarrow\!\!\!\nearrow \mathcal{B}$ respectively to Y and $\overline{Y}$. The DG functor $\overleftrightarrow{\mathcal{A}} \to \mathcal{A} \swarrow\!\!\!\nearrow \mathcal{B} \subset \mathcal{M}or\,\underleftrightarrow{\mathcal{A}}$ is defined to be the composition

$$\overleftrightarrow{\mathcal{A}} := \overleftarrow{\mathcal{A}}' \times_{\mathcal{A}} \overrightarrow{\mathcal{A}}' \to \overleftarrow{\mathcal{A}}_{\mathrm{stup}} \times_{\mathcal{A}} \overrightarrow{\mathcal{A}}_{\mathrm{stup}} \xrightarrow{F} \mathcal{M}or\,\underleftrightarrow{\mathcal{A}},$$

where the DG functors $\overleftarrow{\mathcal{A}}' \to \overleftarrow{\mathcal{A}}_{\mathrm{stup}}$ and $\overrightarrow{\mathcal{A}}' \to \overrightarrow{\mathcal{A}}_{\mathrm{stup}}$ were defined in Remarks 4.11, 4.12 and $F : \overleftarrow{\mathcal{A}}_{\mathrm{stup}} \times_{\mathcal{A}} \overrightarrow{\mathcal{A}}_{\mathrm{stup}} \to \mathcal{M}or\,\underleftrightarrow{\mathcal{A}}$ is the composition DG functor: at the level of objects, if $u = (a, Y, g : a \to Y) \in \mathcal{M}or_{\mathrm{stup}}\,\underrightarrow{\mathcal{A}}$ and $\bar{u} = (\overline{Y}, a, \overline{g} : \overline{Y} \to a) \in \mathcal{M}or_{\mathrm{stup}}\,\underleftarrow{\mathcal{A}}$, $a \in \mathcal{A}$, then $F(\bar{u}, u) = (\overline{Y}, Y, g\overline{g})$; there is no problem to define the DG functor F at the level of morphisms because we are working with the "stupid" versions $\overleftarrow{\mathcal{A}}_{\mathrm{stup}}$, $\overrightarrow{\mathcal{A}}_{\mathrm{stup}}$, $\mathcal{M}or_{\mathrm{stup}}$ (the "non-stupid" composition $\overleftarrow{\mathcal{A}} \times_{\mathcal{A}} \overrightarrow{\mathcal{A}} \to \mathcal{M}or\,\underleftrightarrow{\mathcal{A}}$ is defined as an A_∞-functor rather than as a DG functor).

5. Derived DG functors

We will define a notion of right derived functor in the DG setting modeled on Deligne's definition in the triangulated setting. One can easily pass from right derived DG functors to left ones by considering the dual DG categories.

5.1. *Deligne's definition*

Let $G : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor between triangulated categories and $\mathcal{S} \subset \mathcal{T}$ a triangulated subcategory. Denote by $\mathrm{CohoFunct}(\mathcal{T}')$ the category of k-linear cohomological functors from $(\mathcal{T}')^\circ$ to the category of k-modules. RG is defined to be the functor $\mathcal{T}/\mathcal{S} \to \mathrm{CohoFunct}(\mathcal{T}')$ defined by

$$RG(Y) := \text{“}\varinjlim\text{”}_{(Y \to Z) \in Q_Y} G(Z), \tag{5.1}$$

which is a shorthand for

$$RG(Y)(X) := \varinjlim_{(Y\to Z)\in Q_Y} \mathrm{Hom}\big(X, G(Z)\big), \quad Y \in \mathcal{T},\ X \in \mathcal{T}'. \tag{5.2}$$

Here Q_Y is the filtering category of $\mathcal{T}$-morphisms $f: Y \to Z$ such that $\mathrm{Cone}(f)$ is isomorphic to an object of $\mathcal{S}$.

RG has the following universal property. Let $\pi: \mathcal{T} \to \mathcal{T}/\mathcal{S}$ denote the canonical functor and $\nu: \mathcal{T}' \to \mathrm{CohoFunct}(\mathcal{T}')$ the Yoneda embedding. Let $\Phi: \mathcal{T}/\mathcal{S} \to \mathrm{CohoFunct}(\mathcal{T}')$ be a graded functor (see A.1 for a discussion of the meaning of "graded"). Then there is a canonical isomorphism

$$\mathrm{Hom}(RG, \Phi) = \mathrm{Hom}(\nu G, \Phi\pi) \tag{5.3}$$

functorial in Φ (here Hom is the set of morphisms of graded functors). In particular, if $RG(\mathcal{T}/\mathcal{S}) \subset \mathcal{T}'$ then $RG: \mathcal{T}/\mathcal{S} \to \mathcal{T}'$ is a derived functor in Verdier's sense [56,57].

Let $(\mathcal{T}/\mathcal{S})_G$ be the category of triples (Y, X, φ), where $Y \in \mathcal{T}/\mathcal{S}$, $X \in \mathcal{T}'$, $\varphi: X \xrightarrow{\sim} RG(Y)$. The functor

$$(\mathcal{T}/\mathcal{S})_G \to \mathcal{T}', \qquad (Y, X, \varphi) \mapsto X \tag{5.4}$$

is also denoted by RG. We have an equivalence $(Y, X, \varphi) \mapsto Y$ between $(\mathcal{T}/\mathcal{S})_G$ and a full subcategory of $\mathcal{T}/\mathcal{S}$ (the full subcategory of objects $Y \in \mathcal{T}/\mathcal{S}$ such that $RG(Y)$ is defined as an object of $\mathcal{T}'$).

Remark. Deligne (cf. [10, Definition 1.2.1]) considers RG as a functor from $\mathcal{T}/\mathcal{S}$ to the category of ind-objects $\mathrm{ind}(\mathcal{T}')$ rather than to the category $\mathrm{CohoFunct}(\mathcal{T}')$. In fact, this does not matter. First of all, the image of the functor RG defined by (5.2) is contained in the full subcategory of ind-representable functors $(\mathcal{T}')^\circ \to k\,\mathrm{mod}$, which is canonically identified with $\mathrm{ind}(\mathcal{T}')$ (see [18, §8.2]). This is enough for our purposes, but in fact, since $\mathcal{T}'$ is small, *every* $H \in \mathrm{CohoFunct}(\mathcal{T}')$ is ind-representable by a well-known lemma (see, e.g., [46, Lemma 7.2.4]), which is a version of Brown's theorem [8,9]. Proof: by [18, Theorem 8.3.3] it suffices to check that the category $\mathcal{T}'/H := \{(X, u) \mid X \in \mathcal{T}', u \in H(X)\}$ is filtering.

5.2. Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. Let F be a DG functor from $\mathcal{A}$ to a DG category $\mathcal{A}'$. To define the right derived DG functor RF we use the DG quotient $\mathcal{A} \nearrow \mathcal{B}$ from 4.9. By definition, $RF: \mathcal{A} \nearrow \mathcal{B} \to \underrightarrow{\mathcal{A}}'$ is the restriction of the DG functor $\underrightarrow{F}: \underrightarrow{\mathcal{A}} \to \underrightarrow{\mathcal{A}}'$ to the DG subcategory $\mathcal{A} \nearrow \mathcal{B} \subset \mathcal{B}^\perp \subset \underrightarrow{\mathcal{A}}$. A 2-categorical reformulation of this definition is given in Remark (ii) from E.6.

5.3. Let us show that the definition of RF from 5.2 agrees with Deligne's definition of the right derived functor of a triangulated functor between triangulated categories (see 5.1).

Suppose we are in the situation of 5.2. We have the DG functor $RF: \mathcal{A} \nearrow \mathcal{B} \to \underrightarrow{\mathcal{A}}'$ and the corresponding triangulated functor $(RF)^{\mathrm{tr}}: (\mathcal{A} \nearrow \mathcal{B})^{\mathrm{tr}} \to (\underrightarrow{\mathcal{A}}')^{\mathrm{tr}}$. Using (4.1) we

can rewrite it as $(RF)^{\mathrm{tr}}: \mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to (\underrightarrow{\mathcal{A}}')^{\mathrm{tr}}$. On the other hand, we have the triangulated functor $F^{\mathrm{tr}}: \mathcal{A}^{\mathrm{tr}} \to (\mathcal{A}')^{\mathrm{tr}}$ and its derived functor $RF^{\mathrm{tr}}: \mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to \mathrm{CohoFunct}((\mathcal{A}')^{\mathrm{tr}})$ (see 5.1). Finally, one has the functor $H^0: (\underrightarrow{\mathcal{A}}')^{\mathrm{tr}} \to \mathrm{CohoFunct}((\mathcal{A}')^{\mathrm{tr}})$ defined as follows: a right DG $\mathcal{A}'$-module $M \in \underrightarrow{\mathcal{A}}'$ uniquely extends to a right DG $(\mathcal{A}')^{\text{pre-tr}}$-module $\widetilde{M}$ (cf. Example C.11), and $H^0(M)$ is defined to be the zeroth cohomology of $\widetilde{M}$ (or equivalently $H^0(M)$ is the cohomological functor $N \mapsto \mathrm{Hom}_{(\underrightarrow{\mathcal{A}}')^{\mathrm{tr}}}(N, M)$, $N \in (\mathcal{A}')^{\mathrm{tr}} \subset (\underrightarrow{\mathcal{A}}')^{\mathrm{tr}}$).

Finally, using that $(\mathcal{A}')^{\circ}\text{-DGmod} = ((\mathcal{A}')^{\text{pre-tr}})^{\circ}\text{-DGmod}$ (see Example C.11), one gets the functor $H^0: ((\mathcal{A}')^{\circ}\text{-DGmod})^{\mathrm{tr}} \to \mathrm{CohoFunct}((\mathcal{A}')^{\mathrm{tr}})$.

We are going to construct an isomorphism $RF^{\mathrm{tr}} \xrightarrow{\sim} H^0(RF)^{\mathrm{tr}}$. To this end, consider the diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{A}} & \xrightarrow{\approx} & \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A} \nearrow \mathcal{B} & \hookrightarrow & \underrightarrow{\mathcal{A}} & \to & \underrightarrow{\mathcal{A}}' \end{array} \tag{5.5}$$

(see 4.10 for the definition of $\widetilde{\mathcal{A}}$). Its left square is not commutative, but there is a canonical morphism from the composition $\widetilde{\mathcal{A}} \to \mathcal{A} \hookrightarrow \underrightarrow{\mathcal{A}}$ to the composition $\widetilde{\mathcal{A}} \to \mathcal{A} \nearrow \mathcal{B} \hookrightarrow \underrightarrow{\mathcal{A}}$. So we get a canonical morphism φ from the composition $\widetilde{\mathcal{A}}^{\mathrm{tr}} \to (\mathcal{A}')^{\mathrm{tr}} \to (\underrightarrow{\mathcal{A}}')^{\mathrm{tr}} \to \mathrm{CohoFunct}(\mathcal{A}')$ to the composition $\widetilde{\mathcal{A}}^{\mathrm{tr}} \to (\mathcal{A} \nearrow \mathcal{B})^{\mathrm{tr}} \to (\underrightarrow{\mathcal{A}}')^{\mathrm{tr}} \to \mathrm{CohoFunct}(\mathcal{A}')$. By 4.10 we can identify $\widetilde{\mathcal{A}}^{\mathrm{tr}}$ with $\mathcal{A}^{\mathrm{tr}}$ and $(\mathcal{A} \nearrow \mathcal{B})^{\mathrm{tr}}$ with $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}$, so φ induces a morphism

$$RF^{\mathrm{tr}} \to H^0(RF)^{\mathrm{tr}} \tag{5.6}$$

by the universal property (5.3) of RF^{tr}.

5.4. Proposition. *The morphism* (5.6) *is an isomorphism.*

See Section 9.1 for a proof.

5.5. Define the DG category $(\mathcal{A} \nearrow \mathcal{B})_F$ to be the (naive) fiber product of $\mathcal{A}' \times (\mathcal{A} \nearrow \mathcal{B})$ and $\overrightarrow{\Delta}_{\underrightarrow{\mathcal{A}}'}$ over $\underrightarrow{\mathcal{A}}' \times \underrightarrow{\mathcal{A}}'$, where $\overrightarrow{\Delta}_{\underrightarrow{\mathcal{A}}'}$ is the "diagonal" DG category defined in D.1 and $\mathcal{A} \nearrow \mathcal{B}$ is mapped to $\underrightarrow{\mathcal{A}}'$ by RF. So the objects of $(\mathcal{A} \nearrow \mathcal{B})_F$ are triples (Y, X, φ), where $Y \in \mathcal{A} \nearrow \mathcal{B}$, $X \in \mathcal{A}'$, and $\varphi: X \to RF(Y)$ is a homotopy equivalence. The DG functor $(\mathcal{A} \nearrow \mathcal{B})_F \to \mathcal{A}'$ defined by $(Y, X, \varphi) \mapsto X$ is also called the right derived DG functor of F and denoted by RF.

Now consider the triangulated functor $G = F^{\mathrm{tr}}: \mathcal{A}^{\mathrm{tr}} \to (\mathcal{A}')^{\mathrm{tr}}$. It follows from Proposition 5.4 that $((\mathcal{A} \nearrow \mathcal{B})_F)^{\mathrm{tr}}$ identifies with the triangulated category $(\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}})_G$ from 5.1 and $(RF)^{\mathrm{tr}}: ((\mathcal{A} \nearrow \mathcal{B})_F)^{\mathrm{tr}} \to (\mathcal{A}')^{\mathrm{tr}}$ identifies with Deligne's derived functor $RG: (\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}})_G \to (\mathcal{A}')^{\mathrm{tr}}$.

5.6. The definition of $(\mathcal{A} \nearrow \mathcal{B})_F$ used $\overrightarrow{\Delta}_{\mathcal{A}'}$. There are also versions of $(\mathcal{A} \nearrow \mathcal{B})_F$ using the DG categories $\overleftarrow{\Delta}_{\mathcal{A}'}$ and $\overleftrightarrow{\Delta}_{\mathcal{A}'}$ from D.1. They will be denoted respectively by $(\mathcal{A} \nearrow \mathcal{B})_{\leftarrow F}$ and $(\mathcal{A} \nearrow \mathcal{B})_{\leftrightarrow F}$. E.g., the objects of $(\mathcal{A} \nearrow \mathcal{B})_{\leftarrow F}$ are triples (Y, X, ψ), where $Y \in \mathcal{A} \nearrow \mathcal{B}$, $X \in \mathcal{A}'$, and $\psi : RF(Y) \to X$ is a homotopy equivalence. We have the right derived DG functors $RF : (\mathcal{A} \nearrow \mathcal{B})_{\leftarrow F} \to \mathcal{A}'$ and $RF : (\mathcal{A} \nearrow \mathcal{B})_{\leftrightarrow F} \to \mathcal{A}'$. Sometimes we will write $(\mathcal{A} \nearrow \mathcal{B})_{\to F}$ instead of $(\mathcal{A} \nearrow \mathcal{B})_F$. The DG functors $(\mathcal{A} \nearrow \mathcal{B})_{\to F} \leftarrow (\mathcal{A} \nearrow \mathcal{B})_{\leftrightarrow F} \to (\mathcal{A} \nearrow \mathcal{B})_{\leftarrow F}$ are quasi-equivalences by Lemma D.3, and one has a canonical commutative diagram

$$\begin{array}{ccccc} (\mathcal{A} \nearrow \mathcal{B})_{\to F} & \xleftarrow{\approx} & (\mathcal{A} \nearrow \mathcal{B})_{\leftrightarrow F} & \xrightarrow{\approx} & (\mathcal{A} \nearrow \mathcal{B})_{\leftarrow F} \\ {\scriptstyle RF}\downarrow & & \downarrow{\scriptstyle RF} & & \downarrow{\scriptstyle RF} \\ \mathcal{A} & = & \mathcal{A} & = & \mathcal{A} \end{array} \quad . \tag{5.7}$$

6. Some commutative diagrams

6.1. Uniqueness of DG quotient

Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory. Given a quotient (1.1) of $\mathcal{A}$ modulo $\mathcal{B}$ we will "identify" it with the quotient $\mathcal{A} \xleftarrow{\approx} \widetilde{\mathcal{A}} \xrightarrow{F} \mathcal{A} \nearrow \mathcal{B}$ from 4.10. More precisely, we will construct a canonical commutative diagram of DG categories

$$\begin{array}{ccccc} \widetilde{\mathcal{A}} & \xleftarrow{\approx} & \underline{\mathcal{A}} & \xrightarrow{\approx} & \widetilde{\mathcal{A}} \\ {\scriptstyle F}\downarrow & & \downarrow & & \downarrow{\scriptstyle \xi} \\ \mathcal{A} \nearrow \mathcal{B} & \xleftarrow{\approx} & \underline{\mathcal{C}} & \xrightarrow{\approx} & \mathcal{C} \end{array} \tag{6.1}$$

(the symbols $\xrightarrow{\approx}$, $\xleftarrow{\approx}$ denote quasi-equivalences). To this end, notice that the derived DG functor $R\xi : (\widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}})_\xi \to \mathcal{C}$ defined in 5.5 and the projection $(\widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}})_\xi \to \widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}}$ are quasi-equivalences (here $\widetilde{\mathcal{B}}$ is the preimage of $\mathcal{B}$ in $\widetilde{\mathcal{A}}$). Put $\underline{\mathcal{C}} := (\widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}})_\xi$. Define the DG functor $\underline{\mathcal{C}} \to \mathcal{C}$ to equal $R\xi$ and the DG functor $\underline{\mathcal{C}} \to \mathcal{A} \nearrow \mathcal{B}$ to be the composition $\underline{\mathcal{C}} = (\widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}})_\xi \to \widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}} \to \mathcal{A} \nearrow \mathcal{B}$. We put $\underline{\mathcal{A}} := \overrightarrow{\widetilde{\mathcal{A}}}$, i.e., $\underline{\mathcal{A}}$ is the analog of $\overrightarrow{\mathcal{A}}$ with $(\mathcal{A}, \mathcal{B})$ replaced by $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$. The DG functor $\underline{\mathcal{A}} \to \widetilde{\mathcal{A}}$ is the analog of $\overrightarrow{\mathcal{A}} \to \mathcal{A}$. The DG functor $\underline{\mathcal{A}} \to \overrightarrow{\widetilde{\mathcal{A}}}$ is induced by the DG functors $\widetilde{\mathcal{A}} \to \mathcal{A}$ and $\widetilde{\mathcal{B}} \to \mathcal{B}$. Finally, $\underline{\mathcal{A}} \to \underline{\mathcal{C}}$ is the DG functor $\overrightarrow{\widetilde{\mathcal{A}}} \to \underline{\mathcal{C}}$ defined by $(a, Y, g) \mapsto (Y, \xi(a), \underrightarrow{\xi}(g))$ (here $a \in \widetilde{\mathcal{A}}$, $Y \in \widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}} \subset \underrightarrow{\widetilde{\mathcal{A}}}$, and $g : a \to Y$ is a closed morphism of degree 0 whose cone is homotopy equivalent to an object of $\underrightarrow{\mathcal{B}}$; recall that an object of $\underline{\mathcal{C}}$ is a triple (Y, X, φ), where $Y \in \widetilde{\mathcal{A}} \nearrow \widetilde{\mathcal{B}} \subset \underrightarrow{\widetilde{\mathcal{A}}}$, $X \in \mathcal{C}$, and φ is a homotopy equivalence from X to $R\xi(Y)$, i.e., the image of Y under $\underrightarrow{\xi} : \underrightarrow{\widetilde{\mathcal{A}}} \to \underrightarrow{\mathcal{C}}$).

6.2. More diagrams (to be used in Section 7)

6.2.1. Now let us consider the case that $\widetilde{\mathcal{A}} = \mathcal{A}$ and the DG functor $\widetilde{\mathcal{A}} \to \mathcal{A}$ equals $\mathrm{id}_{\mathcal{A}}$, so our quotient (1.1) is just a DG category $\mathcal{C}$ equipped with a DG functor $\xi : \mathcal{A} \to \mathcal{C}$. Then diagram (6.1) becomes

$$\begin{array}{ccccc} \widetilde{\mathcal{A}} & = & \widetilde{\mathcal{A}} & \xrightarrow{\approx} & \mathcal{A} \\ {\scriptstyle F}\downarrow & & \downarrow & & \downarrow{\scriptstyle \xi}, \quad \underline{\mathcal{C}} := (\mathcal{A} \nearrow \mathcal{B})_{\xi}. \\ \mathcal{A} \nearrow \mathcal{B} & \xleftarrow{\approx} & \underline{\mathcal{C}} & \xrightarrow{\approx} & \mathcal{C} \end{array} \tag{6.2}$$

Here the DG functors $\mathcal{A} \leftarrow \widetilde{\mathcal{A}} \to \mathcal{A} \nearrow \mathcal{B}$ are the same as in (4.2).

In 7.5 we will use a slightly different canonical commutative diagram of DG categories

$$\begin{array}{ccccc} \mathcal{A} \nearrow \mathcal{B} & \xleftarrow{\approx} & \overline{\mathcal{C}} & \xrightarrow{\approx} & \mathcal{C} \\ \cap & & \downarrow & & \downarrow{\scriptstyle \xi^*} \\ \underset{\rightarrow}{\mathcal{A}} & \longleftarrow & \mathcal{A}^\circ\text{-resDGmod} & \xrightarrow{\approx} & \mathcal{A}^\circ\text{-DGmod} \end{array} \tag{6.3}$$

in which ξ^* is defined by $\xi^* c(a) := \mathrm{Hom}(\xi(a), c)$. Here is the construction.

Let us start with the lower row of (6.3). Consider the DG category $\mathcal{M}or(\mathcal{A}^\circ\text{-DGmod})$ (see 2.9 for the definition of $\mathcal{M}or$). Its objects are triples (Q, M, f), where $Q, M \in \mathcal{A}^\circ$-DGmod and $f : Q \to M$ is a closed morphism of degree 0. We define $\mathcal{A}^\circ$-resDGmod $\subset \mathcal{M}or(\mathcal{A}^\circ\text{-DGmod})$ to be the full DG subcategory of triples (Q, M, f) such that $Q \in \underset{\rightarrow}{\mathcal{A}}$ and f is a quasi-isomorphism (so Q is a semi-free resolution of M). In other words, $\mathcal{A}^\circ$-resDGmod is the DG category of *resolved DG $\mathcal{A}^\circ$-modules.* The DG functors $\mathcal{A}^\circ$-resDGmod $\to \underset{\rightarrow}{\mathcal{A}}$ and $\mathcal{A}^\circ$-resDGmod $\to \mathcal{A}^\circ$-DGmod are defined by $(Q, M, f) \mapsto Q$ and $(Q, M, f) \mapsto M$.

We define $\overline{\mathcal{C}}$ to be the DG category $(\mathcal{A} \nearrow \mathcal{B})_{\leftarrow \xi}$ from 5.6. So the objects of $\overline{\mathcal{C}}$ are triples (Y, X, ψ), where $Y \in \mathcal{A} \nearrow \mathcal{B}$, $X \in \mathcal{C}$, and $\psi : R\xi(Y) \to X$ is a homotopy equivalence in $\underset{\rightarrow}{\mathcal{C}}$. The upper row of (6.3) is defined just as the lower row of (6.1).

The DG functor $\overline{\mathcal{C}} \to \mathcal{A}^\circ$-resDGmod $\subset \mathcal{M}or(\mathcal{A}^\circ\text{-DGmod})$ is defined as follows. To $(Y, X, \psi) \in \overline{\mathcal{C}}$ one assigns $(Y, \xi^* X, \chi) \in \mathcal{M}or(\mathcal{A}^\circ\text{-DGmod})$, where $\chi : Y \to \xi^* X$ corresponds to $\psi : R\xi(Y) \to X$ by adjointness. This assignment extends in the obvious way to a DG functor from $\overline{\mathcal{C}}$ to $\mathcal{M}or(\mathcal{A}^\circ\text{-DGmod})$. To show that its image is contained in $\mathcal{A}^\circ$-resDGmod we have to prove that $\chi : Y \to \xi^* X$ is a quasi-isomorphism. This follows from the next lemma.

6.2.2. Lemma. *The natural morphism $Y \to \xi^* \underset{\rightarrow}{\xi}(Y) = \xi^* R\xi(Y)$, $Y \in \mathcal{B}^\perp \subset \underset{\rightarrow}{\mathcal{A}} \subset \mathcal{A}^\circ$-DGmod, is a quasi-isomorphism.*

Proof. We will identify $\mathrm{Ho}(\underset{\rightarrow}{\mathcal{A}})$ with the derived category $D(\mathcal{A})$ of $\mathcal{A}^\circ$-modules (so both Y and $\xi^* \underset{\rightarrow}{\xi}(Y)$ will be considered as objects of $\mathrm{Ho}(\underset{\rightarrow}{\mathcal{A}})$). The essential image of $\mathrm{Ho}(\underset{\rightarrow}{\mathcal{B}})$ in $\mathrm{Ho}(\underset{\rightarrow}{\mathcal{A}})$ will be again denoted by $\mathrm{Ho}(\underset{\rightarrow}{\mathcal{B}})$.

It suffices to show that

$$\mathrm{Cone}\big(Y \to \xi^* \underset{\rightarrow}{\xi}(Y)\big) \in \mathrm{Ho}(\underset{\rightarrow}{\mathcal{B}}) \tag{6.4}$$

for every $Y \in \mathrm{Ho}(\underset{\rightarrow}{\mathcal{A}})$ (then for $Y \in \mathcal{B}^{\perp}$ one has $\mathrm{Cone}(Y \to \xi^* \underset{\rightarrow}{\xi}(Y)) \in \mathrm{Ho}(\underset{\rightarrow}{\mathcal{B}}) \cap \mathrm{Ho}(\mathcal{B}^{\perp}) = 0$). Proposition 1.4 says that (6.4) holds for $Y \in \mathrm{Ho}(\mathcal{A})$. Objects $Y \in \mathrm{Ho}(\underset{\rightarrow}{\mathcal{A}})$ for which (6.4) holds form a triangulated subcategory closed under (infinite) direct sums. So (6.4) holds for all $Y \in \mathrm{Ho}(\underset{\rightarrow}{\mathcal{A}})$ by Lemma 4.4. □

6.2.3. Now let $\underline{\overline{\mathcal{C}}}$ denote the DG category $(\mathcal{A} \nearrow \mathcal{B})_{\leftrightarrow \xi}$ defined in 5.6. Using the quasi-equivalences $\overline{\mathcal{C}} \xleftarrow{\approx} \underline{\overline{\mathcal{C}}} \xrightarrow{\approx} \underline{\mathcal{C}}$ one can "glue" (6.2) and (6.3) and get a canonical commutative diagram of DG categories

$$\begin{array}{ccccc}
\widetilde{\mathcal{A}} & \xleftarrow{\approx} & \widetilde{\mathcal{A}} \times_{\underline{\mathcal{C}}} \underline{\overline{\mathcal{C}}} & \xrightarrow{\approx} & \mathcal{A} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \xi} \\
\mathcal{A} \nearrow \mathcal{B} & \xleftarrow{\approx} & \underline{\overline{\mathcal{C}}} & \xrightarrow{\approx} & \mathcal{C} \\
\cap & & \downarrow & & \downarrow{\scriptstyle \xi^*} \\
\underset{\rightarrow}{\mathcal{A}} & \longleftarrow & \mathcal{A}^{\circ}\text{-resDGmod} & \xrightarrow{\approx} & \mathcal{A}^{\circ}\text{-DGmod}
\end{array} \tag{6.5}$$

(the DG functor $\widetilde{\mathcal{A}} \times_{\underline{\mathcal{C}}} \underline{\overline{\mathcal{C}}} \to \widetilde{\mathcal{A}}$ is a quasi-equivalence by Lemma D.3, and the DG functor $\widetilde{\mathcal{A}} \times_{\underline{\mathcal{C}}} \underline{\overline{\mathcal{C}}} \to \mathcal{A}$ is the composition $\widetilde{\mathcal{A}} \times_{\underline{\mathcal{C}}} \underline{\overline{\mathcal{C}}} \to \widetilde{\mathcal{A}} \to \mathcal{A}$, so it is also a quasi-equivalence).

7. More on derived DG functors

7.1. Let $\xi : \mathcal{A} \to \mathcal{C}$ be a quotient of a DG category $\mathcal{A}$ by a full DG subcategory $\mathcal{B} \subset \mathcal{A}$ (so in (1.1) $\widetilde{\mathcal{A}} = \mathcal{A}$ and the DG functor $\widetilde{\mathcal{A}} \to \mathcal{A}$ equals $\mathrm{id}_{\mathcal{A}}$). Let F be a DG functor from $\mathcal{A}$ to a DG category $\mathcal{A}'$. Under a suitable flatness assumption (e.g., if $\mathcal{C}$ is the DG quotient $\mathcal{A}/\mathcal{B}$ from Section 3 and (3.4) holds), we will define notions of the right derived DG functor of F, which correspond to derived triangulated functors (5.2) and (5.4). They are essentially equivalent to those from 5.2 and 5.5 but are based on $\mathcal{C}$ rather than the DG quotient $\mathcal{A} \nearrow \mathcal{B}$ from 4.9. One can easily pass from right derived DG functors to left ones by considering the dual DG categories.

7.2. Consider the DG functor

$$\xi^* : \mathcal{C} \to \mathcal{A}^{\circ}\text{-DGmod}, \qquad \xi^* c(a) := \mathrm{Hom}\big(\xi(a), c\big). \tag{7.1}$$

From now on we assume that the diagram $\mathcal{C} \xleftarrow{\xi} \mathcal{A} \xrightarrow{F} \mathcal{A}'$ satisfies the following flatness condition:

$$\text{for all } c \in \mathrm{Ob}\,\mathcal{C},\ \text{the morphisms } \xi^* c \otimes_{\mathcal{A}} \mathcal{A}' \to \xi^* c \overset{L}{\otimes}_{\mathcal{A}} \mathcal{A}' \text{ are quasi-isomorphisms.} \quad (7.2)$$

This condition is satisfied if $\mathcal{C}$ is the DG quotient $\mathcal{A}/\mathcal{B}$ from Section 3 and (3.4) holds: in this case the DG $\mathcal{A}^\circ$-modules $\xi^* c$, $c \in \mathcal{C}$, are homotopically flat by Lemma C.15(i).

7.3. We are going to define a DG version of the derived triangulated functor (5.2). As a first step, consider the DG functor

$$\mathbf{R}F : \mathcal{C} \to \left(\mathcal{A}'\right)^\circ\text{-DGmod} \quad (7.3)$$

corresponding to the DG $\mathcal{C} \otimes (\mathcal{A}')^\circ$-module $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}'$ (see C.8). (This is only a first step because the homotopy category of the target of $\mathbf{R}F$ is not the *derived* category of DG $(\mathcal{A}')^\circ$-modules.) The isomorphism $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}' = Hom_{\mathcal{C}} \otimes_{\mathcal{A}} \mathcal{A}'$ (see (C.8)) shows that $\mathbf{R}F = \mathrm{Ind}_F \circ \xi^*$, where $\xi^* : \mathcal{C} \to \mathcal{A}^\circ$-DGmod is defined by (7.1) and $\mathrm{Ind}_F : \mathcal{A}^\circ$-DGmod $\to$ $(\mathcal{A}')^\circ$-DGmod is the induction DG functor (see C.9).

The fiber product of $\mathcal{C}$ and $(\mathcal{A}')^\circ$-resDGmod over $(\mathcal{A}')^\circ$-DGmod will be denoted by $\mathcal{C}_{[F]}$ (see 6.2.1 for the definition of $(\mathcal{A}')^\circ$-resDGmod). The DG functor $\mathcal{C}_{[F]} \to \mathcal{C}$ is a quasi-equivalence. We define the derived DG functor $RF : \mathcal{C}_{[F]} \to \underrightarrow{\mathcal{A}}'$ to be the composition $\mathcal{C}_{[F]} \to (\mathcal{A}')^\circ$-resDGmod $\to \underrightarrow{\mathcal{A}}'$. A 2-categorical reformulation of this definition will be given in Remark (ii) from E.6.

Let $\mathcal{C}_{(F)}$ denote the preimage of $\mathcal{A}' \subset \underrightarrow{\mathcal{A}}'$ under RF (so $\mathcal{C}_{(F)}$ is a full DG subcategory of $\mathcal{C}_{[F]}$). One has $RF : \mathcal{C}_{(F)} \to \mathcal{A}'$.

In 7.4, 7.5 we will show using (7.2) that the above definitions are reasonable: the DG functor $RF : \mathcal{C}_{[F]} \to \underrightarrow{\mathcal{A}}'$ is essentially equivalent to the DG functor RF from 5.2 and therefore agrees with the derived triangulated functor (5.2). There is a similar relation between $RF : \mathcal{C}_{(F)} \to \mathcal{A}'$, the DG functor from 5.5, and the derived triangulated functor (5.4).

Remark. If k is a field or, more generally, if

$$\mathrm{Hom}(U, X) \text{ is a semi-free DG } k\text{-module for all } X \in \mathcal{A},\ U \in \mathcal{B}, \quad (7.4)$$

then the image of $\mathbf{R}F : \mathcal{C} \to (\mathcal{A}')^\circ$-DGmod is contained in the full subcategory $\underrightarrow{\mathcal{A}}'$ of semi-free DG $(\mathcal{A}')^\circ$-modules (in the case $\mathcal{A}' = \mathcal{A}$, $F = \mathrm{id}_{\mathcal{A}}$ this is Lemma C.15(ii), and the general case follows). So if (7.4) holds then one does not have to consider $\mathcal{C}_{[F]}$: one can simply define $RF : \mathcal{C} \to \underrightarrow{\mathcal{A}}'$ to be the DG functor corresponding to $\mathbf{R}F$.

7.4. Assuming (7.2) we will "identify" $RF : \mathcal{C}_{[F]} \to \mathcal{A}'$ with the DG functor $RF : \mathcal{A} \nearrow \mathcal{B} \to \underrightarrow{\mathcal{A}}'$ from 5.2. More precisely, here is a construction of a commutative diagram

$$\begin{array}{ccccccc} \mathcal{A} \nearrow \mathcal{B} & \xleftarrow{\approx} & \mathcal{C}_1 & \xrightarrow{\approx} & \mathcal{C}_{[F]} & \xrightarrow{\approx} & \mathcal{C} \\ {\scriptstyle RF}\downarrow & & \downarrow & & \downarrow{\scriptstyle RF} & & \\ \underrightarrow{\mathcal{A}}' & = & \underrightarrow{\mathcal{A}}' & = & \underrightarrow{\mathcal{A}}' & & \end{array} \quad . \quad (7.5)$$

Put $\mathcal{C}_1 := \mathcal{C}_{[\mathrm{id}_{\mathcal{A}}]}$, so the objects of $\mathcal{C}_1$ are triples (c, Q, f), where $c \in \mathcal{C}$, $Q \in \underrightarrow{\mathcal{A}}$, and $f : Q \to \xi^* c$ is a quasi-isomorphism. The derived DG functor $R\,\mathrm{id}_{\mathcal{A}} : \mathcal{C}_1 \to \mathcal{A}$, i.e., the DG functor $\mathcal{C}_1 \to \underrightarrow{\mathcal{A}}$ defined by $(c, Q, f) \mapsto Q$, induces a quasi-equivalence $\mathcal{C}_1 \xrightarrow{\approx} \mathcal{A} \nearrow \mathcal{B} \subset \underrightarrow{\mathcal{A}}$ (see Proposition 1.5.1). To define the DG functor $\mathcal{C}_1 \to \mathcal{C}_{[F]}$ notice that by the flatness assumption (7.2) the image of the composition

$$\mathcal{C}_1 = \mathcal{C}_{[\mathrm{id}_{\mathcal{A}}]} \to \mathcal{A}^\circ\text{-resDGmod} \hookrightarrow \mathcal{M}or(\mathcal{A}^\circ\text{-DGmod}) \to \mathcal{M}or\big((\mathcal{A}')^\circ\text{-DGmod}\big)$$

is contained in $(\mathcal{A}')^\circ$-resDGmod, so we get a DG functor $\mathcal{C}_1 = \mathcal{C}_{[\mathrm{id}_{\mathcal{A}}]} \to (\mathcal{A}')^\circ$-resDGmod whose composition with the DG functor $(\mathcal{A}')^\circ$-resDGmod $\to (\mathcal{A}')^\circ$-DGmod equals (7.3), i.e., we get a DG functor $\mathcal{C}_1 \to \mathcal{C}_{[F]}$.

7.5. In fact, one can construct a slightly better diagram

$$\begin{array}{ccccccc}
\widetilde{\underrightarrow{\mathcal{A}}} & \xleftarrow{\approx} & \widetilde{\mathcal{A}} & \xrightarrow{\approx} & \mathcal{A} & & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \xi} & & \\
\mathcal{A} \nearrow \mathcal{B} & \xleftarrow{\approx} & \widetilde{\mathcal{C}} & \xrightarrow{\approx} & \mathcal{C}_{[F]} & \xrightarrow{\approx} & \mathcal{C} \\
{\scriptstyle RF}\downarrow & & \downarrow & & \downarrow{\scriptstyle RF} & & \\
\underrightarrow{\mathcal{A}'} & = & \underrightarrow{\mathcal{A}'} & = & \underrightarrow{\mathcal{A}'} & &
\end{array} \qquad . \tag{7.6}$$

To this end, first replace in (7.5) $\mathcal{C}_1$ by the DG category $\overline{\mathcal{C}}$ from (6.3) (the right square of (6.3) defines a DG functor $\overline{\mathcal{C}} \to \mathcal{C}_1$, which is a quasi-equivalence because $\overline{\mathcal{C}} \to \mathcal{C}$ and $\mathcal{C}_1 \to \mathcal{C}$ are). Next, put $\widetilde{\mathcal{C}} := \underline{\overline{\mathcal{C}}}$ (see 6.2.3 for the definition of $\underline{\overline{\mathcal{C}}}$) and replace $\overline{\mathcal{C}}$ by $\widetilde{\mathcal{C}}$. Now the upper two rows of (6.5) yield (7.6) with $\widetilde{\mathcal{A}} := \widetilde{\underrightarrow{\mathcal{A}}} \times_{\underline{\mathcal{C}}} \underline{\overline{\mathcal{C}}}$.

8. Proof of Theorem 3.4

8.1. We can suppose that (3.3) holds (if (3.4) holds replace $\mathcal{A}$ and $\mathcal{B}$ by the dual categories). It suffices to show that Φ is fully faithful (this will imply that $\mathrm{Im}\,\Phi$ is a triangulated subcategory of $(\mathcal{A}/\mathcal{B})^{\mathrm{tr}}$, but on the other hand $\mathrm{Im}\,\Phi \supset \mathcal{A}/\mathcal{B}$, so Φ is essentially surjective). In other words, it suffices to prove that for every $X, Y \in \mathcal{A}^{\text{pre-tr}}$ and every $i \in \mathbb{Z}$ the homomorphism

$$\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}}(X, Y) \to \mathrm{Ext}^i_{(\mathcal{A}/\mathcal{B})^{\mathrm{tr}}}(X, Y) \tag{8.1}$$

is bijective. It is enough to prove this for $X, Y \in \mathcal{A}$.

8.2. By (A.1), the l.h.s. of (8.1) can be computed as follows:

$$\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}}(X, Y) = \varinjlim_{(Y \to Z) \in Q_Y} H^i \,\mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}}(X, Z), \tag{8.2}$$

where Q_Y is the filtering category of $\mathcal{A}^{\mathrm{tr}}$-morphisms $f: Y \to Z$ such that $\mathrm{Cone}(f)$ is $\mathcal{A}^{\mathrm{tr}}$-isomorphic to an object of $\mathcal{B}^{\mathrm{tr}}$.

The r.h.s. of (8.1) can be written as

$$\mathrm{Ext}^i_{(\mathcal{A}/\mathcal{B})^{\mathrm{tr}}}(X, Y) = \varinjlim_{(Y \to Z) \in Q_Y} H^i \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Z). \tag{8.3}$$

To see this, first notice that the DG functor $\mathcal{A}/\mathcal{B} \to (\mathcal{A}/\mathcal{B})^{\text{pre-tr}}$ is fully faithful, so $\mathrm{Ext}^i_{(\mathcal{A}/\mathcal{B})^{\mathrm{tr}}}(X, Y) := H^i \mathrm{Hom}_{(\mathcal{A}/\mathcal{B})^{\text{pre-tr}}}(X, Y) = H^i \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$; then notice that a morphism $Y \to Z$ from Q_Y induces an isomorphism

$$H^i \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = H^i \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Y) \xrightarrow{\sim} H^i \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Z)$$

because $\mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, U)$ is acyclic for every $U \in \mathcal{B}$ (acyclicity is clear since U is homotopy equivalent to 0 as an object of $\mathcal{A}^{\text{pre-tr}}/\mathcal{B}$).

8.3. Consider (8.1) as a morphism from the r.h.s. of (8.2) to the r.h.s. of (8.3). Clearly, it is induced by the morphisms $\alpha_Z : \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}}(X, Z) \to \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Z)$, $Z \in \mathcal{A}^{\text{pre-tr}}$. By (3.2) each α_Z is injective and $L_Z := \mathrm{Coker}\,\alpha_Z$ is the union of an increasing sequence of subcomplexes $0 = (L_Z)_0 \subset (L_Z)_1 \subset \cdots$ such that $(L_Z)_n/(L_Z)_{n-1} = \mathrm{Hom}^n_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Z)$ for $n \geqslant 1$. So to prove that (8.1) is bijective, it suffices to show that

$$\varinjlim_{(Y \to Z) \in Q_Y} H^i \mathrm{Hom}^n_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Z) = 0, \quad n \geqslant 1.$$

For $n \geqslant 1$ the DG functor $Z \mapsto \mathrm{Hom}^n_{\mathcal{A}^{\text{pre-tr}}/\mathcal{B}}(X, Z)$ is a direct sum of DG functors of the form $Z \mapsto F_{X,U} \otimes \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}}(U, Z)$, $U \in \mathcal{B}$, where $F_{X,U}$ is a homotopically flat complex of k-modules. Since

$$\varinjlim_{(Y \to Z) \in Q_Y} H^i \mathrm{Hom}_{\mathcal{A}^{\text{pre-tr}}}(U, Z) = \mathrm{Ext}^i_{\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}}(U, Z) = 0, \quad U \in \mathcal{B}$$

it remains to prove the following lemma.

8.4. Lemma. *Let $\{C_\alpha\}$ be a filtering inductive system of objects of the homotopy category of complexes of k-modules (so each C_α is a complex, to each morphism $\mu : \alpha \to \beta$ there corresponds a morphism $f_\mu : C_\alpha \to C_\beta$ and $f_{\mu\nu}$ is homotopy equivalent to $f_\mu f_\nu$). Suppose that $\varinjlim_\alpha H^i(C_\alpha) = 0$ for all i. Then for every homotopically flat complex F of k-modules*

$$\varinjlim_\alpha H^i(C_\alpha \otimes F) = 0.$$

Remark. This would be obvious if we had a true inductive system of complexes, i.e., if $f_{\mu\nu}$ were equal to $f_\mu f_\nu$ (because in this case

$$\varinjlim_\alpha H^i(C_\alpha) = H^i(C), \qquad \varinjlim_\alpha H^i(C_\alpha \otimes F) = H^i(C \otimes F), \qquad C := \varinjlim_\alpha C_\alpha).$$

If there are countably many α's then Lemma 8.4 is still obvious because we can replace the morphisms f_μ by homotopy equivalent ones so that $f_{\mu\nu} = f_\mu f_\nu$.

The proof of Lemma 8.4 is based on the following lemma due to Spaltenstein [50].

8.5. Lemma. *For every complex F of k-modules there is a quasi-isomorphism $F' \to F$, where F' is a filtering direct limit of finite complexes of finitely generated free k-modules.*

Proof. One can take F' to be a semi-free resolution of F (see Appendix B). Here is a slightly different argument close to the one from [50]. Represent F as a direct limit of bounded above complexes F_n, $n \in \mathbb{N}$. Let $P_n \to F_n$ be a surjective quasi-isomorphism, where P_n is a bounded above complex of free k-modules. The morphism $P_n \to F_{n+1}$ can be lifted to a morphism $P_n \to P_{n+1}$. We can take F' to be the direct limit of the complexes P_n (because each P_n is the union of a filtering family of finite complexes of finitely generated free k-modules). □

8.6. Proof of Lemma 8.4. Let F be as in Lemma 8.4. Choose F' as in Lemma 8.5. Since Lemma 8.4 holds for F' instead of F it suffices to show that the map $H^i(C_\alpha \otimes F') \to H^i(C_\alpha \otimes F)$ is an isomorphism. As $\mathrm{Cone}(F' \to F)$ is homotopically flat and acyclic this follows from [50, Proposition 5.8]: if a complex C is homotopically flat and acyclic then $C \otimes C'$ is acyclic for every complex C' (proof: by Lemma 8.5 one may assume that C' is either homotopically flat or acyclic). □

9. Proofs of Propositions 1.4 and 5.4

9.1. Proof of Proposition 5.4. Let $Y \in \mathrm{Ob}\,\mathcal{A}$. Then

$$RF^{\mathrm{tr}}(Y) = \text{“}\varinjlim\text{”}_{(Y \to Z) \in Q_Y} F^{\mathrm{tr}}(Z). \tag{9.1}$$

Here Q_Y is the filtering category of $\mathcal{A}^{\mathrm{tr}}$-morphisms $g : Y \to Z$ such that $\mathrm{Cone}(g)$ is isomorphic to an object of $\mathcal{B}^{\mathrm{tr}}$.

To compute $RF^{\mathrm{tr}}(Y)$ choose a closed morphism $f : P \to Y$ of degree 0 with $P \in \underset{\to}{\mathcal{B}}$, $\mathrm{Cone}(f) \in \mathcal{B}^\perp$ (i.e., choose a semi-free resolution of the DG $\mathcal{B}^\circ$-module $b \mapsto \mathrm{Hom}(b, Y)$, $b \in \mathcal{B}$). Then

$$H^0(RF)^{\mathrm{tr}}(Y) = \text{“}\varinjlim\text{”}_{(W \to P) \in Q'_P} F^{\mathrm{tr}}\big(\mathrm{Cone}(W \to Y)\big), \tag{9.2}$$

where Q'_P is the filtering category of $\underset{\to}{\mathcal{B}}$-morphisms $W \to P$ with $W \in \mathcal{B}^{\text{pre-tr}}$. We have the functor $\Phi : Q'_P \to Q_Y$ that sends $h : W \to P$ to $g : Y \to \mathrm{Cone}(fh)$, and (5.6) is the morphism from the r.h.s. of (9.1) to the r.h.s. of (9.2) corresponding to Φ. It remains to prove the following lemma.

9.2. Lemma. *Let $f : P \to Y$ be a closed morphism of degree* 0 *with $Y \in \mathcal{A}$, $P \in \underrightarrow{\mathcal{B}}$,* $\mathrm{Cone}(f) \in \mathcal{B}^{\perp}$. *Then the above functor $\Phi : Q'_P \to Q_Y$ is cofinal.*

Proof. By the definition of cofinality (see [18, §8.1]), we have to show that for every $(g : Y \to Z) \in Q_Y$ there exists $(W \to P) \in Q'_P$ such that the $\mathcal{A}^{\mathrm{tr}}$-morphism $Y \to \mathrm{Cone}(W \to Y)$ can be factored through g. There is a distinguished triangle $V \xrightarrow{\psi} Y \xrightarrow{g} Z \to V[1]$, $V \in \mathcal{B}^{\mathrm{tr}}$, so it suffices to show that ψ is in the image of the composition

$$\varinjlim_{(W\to P)\in Q'_P} \mathrm{Hom}_{\mathcal{A}^{\mathrm{tr}}}(V, W) \to \mathrm{Hom}_{\underrightarrow{\mathcal{A}}^{\mathrm{tr}}}(V, P) \to \mathrm{Hom}_{\mathcal{A}^{\mathrm{tr}}}(V, Y). \tag{9.3}$$

This is clear because both maps in (9.3) are bijective (the second one is bijective because $V \in \mathcal{B}^{\mathrm{tr}}$ and $\mathrm{Cone}(f : P \to Y) \in \mathcal{B}^{\perp}$). □

9.3. Proof of Proposition 1.4. We will use the convention of 4.1: $\underrightarrow{\mathcal{B}}$ is identified with its essential image under the induction DG functor $\underrightarrow{\mathcal{B}} \to \underrightarrow{\mathcal{A}}$.

To prove that (i) ⇒ (ii), choose a closed morphism $f : P \to Y$ of degree 0 with $P \in \underrightarrow{\mathcal{B}} \subset \underrightarrow{\mathcal{A}}$, $\mathrm{Cone}(f) \in \mathcal{B}^{\perp}$ (i.e., choose a semi-free resolution of the DG $\mathcal{B}^{\circ}$-module $b \mapsto \mathrm{Hom}(b, Y)$, $b \in \mathcal{B}$). It suffices to show that (1.4) is quasi-isomorphic to $P[1]$. To this end, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(X, Y) & \xrightarrow{u_X} & \varinjlim \mathrm{Hom}\big(X, \mathrm{Cone}(W \to Y)\big) \\ \downarrow{\scriptstyle v_X} & & \downarrow{\scriptstyle \alpha_X} \\ \mathrm{Hom}\big(\xi(X), \xi(Y)\big) & \xrightarrow{\beta_X} & \varinjlim \mathrm{Hom}\big(\xi(X), \xi\big(\mathrm{Cone}(W \to Y)\big)\big) \end{array} \tag{9.4}$$

in which the direct limits are over $(W \to P) \in Q'_P$ (see 9.1 for the definition of Q'_P). Objects of $\xi(\mathcal{B})$ are homotopic to zero, so β_X is a quasi-isomorphism. By (A.1) and Lemma 9.2, α_X is also a quasi-isomorphism. So the DG $\mathcal{A}^{\circ}$-module $X \mapsto \mathrm{Cone}(v_X)$ is quasi-isomorphic to the DG $\mathcal{A}^{\circ}$-module $X \mapsto \mathrm{Cone}(u_X)$, i.e., to $P[1]$.

To prove that (ii) ⇒ (i), consider again the commutative diagram (9.4). The DG $\mathcal{A}^{\circ}$-module $X \mapsto \mathrm{Cone}(u_X)$ is quasi-isomorphic to $P[1]$, and β_X is a quasi-isomorphism. So if the DG $\mathcal{A}^{\circ}$-module $X \mapsto \mathrm{Cone}(v_X)$ is quasi-isomorphic to an object of $\underrightarrow{\mathcal{B}} \subset \underrightarrow{\mathcal{A}}$ then the DG $\mathcal{A}^{\circ}$-module

$$X \mapsto \mathrm{Cone}(\alpha_X), \quad X \in \mathcal{A} \tag{9.5}$$

is quasi-isomorphic to some $M \in \underrightarrow{\mathcal{B}} \subset \underrightarrow{\mathcal{A}}$. Clearly, M is quasi-isomorphic to the restriction of (9.5) to $\mathcal{B}$. By (A.1) and Lemma 9.2 one has

$$\varinjlim_{(W\to P)\in Q'_P} H^i \mathrm{Hom}\big(X, \mathrm{Cone}(W \to Y)\big) = \mathrm{Ext}^i_{\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}}(X, Y), \quad X, Y \in \mathcal{A}.$$

So the restriction of (9.5) to $\mathcal{B}$ is acyclic. Therefore, α_X is a quasi-isomorphism for all $X \in \mathcal{A}$. So the canonical map $\mathrm{Ext}^i_{\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}}}(X, Y) \to \mathrm{Ext}^i_{\mathcal{C}^{\mathrm{tr}}}(\xi(X), \xi(Y))$ is an isomorphism

for all $X, Y \in \mathcal{A}$, i.e., the functor $\mathcal{A}^{\mathrm{tr}}/\mathcal{B}^{\mathrm{tr}} \to \mathcal{C}^{\mathrm{tr}}$ induced by ξ is fully faithful. Its essential image is a triangulated subcategory containing $\mathrm{Ho}(\mathcal{C})$, so it equals $\mathcal{C}^{\mathrm{tr}}$. □

10. Proofs of Propositions 1.5.1, 4.6, and 4.7

10.1. Proof of Proposition 4.7. Identify $\underrightarrow{\mathcal{A}}^{\mathrm{tr}} = \mathrm{Ho}(\underrightarrow{\mathcal{A}})$ with $D(\mathcal{A})$ and $\underrightarrow{\mathcal{B}}^{\mathrm{tr}} = \mathrm{Ho}(\underrightarrow{\mathcal{B}})$ with $D(\mathcal{B})$. Then the embedding $\underrightarrow{\mathcal{B}}^{\mathrm{tr}} \to \underrightarrow{\mathcal{A}}^{\mathrm{tr}}$ identifies with the derived induction functor, so it has a right adjoint, namely the restriction functor. This proves (i). By adjointness, $(\underrightarrow{\mathcal{B}}^{\mathrm{tr}})^{\perp} \subset \mathrm{Ho}(\underrightarrow{\mathcal{A}})$ is the kernel of the restriction functor, which proves (ii). Statement (iii) follows from (i) and (ii). To prove (iv) apply Lemma A.5 in the following situation: $\mathcal{T}_0 = \mathcal{A}^{\mathrm{tr}}$, $\mathcal{T} = \underrightarrow{\mathcal{A}}^{\mathrm{tr}}$, $\mathcal{Q}_0 = \mathcal{B}^{\mathrm{tr}}$, $\mathcal{Q} = \underrightarrow{\mathcal{B}}^{\mathrm{tr}}$. □

10.2. Proof of Proposition 1.5.1. (a) is a particular case of Proposition 4.6(ii). Here is a direct proof of (a). As ξ is essentially surjective it suffices to show that the morphism $f : \mathrm{Ext}^n(\xi(a), c) \to \mathrm{Ext}^n(\xi^*\xi(a), \xi^* c)$ is an isomorphism for every $a \in \mathcal{A}$ and $c \in \mathcal{C}$. Decompose f as $\mathrm{Ext}^n(\xi(a), c) = \mathrm{Ext}^n(a, \xi^* c) \xrightarrow{f'} \mathrm{Ext}^n(\xi^*\xi(a), \xi^* c)$, where f' comes from the morphism $\varphi : a \to \xi^*\xi(a)$. By Proposition 1.4(ii), there is a distinguished triangle

$$L\,\mathrm{Ind}(N) \longrightarrow a \xrightarrow{\varphi} \xi^*\xi(a) \longrightarrow L\,\mathrm{Ind}(N)[1], \quad N \in D(\mathcal{B}), \tag{10.1}$$

where $L\,\mathrm{Ind} : D(\mathcal{B}) \to D(\mathcal{A})$ is the derived induction functor $L\,\mathrm{Ind} : D(\mathcal{B}) \to D(\mathcal{A})$. As $\xi^* c$ is annihilated by the restriction functor $\mathrm{Res} : D(\mathcal{A}) \to D(\mathcal{B})$, we see that $\mathrm{Ext}^n(L\,\mathrm{Ind}(N), \xi^* c) = 0$, so f' is an isomorphism.

Applying Res to (10.1) and using the equalities $\mathrm{Res}\,\xi^* = 0$, $\mathrm{Res}\,L\,\mathrm{Ind} = \mathrm{id}_{D(\mathcal{B})}$, we get $N = \mathrm{Res}\,a$ and $\xi^*\xi(a) \simeq \mathrm{Cone}(L\,\mathrm{Ind}\,\mathrm{Res}\,a \to a)$. This implies (b). □

10.3. Proof of Proposition 4.6. The derived category of $\mathcal{A}^{\circ}$-modules identifies with $\mathrm{Ho}(\underrightarrow{\mathcal{A}})$. The derived induction functor $I : \mathrm{Ho}(\underrightarrow{\mathcal{A}}) \to \mathrm{Ho}(\underrightarrow{\mathcal{C}})$ is left adjoint to the restriction functor $R : \mathrm{Ho}(\underrightarrow{\mathcal{C}}) \to \mathrm{Ho}(\underrightarrow{\mathcal{A}})$.

By Proposition 4.7 we can identify $\mathrm{Ho}(\underrightarrow{\mathcal{A}})/\mathrm{Ho}(\underrightarrow{\mathcal{B}})$ with $\mathrm{Ho}(\mathcal{B}^{\perp}) = (\mathrm{Ho}(\underrightarrow{\mathcal{B}}))^{\perp}$. Clearly, $R(\mathrm{Ho}(\underrightarrow{\mathcal{C}})) \subset \mathrm{Ho}(\mathcal{B}^{\perp})$. Let $i : \mathrm{Ho}(\mathcal{B}^{\perp}) \to \mathrm{Ho}(\underrightarrow{\mathcal{C}})$ and $r : \mathrm{Ho}(\underrightarrow{\mathcal{C}}) \to \mathrm{Ho}(\mathcal{B}^{\perp})$ be the functors corresponding to I and R. It suffices to show that they are quasi-inverse equivalences. Clearly, i is left adjoint to r. So we have the adjunction morphisms $\mathrm{id} \to ri$, $ir \to \mathrm{id}$, and we have to show that they are isomorphisms. By Lemma 6.2.2, the morphism $\mathrm{id} \to ri$ is an isomorphism. Therefore, the natural morphism $r \to rir$ is an isomorphism, so the morphism $rir \to r$ is an isomorphism (because the composition $r \to rir \to r$ equals id), and finally the morphism $ir \to \mathrm{id}$ is an isomorphism (because r is conservative, i.e., if f is a morphism in $\mathrm{Ho}(\underrightarrow{\mathcal{C}})$ such that $r(f)$ is an isomorphism then f is an isomorphism). □

11. Proofs of Proposition 1.6.3 and Theorem 1.6.2

11.1. Proof of Proposition 1.6.3. Let M_Y denote the DG $\mathcal{A}^{\circ}$-module (1.4). Replacing $\xi : \mathcal{A} \to \mathcal{C}$ by $\xi \otimes \mathrm{id}_{\mathcal{K}} : \mathcal{A} \otimes \mathcal{K} \to \mathcal{C} \otimes \mathcal{K}$ one gets a similar DG $\mathcal{A}^{\circ} \otimes \mathcal{K}^{\circ}$-module $M_{Y \otimes Z}$

for every $Z \in \mathcal{K}$. Clearly, $M_{Y\otimes Z} = M_Y \otimes h_Z$, where h_Z is the image of Z under the Yoneda embedding $\mathcal{K} \hookrightarrow \mathcal{K}^\circ$-DGmod. As $\mathcal{K}$ is homotopically flat over k property 1.4(ii) for $\xi : \mathcal{A} \to \mathcal{C}$ implies property 1.4(ii) for $\xi \otimes \mathrm{id}_{\mathcal{K}} : \mathcal{A} \otimes \mathcal{K} \to \mathcal{C} \otimes \mathcal{K}$. It remains to use Proposition 1.4. □

11.2. Proof of Theorem 1.6.2. A pair $(\mathcal{C}, \xi)$ satisfying Theorem 1.6.2(ii) is clearly unique in the sense of *DGcat*, and in Sections 3, 4 we proved the existence of DG quotient, i.e., the existence of a pair $(\mathcal{C}, \xi)$ satisfying Theorem 1.6.2(i). So it remains to show that (i) $\Rightarrow$ (ii).

We will use the definition of $T(\mathcal{A}, \mathcal{K})$ from E.1–E.4. One can assume that $\mathcal{K}$ is homotopically flat over k. So $T(\mathcal{A}, \mathcal{K}) \subset D(\mathcal{A}^\circ \otimes \mathcal{K})$, $T(\mathcal{B}, \mathcal{K}) \subset D(\mathcal{B}^\circ \otimes \mathcal{K})$, $T(\mathcal{C}, \mathcal{K}) \subset D(\mathcal{C}^\circ \otimes \mathcal{K})$. We can also assume that $\xi \in T(\mathcal{A}, \mathcal{C})$ comes from a DG functor $\xi : \mathcal{A} \to \mathcal{C}$ (otherwise replace $\mathcal{A}$ by one of its semi-free resolutions and apply Proposition E.7.2). So if Theorem 1.6.2(i) holds, one can apply Proposition 1.6.3 and 4.6. We see that the restriction functor $D(\mathcal{C}^\circ \otimes \mathcal{K}) \to D(\mathcal{A}^\circ \otimes \mathcal{K})$ is fully faithful, and its essential image consists precisely of objects of $D(\mathcal{A}^\circ \otimes \mathcal{K})$ annihilated by the restriction functor $D(\mathcal{A}^\circ \otimes \mathcal{K}) \to D(\mathcal{B}^\circ \otimes \mathcal{K})$. Property (ii) follows. □

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Appendix A. Triangulated categories

A.1. Categories with $\mathbb{Z}$-action and graded categories

Let C be a category with a weak action of $\mathbb{Z}$, i.e., a monoidal functor from $\mathbb{Z}$ to the monoidal category $\mathrm{Funct}(C, C)$ of functors $C \to C$ (here $\mathbb{Z}$ is viewed as a monoidal category: $\mathrm{Mor}(m, n) := \emptyset$ if $m \neq n$, $\mathrm{Mor}(n, n) := \{\mathrm{id}_n\}$, $m \otimes n := m + n$ for $m, n \in \mathbb{Z}$). For $c_1, c_2 \in C$ put $\mathrm{Ext}^n(c_1, c_2) := \mathrm{Mor}(c_1, F_n(c_2))$, where $F_n : \mathcal{C} \to \mathcal{C}$ is the functor corresponding to $n \in \mathbb{Z}$. Using the isomorphism $F_m F_n \xrightarrow{\sim} F_{m+n}$, one gets the composition map $\mathrm{Ext}^m(c_1, c_2) \times \mathrm{Ext}^n(c_2, c_3) \to \mathrm{Ext}^{m+n}(c_1, c_3)$, so $\mathcal{C}$ becomes a $\mathbb{Z}$-graded category. This $\mathbb{Z}$-graded category has an additional property: for every $n \in \mathbb{Z}$ and $c \in C$ there exists an object $c[n] \in C$ with an isomorphism $c[n] \xrightarrow{\sim} c$ of degree n. Every $\mathbb{Z}$-graded category C with this property comes from an essentially unique weak action of $\mathbb{Z}$ on C.

Suppose that each of the categories C and C' is equipped with a weak action of $\mathbb{Z}$. Consider C and C' as graded categories. Then a graded functor $C \to C'$ (i.e., a functor between the corresponding graded categories) is the same as a functor $\Phi : C \to C'$

equipped with an isomorphism $\Phi\Sigma \xrightarrow{\sim} \Sigma'\Phi$, where $\Sigma \in \mathrm{Funct}(C, C)$ and $\Sigma' \in \mathrm{Funct}(C', C')$ are the images of $1 \in \mathbb{Z}$.

An additive $\mathbb{Z}$-graded category C is considered as a plain (non-graded) category by considering elements of $\bigoplus_n \mathrm{Ext}^n(c_1, c_2)$ (rather than those of $\bigsqcup_n \mathrm{Ext}^n(c_1, c_2)$) as morphisms $c_1 \to c_2$.

All this applies, in particular, to triangulated categories.

A.2. Quotients

The quotient $\mathcal{T}/\mathcal{T}'$ of a triangulated category $\mathcal{T}$ by a triangulated subcategory $\mathcal{T}'$ is defined to be the localization of $\mathcal{T}$ by the multiplicative set S of morphisms f such that $\mathrm{Cone}(f)$ is isomorphic to an object of $\mathcal{T}'$. The category $\mathcal{T}/\mathcal{T}'$ has a canonical triangulated structure; by definition, the distinguished triangles of $\mathcal{T}/\mathcal{T}'$ are those isomorphic to the images of the distinguished triangles of $\mathcal{T}$. This is due to Verdier [56,57].

He also proved in [56,57] that for every $Y \in \mathrm{Ob}\,\mathcal{T}$ the category Q_Y of $\mathcal{T}$-morphisms $f : Y \to Z$ such that $\mathrm{Cone}(f)$ is isomorphic to an object of $\mathcal{T}'$ is filtering, and for every $Y \in \mathrm{Ob}\,\mathcal{T}$ one has an isomorphism

$$\varinjlim_{(Y \to Z) \in Q_Y} \mathrm{Ext}^i_{\mathcal{T}}(X, Z) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{T}/\mathcal{T}'}(X, Y). \tag{A.1}$$

A.3. Remarks.

(i) Verdier requires $\mathcal{T}'$ to be thick (épaisse), which means according to [57] that an object of $\mathcal{T}$ which is (isomorphic to) a direct summand of an object $\mathcal{T}'$ belongs to $\mathcal{T}'$. But the statements from A.2 hold without the thickness assumption because in [57, §II.2.2] (or in [56, Chapter 1, §2.3]) the multiplicative set S is not required to be saturated (by [57, Proposition 2.1.8] thickness of $\mathcal{T}'$ is equivalent to saturatedness of S).

(ii) $\mathcal{T}/\mathcal{T}' = \mathcal{T}/\mathcal{T}''$, where $\mathcal{T}'' \subset \mathcal{T}$ is the smallest thick subcategory containing $\mathcal{T}'$. So according to [57] an object of $\mathcal{T}$ has zero image in $\mathcal{T}/\mathcal{T}'$ if and only if it belongs to $\mathcal{T}''$.

(iii) The definitions of thickness from [56,57] are equivalent: if $\mathcal{T}' \subset \mathcal{T}$ is thick in the sense of [57] then according to [57] $\mathcal{T}'$ is the set of objects of $\mathcal{T}$ whose image in $\mathcal{T}/\mathcal{T}'$ is zero, so $\mathcal{T}'$ is thick in the sense of [56]. Direct proofs of the equivalence can be found in [49, Proposition 1.3, p. 305] and [45, Criterion 1.3, p. 390].

A.4. Let $\mathcal{Q}$ be a triangulated subcategory of a triangulated category $\mathcal{T}$. Let $\mathcal{Q}^\perp \subset \mathcal{T}$ be the right orthogonal complement of $\mathcal{Q}$, i.e., $\mathcal{Q}^\perp$ is the full subcategory of $\mathcal{T}$ formed by objects X of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(Y, X) = 0$ for all $Y \in \mathrm{Ob}\,\mathcal{Q}$. Then the morphism $\mathrm{Hom}_{\mathcal{T}}(Y, X) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{Q}}(Y, X)$ is an isomorphism for all $X \in \mathrm{Ob}\,\mathcal{Q}$, $Y \in \mathrm{Ob}\,\mathcal{T}$ (see [56, Chapter I, §6] and [57, Proposition II.2.3.3]). In particular, the functor $\mathcal{Q}^\perp \to \mathcal{T}/\mathcal{Q}$ is fully faithful. This is a particular case ($\mathcal{T}_0 = \mathcal{Q}^\perp$, $\mathcal{Q}_0 = 0$) of the following lemma.

A.5. Lemma. *Let $\mathcal{Q}$, $\mathcal{T}_0$, $\mathcal{Q}_0$ be triangulated subcategories of a triangulated category $\mathcal{T}$, $\mathcal{Q}_0 \subset \mathcal{Q} \cap \mathcal{T}_0$. Suppose that every morphism from an object of $\mathcal{T}_0$ to an object of $\mathcal{Q}$ factors through an object of $\mathcal{Q}_0$. Then the functor $\mathcal{T}_0/\mathcal{Q}_0 \to \mathcal{T}/\mathcal{Q}$ is fully faithful.*

Proof. The functor $\mathcal{T}_0/\mathcal{Q}_0 \to \mathcal{T}/\mathcal{Q}_0$ is fully faithful by (A.1). Our factorization condition implies that $\mathrm{Hom}_{\mathcal{T}/\mathcal{Q}_0}(X, Y) = 0$ for all $X \in \mathrm{Ob}\,\mathcal{T}_0$, $Y \in \mathrm{Ob}\,Q$. In other words, $\mathcal{T}_0/\mathcal{Q}_0$ is contained in the right orthogonal complement of $\mathcal{Q}/\mathcal{Q}_0$ in $\mathcal{T}/\mathcal{T}_0$, so by A.4 the functor $\mathcal{T}_0/\mathcal{Q}_0 \to (\mathcal{T}/\mathcal{Q}_0)/(\mathcal{Q}/\mathcal{Q}_0) = \mathcal{T}/\mathcal{Q}$ is fully faithful. □

A.6. Admissible subcategories

Suppose that a triangulated subcategory $\mathcal{Q} \subset \mathcal{T}$ is strictly full ("strictly" means that every object of $\mathcal{T}$ isomorphic to an object of $\mathcal{Q}$ belongs to $\mathcal{Q}$). Let $\mathcal{Q}^{\perp} \subset \mathcal{T}$ (respectively ${}^{\perp}\mathcal{Q} \subset \mathcal{T}$) be the right (respectively left) orthogonal complement of $\mathcal{Q}$, i.e., the full subcategory of $\mathcal{T}$ formed by objects X of $\mathcal{T}$ such that $\mathrm{Hom}(Y, X) = 0$ (respectively $\mathrm{Hom}(X, Y) = 0$) for all $Y \in \mathrm{Ob}\,\mathcal{Q}$. According to [5, §1], $\mathcal{Q}$ is said to be *right-admissible* if for each $X \in \mathcal{T}$ there exists a distinguished triangle $X' \to X \to X'' \to X'[1]$ with $X' \in \mathcal{Q}$ and $X'' \in \mathcal{Q}^{\perp}$ (such a triangle is unique up to unique isomorphism). As $\mathcal{Q}^{\perp}$ is thick, $\mathcal{Q}$ is right-admissible if and only if the functor $\mathcal{Q} \to \mathcal{T}/\mathcal{Q}^{\perp}$ is essentially surjective. $\mathcal{Q}$ is said to be *left-admissible* if $\mathcal{Q}^{\circ} \subset \mathcal{T}^{\circ}$ is right-admissible. There is a one-to-one correspondence between right-admissible subcategories $\mathcal{Q} \subset \mathcal{T}$ and left-admissible subcategories $\mathcal{Q}' \subset \mathcal{T}$, namely $\mathcal{Q}' = \mathcal{Q}^{\perp}$, $\mathcal{Q} = {}^{\perp}\mathcal{Q}'$. According to [5, §1] and [56, Chapter 1, §2.6] right-admissibility is equivalent to each of the following conditions:

(a) $\mathcal{Q}$ is thick and the functor $\mathcal{Q}^{\perp} \to \mathcal{T}/\mathcal{Q}$ is essentially surjective (and therefore an equivalence);
(b) the inclusion functor $\mathcal{Q} \hookrightarrow \mathcal{T}$ has a right adjoint;
(c) $\mathcal{Q}$ is thick and the functor $\mathcal{T} \to \mathcal{T}/\mathcal{Q}$ has a right adjoint;
(d) $\mathcal{T}$ is generated by $\mathcal{Q}$ and $\mathcal{Q}^{\perp}$ (i.e., if $\mathcal{T}' \subset \mathcal{T}$ is a strictly full triangulated subcategory containing $\mathcal{Q}$ and $\mathcal{Q}^{\perp}$ then $\mathcal{T}' = \mathcal{T}$).

Remark. A left or right adjoint of a triangulated functor is automatically triangulated (see [27] or [5, Proposition 1.4]).

Appendix B. Semi-free resolutions

B.1. Definition. A DG R-module F over a DG ring R is *free* if it is isomorphic to a direct sum of DG modules of the form $R[n]$, $n \in \mathbb{Z}$. A DG R-module F is *semi-free* if the following equivalent conditions hold:

(1) F can be represented as the union of an increasing sequence of DG submodules F_i, $i = 0, 1, \ldots$, so that $F_0 = 0$ and each quotient F_i/F_{i-1} is free;

(2) F has a homogeneous R-module basis B with the following property: for a subset $S \subset B$ let $\delta(S)$ be the smallest subset $T \subset B$ such that $d(S)$ is contained in the R-linear span of T, then for every $b \in B$ there is an $n \in \mathbb{N}$ such that $\delta^n(\{b\}) = \emptyset$.

A complex of k-modules is semi-free if it is semi-free as a DG k-module.

B.2. Remarks.

(i) A bounded above complex of free k-modules is semi-free.
(ii) Semi-free DG modules were explicitly introduced in [2] (according to the terminology of [2], a DG module over a DG algebra R is free if it is freely generated, as an R-module, by homogeneous elements e_α such that $\mathrm{d}e_\alpha = 0$, so semi-free is weaker than free). In fact, the notion of semi-free DG module had been known to topologists long before [2] (see, e.g., [16]). Semi-free DG modules are also called "cell DG modules" (Kriz–May [35]) and "standard cofibrant DG modules" (Hinich [19]). In fact, Hinich shows in [19, Sections 2, 3] that DG modules over a fixed DG algebra form a closed model category with weak equivalences being quasi-isomorphisms and fibrations being surjective maps. He shows, that a DG module C is cofibrant (i.e., the morphism $0 \to C$ is cofibrant) if and only if it is a direct summand of a semi-free DG module.
(iii) As noticed in [1,19], a semi-free DG module F is *homotopically projective*, which means that for every acyclic DG module N every morphism $f : F \to N$ is homotopic to 0 (we prefer to use the name "homotopically projective" instead of Spaltenstein's name "K-projective"). Indeed, if $\{F_i\}$ is a filtration on F satisfying the condition from B.1, then every homotopy between $f|_{F_{i-1}}$ and 0 can be extended to a homotopy between $f|_{F_i}$ and 0. This also follows from Lemma 4.4 applied to the triangulated subcategory $\mathcal{T}_N$ of semi-free DG R-modules F such that the complex $\mathrm{Hom}(F, N)$ is acyclic ($\mathcal{T}_N$ is closed under arbitrary direct sums and contains R).
(iv) By (iii) and Lemma B.3, the functor from the homotopy category of semi-free DG R-modules to the derived category of R-modules is an equivalence.

B.3. Lemma. *For every DG module M over a DG algebra R there is a quasi-isomorphism $f : F \to M$ with F a semi-free DG R-module. One can choose f to be surjective.*

The pair (F, f) is constructed in [2] as the direct limit of (F_i, f_i) where $0 = F_0 \hookrightarrow F_1 \hookrightarrow F_2 \hookrightarrow \cdots$, each quotient F_i/F_{i-1} is free, $f_i : F_i \to M$, $f_i|_{M_{i-1}} = f_{i-1}$. Given F_{i-1} and $f_{i-1} : F_{i-1} \to M$, one finds a morphism $\pi : P \to \mathrm{Cone}(f_{i-1})[-1]$ such that P is free and π induces an epimorphism of the cohomology groups. π defines a morphism $f_i : F_i := \mathrm{Cone}(P \to F_{i-1}) \to M$ such that $f_i|_{F_{i-1}} = f_{i-1}$. The map $\mathrm{Cone}(f_{i-1}) \to \mathrm{Cone}(f_i)$ induces a zero map of the cohomology groups, so $\mathrm{Cone}(f)$ is acyclic, i.e., f is a quasi-isomorphism.

Remark. One can reformulate the above proof of the lemma without using the "linear" word "cone" (it suffices to replace "category" by "module" in the proof of Lemma A.5).

B.4. Hinich [19] proved a version of Lemma B.3 for DG algebras, i.e., DG categories with one object. The case of a general DG category is similar.

Definition. Let $\mathcal{A}$ be a DG category $\mathcal{A}$ equipped with a DG functor $\mathcal{K} \to \mathcal{A}$. We say that $\mathcal{A}$ is *semi-free over* $\mathcal{K}$ if $\mathcal{A}$ can be represented as the union of an increasing sequence of DG subcategories $\mathcal{A}_i$, $i = 0, 1, \ldots$, so that $\operatorname{Ob}\mathcal{A}_i = \operatorname{Ob}\mathcal{A}$, $\mathcal{K}$ maps isomorphically onto $\mathcal{A}_0$, and for every $i > 0$ $\mathcal{A}_i$ as a graded k-category over $\mathcal{A}_{i-1}$ (i.e., with forgotten differentials in the Hom complexes) is freely generated over $\mathcal{A}_{i-1}$ by a family of homogeneous morphisms f_α such that $\mathrm{d}f_\alpha \in \operatorname{Mor}\mathcal{A}_{i-1}$.

Definition. A DG category $\mathcal{A}$ is *semi-free* if it is semi-free over $\mathcal{A}_{\mathrm{discr}}$, where $\mathcal{A}_{\mathrm{discr}}$ is the DG category with $\operatorname{Ob}\mathcal{A}_{\mathrm{discr}} = \operatorname{Ob}\mathcal{A}$ such that the endomorphism DG algebra of each object of $\mathcal{A}_{\mathrm{discr}}$ equals k and $\operatorname{Hom}_{\mathcal{A}_{\mathrm{discr}}}(X, Y) = 0$ if X, Y are different objects of $\mathcal{A}_{\mathrm{discr}}$.

Remarks.

(1) Semi-free DG categories with one object were introduced in [19] under the name of "standard cofibrant" DG algebras. In fact, Hinich shows in [19, Sections 2, 4] that DG algebras form a closed model category with weak equivalences being quasi-isomorphisms and fibrations being surjective maps. He shows that a DG algebra R is cofibrant (i.e., the morphism $k \to C$ is cofibrant) if and only if R is a retract of a semi-free DG algebra.
(2) $\mathbb{Z}_-$-graded semi-free DG algebras were considered as early as 1957 by Tate [55], and $\mathbb{Z}_+$-graded ones were considered in 1973 by Sullivan [53,54]. Hinich [19] explained following [1,50] that it is easy and natural to work with DG algebras without boundedness conditions.

B.5. Lemma. *For every DG category $\mathcal{A}$ there exists a semi-free DG category $\widetilde{\mathcal{A}}$ with $\operatorname{Ob}\widetilde{\mathcal{A}} = \operatorname{Ob}\mathcal{A}$ and a functor $\Psi : \widetilde{\mathcal{A}} \to \mathcal{A}$ such that $\Psi(X) = X$ for every $X \in \operatorname{Ob}\widetilde{\mathcal{A}}$ and Ψ induces a surjective quasi-isomorphism $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(\Psi(X), \Psi(Y))$ for every $X, Y \in \widetilde{\mathcal{A}}$.*

The proof is the same as for DG algebras [19, Sections 2, 4] and similar to that of Lemma B.3. $(\widetilde{\mathcal{A}}, \Psi)$ is constructed as the direct limit of $(\widetilde{\mathcal{A}}_i, \Psi_i)$ where $\operatorname{Ob}\widetilde{\mathcal{A}}_i = \operatorname{Ob}\mathcal{A}$, $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \cdots$, $\Psi_i : \widetilde{\mathcal{A}}_i \to \mathcal{A}$, $\Psi_i|_{\widetilde{\mathcal{A}}_{i-1}} = \Psi_{i-1}$, and the following conditions are satisfied:

(i) $\mathcal{A}_0$ is the discrete k-category;
(ii) for every $i > 0$ $\mathcal{A}_i$ as a graded k-category is freely generated over $\mathcal{A}_{i-1}$ by a family of homogeneous morphisms f_α such that $\mathrm{d}f_\alpha \in \operatorname{Mor}\mathcal{A}_{i-1}$;
(iii) for every $i > 0$ and $X, Y \in \operatorname{Ob}\mathcal{A}$ the morphism $\operatorname{Hom}_{\mathcal{A}_i}(X, Y) \to \operatorname{Hom}_{\mathcal{A}}(\Psi(X), \Psi(Y))$ is surjective and induces a surjective map between the sets of the cocycles;
(iv) for every $i > 0$ and $X, Y \in \operatorname{Ob}\mathcal{A}$ every cocycle $f \in \operatorname{Hom}_{\mathcal{A}_i}(X, Y)$ whose image in $\operatorname{Hom}_{\mathcal{A}}(\Psi(X), \Psi(Y))$ is a coboundary becomes a coboundary in $\operatorname{Hom}_{\mathcal{A}_{i+1}}(X, Y)$.

One constructs $(\widetilde{\mathcal{A}}_i, \Psi_i)$ by induction. Notice that (iii) holds for all i if it holds for $i = 1$, so after $(\widetilde{\mathcal{A}}_1, \Psi_1)$ is constructed one only has to kill cohomology classes by adding new morphisms.

B.6. Lemma. *If a DG functor $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ is a surjective quasi-equivalence* (*i.e., if π induces a surjection* $\mathrm{Ob}\,\widetilde{\mathcal{C}} \to \mathrm{Ob}\,\mathcal{C}$ *and surjective quasi-isomorphisms between the* Hom *complexes*) *then every DG functor from a semi-free DG category $\mathcal{A}$ to $\mathcal{C}$ lifts to a DG functor $\mathcal{A} \to \widetilde{\mathcal{C}}$. More generally, for every commutative diagram*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\Phi} & \widetilde{\mathcal{C}} \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{R} & \xrightarrow{\Psi} & \mathcal{C} \end{array}$$

such that $\mathcal{R}$ is semi-free over $\mathcal{K}$ and π is a surjective quasi-equivalence there exists a DG functor $\widetilde{\Psi} : \mathcal{R} \to \widetilde{\mathcal{C}}$ such that $\pi\widetilde{\Psi} = \Psi$ and $\widetilde{\Psi}\nu = \Phi$.

Remark. This is one of the closed model category axioms checked in [19].

Proof. Use the following fact: if $f : A \to B$ is a surjective quasi-isomorphism of complexes, $a \in A$, $b \in B$, $f(a) = \mathrm{d}b$, and $\mathrm{d}a = 0$ then there is an $a' \in A$ such that $f(a') = b$ and $a = \mathrm{d}a'$. □

Appendix C. DG modules over DG categories

Additive functors from a preadditive category $\mathcal{A}$ to the category of abelian groups are often called $\mathcal{A}$-modules (see [42]). We are going to introduce a similar terminology in the DG setting. The definitions below are similar to those of Mitchell [41].

C.1. Let $\mathcal{A}$ be a DG category. A *left DG $\mathcal{A}$-module* is a DG functor from $\mathcal{A}$ to the DG category of complexes of k-modules. Sometimes left DG $\mathcal{A}$-modules will be called simply DG $\mathcal{A}$-modules. If $\mathcal{A}$ has a single object U with $\mathrm{End}_{\mathcal{A}} U = R$ then a DG $\mathcal{A}$-module is the same as a DG R-module. A *right DG $\mathcal{A}$-module* is a left DG module over the dual DG category $\mathcal{A}^\circ$. The DG category of DG $\mathcal{A}$-modules is denoted by $\mathcal{A}$-DGmod. In particular, k-DGmod is the DG category of complexes of k-modules.

C.2. Let $\mathcal{A}$ be a DG category. Then the complex

$$\mathrm{Alg}_{\mathcal{A}} := \bigoplus_{X,Y \in \mathrm{Ob}\,\mathcal{A}} \mathrm{Hom}(X, Y)$$

has a natural DG algebra structure (interpret elements of $\mathrm{Alg}_{\mathcal{A}}$ as matrices (f_{XY}), $f_{XY} \in \mathrm{Hom}(Y, X)$, whose rows and columns are labeled by $\mathrm{Ob}\,\mathcal{A}$). The DG algebra $\mathrm{Alg}_{\mathcal{A}}$ has

the following property: every finite subset of $\mathrm{Alg}_{\mathcal{A}}$ is contained in $e\,\mathrm{Alg}_{\mathcal{A}}\,e$ for some idempotent $e \in \mathrm{Alg}_{\mathcal{A}}$ such that $de = 0$ and $\deg e = 0$. We say that a module M over $\mathrm{Alg}_{\mathcal{A}}$ is *quasi-unital* if every element of M belongs to eM for some idempotent $e \in \mathrm{Alg}_{\mathcal{A}}$ (which may be assumed closed of degree 0 without loss of generality). If Φ is a DG $\mathcal{A}$-module then $M_\Phi := \bigoplus_{X\in \mathrm{Ob}\,\mathcal{A}} \Phi(X)$ is a DG module over $\mathrm{Alg}_{\mathcal{A}}$ (to define multiplication write elements of $\mathrm{Alg}_{\mathcal{A}}$ as matrices and elements of M_Φ as columns). Thus, we get a DG equivalence between the DG category of DG $\mathcal{A}$-modules and that of quasi-unital DG modules over $\mathrm{Alg}_{\mathcal{A}}$.

C.3. Let $F : \mathcal{A} \to k$-DGmod be a left DG $\mathcal{A}$-module and $G : \mathcal{A} \to k$-DGmod a right DG $\mathcal{A}$-module. A *DG pairing* $G \times F \to C$, $C \in k$-DGmod, is a DG morphism from the DG bifunctor $(X, Y) \mapsto \mathrm{Hom}(X, Y)$ to the DG bifunctor $(X, Y) \mapsto \mathrm{Hom}(G(Y) \otimes F(X), C)$. It can be equivalently defined as a DG morphism $F \to \mathrm{Hom}(G, C)$ or as a DG morphism $G \to \mathrm{Hom}(F, C)$, where $\mathrm{Hom}(G, C)$ is the DG functor $X \mapsto \mathrm{Hom}(G(X), C)$, $X \in \mathcal{A}$. There is a universal DG pairing $G \times F \to C_0$. We say that C_0 is the *tensor product* of G and F, and we write $C_0 = G \otimes_{\mathcal{A}} F$. Explicitly, $G \otimes_{\mathcal{A}} F$ is the quotient of $\bigoplus_{X\in\mathcal{A}} G(X) \otimes F(X)$ by the following relations: for every morphism $f : X \to Y$ in $\mathcal{A}$ and every $u \in G(Y)$, $v \in F(X)$ one should identify $f^*(u) \otimes v$ and $u \otimes f_*(v)$. In terms of [39, §IX.6], $G \otimes_{\mathcal{A}} F = \int^X G(X) \otimes F(X)$, i.e., $G \otimes_{\mathcal{A}} F$ is the coend of the functor $\mathcal{A}^\circ \times \mathcal{A} \to k$-DGmod defined by $(Y, X) \mapsto G(Y) \otimes F(X)$. In terms of C.2, a DG pairing $G \times F \to C$ is the same as a DG pairing $M_G \times M_F \to C$, so $G \otimes_{\mathcal{A}} F = M_G \otimes_{\mathrm{Alg}_{\mathcal{A}}} M_F$.

C.4. Example. For every $Y \in \mathcal{A}$ one has the right DG $\mathcal{A}$-module h_Y and the left DG $\mathcal{A}$-module $\tilde{h}_Y$ defined by $h_Y(Z) := \mathrm{Hom}(Z, Y)$, $\tilde{h}_Y(Z) := \mathrm{Hom}(Y, Z)$, $Z \in \mathcal{A}$. One has the canonical isomorphisms

$$G \otimes_{\mathcal{A}} \tilde{h}_Y = G(Y), \tag{C.1}$$

$$h_Y \otimes_{\mathcal{A}} F = F(Y) \tag{C.2}$$

induced by the maps $G(Z) \otimes \mathrm{Hom}(Y, Z) \to G(Y)$, $\mathrm{Hom}(Z, Y) \otimes F(Z) \to F(Y)$, $Z \in \mathcal{A}$.

C.5. Given DG categories $\mathcal{A}, \mathcal{B}, \overline{\mathcal{B}}$, a DG $\mathcal{A} \otimes \mathcal{B}$-module F, and a DG $(\mathcal{A}^\circ \otimes \overline{\mathcal{B}})$-module G, one defines the DG $\overline{\mathcal{B}} \otimes \mathcal{B}$-module $G \otimes_{\mathcal{A}} F$ as follows. We consider F as a DG functor from $\mathcal{B}$ to the DG category of DG $\mathcal{A}$-modules, so $F(X)$ is a DG $\mathcal{A}$-module for every $X \in \mathcal{B}$. Quite similarly, $G(Y)$ is a DG $(\mathcal{A})^\circ$-module for every $Y \in \overline{\mathcal{B}}$. Now $G \otimes_{\mathcal{A}} F$ is the DG functor $Y \otimes X \mapsto G(Y) \otimes_{\mathcal{A}} F(X)$, $X \in \mathcal{B}$, $Y \in \overline{\mathcal{B}}$.

C.6. Denote by $Hom_{\mathcal{A}}$ the DG $\mathcal{A} \otimes \mathcal{A}^\circ$-module $(X, Y) \mapsto \mathrm{Hom}(Y, X)$, $X, Y \in \mathcal{A}$. E.g., if $\mathcal{A}$ has a single object and R is its DG algebra of endomorphisms then $Hom_{\mathcal{A}}$ is the DG R-bimodule R. For any DG category $\mathcal{A}$ the isomorphisms (C.1) and (C.2) induce canonical isomorphisms

$$Hom_{\mathcal{A}} \otimes_{\mathcal{A}} F = F, \qquad G \otimes_{\mathcal{A}} Hom_{\mathcal{A}} = G \tag{C.3}$$

for every left DG $\mathcal{A}$-module F and right DG $\mathcal{A}$-module G (the meaning of $Hom_{\mathcal{A}} \otimes_{\mathcal{A}} F$ and $G \otimes_{\mathcal{A}} Hom_{\mathcal{A}}$ was explained in C.5). The isomorphisms (C.3) are clear from the point of view of C.2 because $M_{Hom_{\mathcal{A}}}$ is $\mathrm{Alg}_{\mathcal{A}}$ considered as a DG bimodule over itself.

C.7. A left or right DG $\mathcal{A}$-module $F : \mathcal{A} \to k$-DGmod is said to be *acyclic* if the complex $F(X)$ is acyclic for every $X \in \mathcal{A}$. A left DG $\mathcal{A}$-module F is said to be *homotopically flat* if $G \otimes_{\mathcal{A}} F$ is acyclic for every acyclic right DG $\mathcal{A}$-module G. A right DG $\mathcal{A}$-module is said to be homotopically flat if it is homotopically flat as a left DG $\mathcal{A}^\circ$-module. It follows from (C.1) and (C.2) that h_Y and $\tilde{h}_Y$ are homotopically flat.

C.8. Let $\mathcal{A}$ be a DG category. A DG $\mathcal{A}$-module is said to be *free* if it is isomorphic to a direct sum of complexes of the form $\tilde{h}_X[n]$, $X \in \mathcal{A}$, $n \in \mathbb{Z}$. The notion of semi-free DG $\mathcal{A}$-module is quite similar to that of semi-free module over a DG algebra (see Definition B.1): an $\mathcal{A}$-module Φ is said to be *semi-free* if it can be represented as the union of an increasing sequence of DG submodules Φ_i, $i = 0, 1, \ldots$, so that $\Phi_0 = 0$ and each quotient Φ_i/Φ_{i-1} is free. Clearly, a semi-free DG $\mathcal{A}$-module is homotopically flat. For every DG $\mathcal{A}$-module Φ_i there is a quasi-isomorphism $F \to \Phi$ such that F is a semi-free DG $\mathcal{A}$-module; this is proved just as in the case that $\mathcal{A}$ has a single object (see Lemma B.3). Just as in Remarks B.2, one shows that a semi-free DG $\mathcal{A}$-module is homotopically projective (i.e., the complex $\mathrm{Hom}(F, N)$ is acyclic for every acyclic DG $\mathcal{A}$-module N) and that the functor from the homotopy category of semi-free DG $\mathcal{A}$-modules to the derived category $D(\mathcal{A}^\circ)$ of $\mathcal{A}$-modules is an equivalence.

C.9. Let $F : \mathcal{A} \to \mathcal{A}'$ be a DG functor between DG categories. Then we have the *restriction* DG functor $\mathrm{Res}_F : \mathcal{A}'$-DGmod $\to \mathcal{A}$-DGmod, which maps a DG $\mathcal{A}$-module $\Psi : \mathcal{A}' \to k$-DGmod to $\Psi \circ F$. Sometimes instead of $\mathrm{Res}_F \Psi$ we write Ψ or "Ψ considered as a DG $\mathcal{A}$-module."

We define the *induction* functor $\mathrm{Ind}_F : \mathcal{A}$-DGmod $\to \mathcal{A}'$-DGmod by

$$\mathrm{Ind}_F \Phi(Y) = (\mathrm{Res}_F h_Y) \otimes_{\mathcal{A}} \Phi, \quad Y \in \mathcal{A}'. \tag{C.4}$$

or equivalently by

$$\mathrm{Ind}_F \Phi := Hom_{\mathcal{A}'} \otimes_{\mathcal{A}} \Phi \tag{C.5}$$

(according to C.6, $Hom_{\mathcal{A}'}$ is a DG $\mathcal{A}' \otimes (\mathcal{A}')^\circ$-module, but in (C.5) we consider it as a DG $\mathcal{A}' \otimes \mathcal{A}^\circ$-module). Usually we write $\mathcal{A}' \otimes_{\mathcal{A}} \Phi$ instead of $Hom_{\mathcal{A}'} \otimes_{\mathcal{A}} \Phi = \mathrm{Ind}_F \Phi$.

The DG functor Ind_F is left adjoint to Res_F. Indeed, for every DG $\mathcal{A}'$-module Ψ the complex $\mathrm{Hom}_{\mathcal{A}'\text{-DGmod}}(Hom_{\mathcal{A}'} \otimes_{\mathcal{A}} \Phi, \Psi)$ is canonically isomorphic to $\mathrm{Hom}_{\mathcal{A}\text{-DGmod}}(\Phi, \mathrm{Hom}_{\mathcal{A}'\text{-DGmod}}(Hom_{\mathcal{A}'}, \Psi))$, and the DG $\mathcal{A}'$-module $\mathrm{Hom}_{\mathcal{A}'\text{-DGmod}}(Hom_{\mathcal{A}'}, \Psi))$ is canonically isomorphic to Ψ.

In terms of C.2, the DG functors Res_F and Ind_F correspond to the usual restriction and induction for the DG algebra morphism $\mathrm{Alg}_{\mathcal{A}} \to \mathrm{Alg}_{\mathcal{A}'}$ corresponding to Φ.

Similar definitions and conventions apply to right DG modules (in this case we have $\mathrm{Ind}_F \Phi(Y) = \Phi \otimes_{\mathcal{A}} (\mathrm{Res}_F \tilde{h}_Y)$, $\Phi \otimes_{\mathcal{A}} \mathcal{A}' := \Phi \otimes_{\mathcal{A}} Hom_{\mathcal{A}'} = \mathrm{Ind}_F \Phi$).

C.10. Example. There is a canonical isomorphism

$$\mathrm{Ind}_F \tilde{h}_X = \tilde{h}_{F(X)}, \quad X \in \mathcal{A}, \tag{C.6}$$

where $\tilde{h}_X(Y) := \mathrm{Hom}_{\mathcal{A}}(X, Y)$, $Y \in \mathcal{A}$. This follows either from (C.4) and (C.1) or equivalently from (C.5) and (C.3) (or from the fact that Ind_F is the DG functor left adjoint to Res_F). Quite similarly, there is a canonical isomorphism $\mathrm{Ind}_F h_X = h_{F(X)}$, which means that the following diagram is commutative up to isomorphism:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}^\circ\text{-DGmod} \\ \downarrow & & \downarrow \\ \mathcal{A}' & \longrightarrow & (\mathcal{A}')^\circ\text{-DGmod}. \end{array} \tag{C.7}$$

The horizontal arrows of (C.7) are the Yoneda embeddings defined by $X \mapsto h_X$, the left vertical arrow is F, and the right one is the induction functor.

C.11. Example. Let $\mathcal{A}$ be a DG category and $F : \mathcal{A} \to \mathcal{A}^{\text{pre-tr}}$ the embedding. Then $\mathrm{Res}_F : \mathcal{A}^{\text{pre-tr}}\text{-DGmod} \to \mathcal{A}\text{-DGmod}$ is a DG equivalence. So $\mathrm{Ind}_F : \mathcal{A}\text{-DGmod} \to \mathcal{A}^{\text{pre-tr}}\text{-DGmod}$ is a quasi-inverse DG equivalence.

C.12. Derived induction

As explained (e.g., [6, §10]), in the situation of C.9 the functor $\mathrm{Ind}_F : \mathrm{Ho}(\mathcal{A}^\circ\text{-DGmod}) \to \mathrm{Ho}((\mathcal{A}')^\circ\text{-DGmod})$ has a left derived functor $L\,\mathrm{Ind}_F : D(\mathcal{A}) \to D(\mathcal{A}')$, which is called *derived induction*. Derived induction is left adjoint to the obvious restriction functor $D(\mathcal{A}') \to D(\mathcal{A})$.

By C.8 one can identify $D(\mathcal{A})$ with $\mathrm{Ho}(\underset{\to}{\mathcal{A}})$, where $\underset{\to}{\mathcal{A}}$ is the DG category of semi-free DG $\mathcal{A}^\circ$-modules. Derived induction viewed as a functor $\mathrm{Ho}(\underset{\to}{\mathcal{A}}) \to \mathrm{Ho}(\underset{\to}{\mathcal{A}'})$ is the obvious induction functor. Restriction viewed as a functor $\mathrm{Ho}(\underset{\to}{\mathcal{A}'}) \to \mathrm{Ho}(\underset{\to}{\mathcal{A}})$ sends a semi-free DG $(\mathcal{A}')^\circ$-module to a semi-free resolution of its restriction to $\mathcal{A}^\circ$.

C.13. Given DG algebras A, C, A', and DG morphisms $C \leftarrow A \to A'$, one has the DG $C \otimes (A')^\circ$-module $C \otimes_A A'$. Quite similarly, given DG categories $\mathcal{A}, \mathcal{C}, \mathcal{A}'$, and DG functors $F : \mathcal{A} \to \mathcal{A}'$, $G : \mathcal{A} \to \mathcal{C}$, one defines the DG $\mathcal{C} \otimes (\mathcal{A}')^\circ$-module $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}'$ by

$$\begin{aligned} \mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}' &:= \mathit{Hom}_{\mathcal{C}} \otimes_{\mathcal{A}} \mathit{Hom}_{\mathcal{A}'} = \mathcal{C} \otimes_{\mathcal{A}} \mathit{Hom}_{\mathcal{A}'} = \mathit{Hom}_{\mathcal{C}} \otimes_{\mathcal{A}} \mathcal{A}' \\ &= \mathcal{C} \otimes_{\mathcal{A}} \mathit{Hom}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}', \end{aligned} \tag{C.8}$$

where $\mathit{Hom}_{\mathcal{C}}$ is considered as a $\mathcal{C} \otimes \mathcal{A}^\circ$-module and $\mathit{Hom}_{\mathcal{A}'}$ as an $\mathcal{A} \otimes (\mathcal{A}')^\circ$-module. In other words, $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}'$ is the DG functor $\mathcal{C} \times (\mathcal{A}')^\circ \to k$-DGmod defined by

$$(X, Y) \mapsto \int^Z \mathrm{Hom}\big(F(Z), Y\big) \otimes \mathrm{Hom}\big(X, G(Z)\big), \quad X \in \mathrm{Ob}\,\mathcal{C},\ Y \in \mathrm{Ob}\,\mathcal{A}',$$

where the $\int$ symbol denotes the coend (see C.3), so the above "integral" is the tensor product of the right $\mathcal{A}$-module $Z \mapsto \mathrm{Hom}(F(Z), Y)$ and the left $\mathcal{A}$-module $Z \mapsto \mathrm{Hom}(X, G(Z))$. In terms of C.2, the DG module over $\mathrm{Alg}_{\mathcal{C}} \otimes (\mathrm{Alg}_{\mathcal{A}'})^{\circ}$ corresponding to $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{A}'$ equals $\mathrm{Alg}_{\mathcal{C}} \otimes_{\mathrm{Alg}_{\mathcal{A}}} \mathrm{Alg}_{\mathcal{A}'}$.

C.14. Given a DG functor $F : \mathcal{A} \to \mathcal{A}'$ we say that $\mathcal{A}'$ is *right F-flat* (or *right homotopically flat over $\mathcal{A}$*) if the right $\mathcal{A}$-module $\mathrm{Res}_F h_X$ is homotopically flat for all $X \in \mathcal{A}'$; here $h_X(Y) := \mathrm{Hom}(Y, X)$, $X, Y \in \mathcal{A}'$. We say that $\mathcal{A}'$ is *right module-semifree over $\mathcal{A}$* if the right DG $\mathcal{A}$-modules $\mathrm{Res}_F h_X$, $X \in \mathcal{A}'$, are semi-free. $\mathcal{A}'$ is said to be *left F-flat* (or *left homotopically flat over $\mathcal{A}$*) if the left $\mathcal{A}$-module $\mathrm{Res}_F \tilde{h}_X$ is homotopically flat for all $X \in \mathcal{A}'$; here $\tilde{h}_X(Y) := \mathrm{Hom}(X, Y)$, $X, Y \in \mathcal{A}'$. If $\mathcal{A}'$ is right homotopically flat over $\mathcal{A}$ then the induction functor Ind_F maps acyclic left DG $\mathcal{A}$-modules to acyclic left DG $\mathcal{A}'$-modules. The previous sentence remains true if "left" and "right" are interchanged.

C.15. Lemma. *Let $\mathcal{A}$ be a DG category and $\mathcal{B} \subset \mathcal{A}$ a full DG subcategory.*

(i) *If* (3.4) *holds then $\mathcal{A}/\mathcal{B}$ is right homotopically flat over $\mathcal{A}$.*
(ii) *If* (7.4) *holds then $\mathcal{A}/\mathcal{B}$ is right module-semifree over $\mathcal{A}$.*

Proof. We will only prove (i) (the proof of (ii) is similar). We have to show that for every $Y \in \mathcal{A}$ the functor $\Psi_Y : \mathcal{A}^{\circ} \to k$-DGmod defined by $\Psi_Y(X) = \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ is a homotopically flat right $\mathcal{A}$-module. By (3.2), there is a filtration $\Psi_Y = \bigcup_n \Psi_Y^n$, $\Psi_Y^n \subset \Psi_Y^{n+1}$, such that $\Psi_Y^0 = h_Y$ and $\Psi_Y^n / \Psi_Y^{n-1} = \bigoplus_{U \in \mathcal{B}} C_U^n \otimes h_U$ for every $n > 0$, where C_U^n is the direct sum of complexes

$$\mathrm{Hom}_{\mathcal{A}}(U_1, U_2) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(U_{n-1}, U_n) \otimes \mathrm{Hom}_{\mathcal{A}}(U_n, Y), \quad U_i \in \mathcal{B},\ U_1 = U.$$

It remains to notice that for every $Y \in \mathcal{A}$ the right $\mathcal{A}$-module h_Y is homotopically flat (see C.7) and by (3.4) the complexes C_U^n are homotopically flat. □

C.16. Quasi-representability

Let $\mathcal{A}$ be a DG category. We have the DG functor from $\mathcal{A}$ to the DG category of DG $\mathcal{A}^{\circ}$-modules defined by $X \mapsto h_X$.

C.16.1. Definition. A DG $\mathcal{A}^{\circ}$-module Φ is *quasi-representable* if there is a quasi-isomorphism $f : h_X \to \Phi$ for some $X \in \mathcal{A}$.

Remark. By C.8, for every DG $\mathcal{A}^{\circ}$-module Φ there exists a semi-free resolution $\pi : \overline{\Phi} \to \Phi$ (i.e., $\overline{\Phi}$ is semi-free and π is a quasi-isomorphism), and the homotopy class of $\overline{\Phi}$ does not depend on the choice of $(\overline{\Phi}, \pi)$. So Φ is quasi-representable if and only if this class contains h_X for some $X \in \mathcal{A}$.

C.16.2. Lemma. *Φ is quasi-representable if and only if the graded functor $H^{\cdot}\Phi$: $(\mathrm{Ho}^{\cdot}(\mathcal{A}))^{\circ} \to \{$graded k-modules$\}$ is representable.*

Proof. We only have to prove the "if" statement. Suppose $H^{\cdot}\Phi$ is represented by (X, u), $X \in \mathrm{Ob}\,\mathcal{A}$, $u \in H^0\Phi(X)$. Our u is the cohomology class of some $\tilde{u} \in \Phi(X)$ such that $\mathrm{d}\tilde{u} = 0$, $\deg \tilde{u} = 0$. Then $\tilde{u}$ defines a closed morphism $f : h_X \to \Phi$ of degree 0 such that for every $Y \in \mathcal{A}$ the morphism $H^{\cdot}h_X(Y) \to H^{\cdot}\Phi(Y)$ is an isomorphism, so f is a quasi-isomorphism. □

C.16.3. Let $\mathcal{A}' \subset \mathcal{A}^\circ$-DGmod be the full DG subcategory of quasi-representable DG modules. We have the DG functors $\mathcal{A} \leftarrow \mathcal{A}'' \xrightarrow{\pi} \mathcal{A}'$, where $\mathcal{A}''$ is the DG category whose objects are triples consisting of an object $Y \in \mathcal{A}$, a DG $\mathcal{A}^\circ$-module Ψ, and a quasi-isomorphism $h_Y \to \Psi$ (more precisely, $\mathcal{A}''$ is the full DG subcategory of the DG category $\mathcal{A}^\circ$-resDGmod from 6.2.1 which is formed by these triples). Clearly, π is a surjective quasi-equivalence.

C.16.4. Quasi-corepresentability

We say that a DG $\mathcal{A}$-module Φ is *quasi-corepresentable* if there is a quasi-isomorphism $f : \tilde{h}_X \to \Phi$ for some $X \in \mathcal{A}$, i.e., if Φ is representable as a DG $(\mathcal{A}^\circ)^\circ$-module.

Appendix D. The diagonal DG categories

D.1. Given topological spaces M', M'' mapped to a space M, one has the "homotopy fiber product" $(M' \times M'') \times_{M\times M} \Delta^h_M$, where Δ^h_M is the "homotopy diagonal," i.e., the space of paths $[0, 1] \to M$ ($\gamma \in \Delta^h_M$ is mapped to $(\gamma(0), \gamma(1)) \in M \times M$). In the same spirit, given a DG category $\mathcal{C}$ it is sometimes useful to replace the naive diagonal $\Delta_\mathcal{C} \subset \mathcal{C} \times \mathcal{C}$ by one of the following DG categories $\overrightarrow{\Delta}_\mathcal{C}$, $\overleftarrow{\Delta}_\mathcal{C}$, $\overleftrightarrow{\Delta}_\mathcal{C}$, each of them equipped with a DG functor to $\mathcal{C} \times \mathcal{C}$. We define $\overrightarrow{\Delta}_\mathcal{C}$ to be the full DG subcategory of the DG category $\mathcal{M}or\mathcal{C}$ from 2.9 that consists of triples (X, Y, f) such that f is a homotopy equivalence; the DG functor $\overrightarrow{\Delta}_\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is defined by $(X, Y, f) \mapsto (X, Y)$. We define $\overleftarrow{\Delta}_\mathcal{C}$ to be the same full DG subcategory of $\mathcal{M}or\mathcal{C}$, but the DG functor $\overleftarrow{\Delta}_\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is defined by $(X, Y, f) \mapsto (Y, X)$.

Finally, define $\overleftrightarrow{\Delta}_\mathcal{C}$ to be the DG category A_∞-funct$(\mathbf{I}_2, \mathcal{C})$ of *A_∞-functors* $\mathbf{I}_2 \to \mathcal{C}$, where $\mathbf{I}_n$ denotes the k-category freely generated by the category $\mathbf{J}_n$ with objects $1, \ldots, n$ and precisely one morphism with any given source and target. Here the word "A_∞-functor" is understood in the "strictly unital" sense (cf. [24, §3.5] or [36, §3.1]; according to [31, 33,36,37], there are several versions of the notion of A_∞-functor which differ on how an A_∞ analog of the axiom $F(\mathrm{id}) = \mathrm{id}$ in the definition of usual functor is formulated; the difference is inessential for our purposes and for any reasonable purpose). So an A_∞-functor $\mathbf{I}_2 \to \mathcal{C}$ is a DG functor $\mathcal{D}_2 \to \mathcal{C}$, where $\mathcal{D}_2$ is a certain DG category with $\mathrm{Ob}\,\mathcal{D}_2 = \{1, 2\}$, which is freely generated (as a graded k-category, i.e., after one forgets the differential) by morphisms $f_{12} : 1 \to 2$ and $f_{21} : 2 \to 1$ of degree 0, morphisms $f_{121} : 1 \to 1$ and $f_{212} : 2 \to 2$ of degree -1, morphisms $f_{1212} : 1 \to 2$ and $f_{2121} : 2 \to 1$ of degree -2, etc. One has $\mathrm{d}f_{12} = 0 = \mathrm{d}f_{21}$, $\mathrm{d}f_{121} = f_{21}f_{12} - 1$, $\mathrm{d}f_{212} = f_{12}f_{21} - 1$, and we do not need explicit formulas for the differential of f_{1212}, f_{2121}, etc.

D.2. Let e_{ij} be the unique $\mathbf{J}_2$-morphism $i \to j$, $i, j \in \{1, 2\}$. Let $\mathbf{I}'_2 \subset \mathbf{I}_2$ denote the k-subcategory generated by e_{12}. Then A_∞-funct$(\mathbf{I}'_2, \mathcal{C})$ identifies with $\mathcal{M}or\mathcal{C}$, so we get a canonical DG functor $\overleftrightarrow{\Delta}_{\mathcal{C}} \to \overrightarrow{\Delta}_{\mathcal{C}} \subset \mathcal{M}or\mathcal{C}$. There is a similar DG functor $\overleftrightarrow{\Delta}_{\mathcal{C}} \to \overleftarrow{\Delta}_{\mathcal{C}}$.

D.3. Lemma. *For every DG category $\mathcal{K}$ equipped with a DG functor $\mathcal{K} \to \overrightarrow{\Delta}_{\mathcal{C}}$ the DG functor $\mathcal{K} \times_{\overrightarrow{\Delta}_{\mathcal{C}}} \overleftrightarrow{\Delta}_{\mathcal{C}} \to \mathcal{K}$ is a quasi-equivalence. Same is true if $(\overleftrightarrow{\Delta}_{\mathcal{C}}, \overrightarrow{\Delta}_{\mathcal{C}})$ is replaced by $(\overrightarrow{\Delta}_{\mathcal{C}}, \mathcal{C})$, $(\overleftrightarrow{\Delta}_{\mathcal{C}}, \overleftarrow{\Delta}_{\mathcal{C}})$ $(\overleftrightarrow{\Delta}_{\mathcal{C}}, \overrightarrow{\Delta}_{\mathcal{C}})$, or $(\overleftarrow{\Delta}_{\mathcal{C}}, \mathcal{C})$.*

In other words, the lemma says that the DG functors $\overleftrightarrow{\Delta}_{\mathcal{C}} \to \overrightarrow{\Delta}_{\mathcal{C}} \to \mathcal{C}$ are quasi-equivalences and this remains true after any "base change" in the sense of 2.8.

Proof. The DG functors $\overleftrightarrow{\Delta}_{\mathcal{C}} \to \overrightarrow{\Delta}_{\mathcal{C}} \to \mathcal{C}$ induce surjections of Hom complexes (this follows from the definition of these complexes, see [31,33,36,37]). So it suffices to show that they are quasi-equivalences and induce surjections $\mathrm{Ob}\,\overleftrightarrow{\Delta}_{\mathcal{C}} \to \mathrm{Ob}\,\overrightarrow{\Delta}_{\mathcal{C}} \to \mathrm{Ob}\,\mathcal{C}$. Both statements are clear for $\overrightarrow{\Delta}_{\mathcal{C}} \to \mathrm{Ob}\,\mathcal{C}$. The DG functor $F : \overleftrightarrow{\Delta}_{\mathcal{C}} \to \mathcal{C}$ is the DG functor

$$A_\infty\text{-funct}(\mathbf{I}_2, \mathcal{C}) \to A_\infty\text{-funct}(\mathbf{I}_1, \mathcal{C})$$

that comes from a functor $i : \mathbf{I}_1 \to \mathbf{I}_2$ induced by an embedding $\mathbf{I}_1 \hookrightarrow \mathbf{I}_2$. F is a quasi-equivalence because i is an equivalence (more generally, if all the Hom complexes of DG categories $\mathcal{A}_1, \mathcal{A}_2$ are semi-free DG k-modules then a quasi-equivalence $\mathcal{A}_1 \xrightarrow{\approx} \mathcal{A}_2$ induces a quasi-equivalence $A_\infty\text{-funct}(\mathcal{A}_2, \mathcal{C}) \xrightarrow{\approx} A_\infty\text{-funct}(\mathcal{A}_1, \mathcal{C})$: this follows from E.7.4 because the functor $T(\mathcal{A}_2, \mathcal{C}) \to T(\mathcal{A}_1, \mathcal{C})$ is an equivalence).

Finally, let us prove the surjectivity of the map $\mathrm{Ob}\,\overleftrightarrow{\Delta}_{\mathcal{C}} \to \mathrm{Ob}\,\overrightarrow{\Delta}_{\mathcal{C}}$ essentially following [31] (where a slightly weaker statement is formulated). We will prove a formally more general statement. Let e_{ij} and $\mathbf{I}'_2 \subset \mathbf{I}_2$ have the same meaning as in D.2. Suppose that the embedding $\mathbf{I}'_2 \hookrightarrow \mathbf{I}_2$ (considered as a DG functor between DG categories) is decomposed as $\mathbf{I}'_2 \hookrightarrow \mathcal{R} \to \mathbf{I}_2$, where $\mathrm{Ob}\,\mathcal{R} = \mathrm{Ob}\,\mathbf{I}_2 = \mathbf{I}'_2 = \{1, 2\}$ and $\mathcal{R}$ is semi-free over $\mathbf{I}'_2$ (see B.4). Let $F : \mathbf{I}'_2 \to \mathcal{C}$ be a DG functor such that $F(e_{12})$ is a homotopy equivalence. Then we will show that F extends to a DG functor $G : \mathcal{R} \to \mathcal{C}$ (to prove the surjectivity of the map $\mathrm{Ob}\,\overleftrightarrow{\Delta}_{\mathcal{C}} \to \mathrm{Ob}\,\overrightarrow{\Delta}_{\mathcal{C}}$ put $\mathcal{R} = \mathcal{D}_2$). We will do this by decomposing F as

$$\mathbf{I}'_2 \xrightarrow{\Phi} \mathcal{R}' \to \mathcal{C}, \qquad \mathrm{Ho}^\cdot(\mathcal{R}') = \mathbf{I}_2 \tag{D.1}$$

(here the equality $\mathrm{Ho}^\cdot(\mathcal{R}') = \mathbf{I}_2$ means that the functor $\mathbf{I}'_2 = \mathrm{Ho}^\cdot(\mathbf{I}'_2) \to \mathcal{R}'$ extends to an isomorphism $\mathbf{I}_2 \xrightarrow{\sim} \mathcal{R}'$). Such a decomposition allows to extend F to a DG functor $G : \mathcal{R} \to \mathcal{C}$: first reduce to the case that all Ext^n groups in $\mathcal{R}'$ vanish for $n > 0$ (otherwise replace $\mathcal{R}'$ by a suitable DG subcategory), then one has a commutative diagram

$$\begin{array}{ccc} \mathbf{I}'_2 & \xrightarrow{\Phi} & \mathcal{R}' \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{R} & \longrightarrow & \mathbf{I}_2 \end{array}$$

with π being a surjective quasi-equivalence, and it remains to decompose Φ as $\mathbf{I}'_2 \xrightarrow{\nu} \mathcal{R} \to \mathcal{R}'$ by applying Lemma B.6.

Here are two ways to construct a decomposition (D.1). The first way is, essentially, to construct an $\mathcal{R}'$ independent on $\mathcal{C}$ and $F:\mathbf{I}\to\mathcal{C}$ by slightly modifying $\mathbf{I}'_2$. The second construction seems simpler to me, but it gives an $\mathcal{R}'$ which depends on $\mathcal{C}$ and $F:\mathbf{I}\to\mathcal{C}$.

(i) Our $\mathbf{I}'_2$ equals the DG category $\mathcal{A}_0$ from 3.7.1. Let $\mathcal{R}'$ be the DG category $(\mathcal{A}/\mathcal{B})_0 \subset \mathcal{A}/\mathcal{B}$ from 3.7.1. One gets a DG functor $\mathcal{R}' := (\mathcal{A}/\mathcal{B})_0 \to \mathcal{C}$ and, in fact, a DG functor $\mathcal{A}/\mathcal{B} \to \mathcal{C}^{\text{pre-tr}}$ as follows. First extend $F:\mathcal{A}_0 := \mathbf{I}' \to \mathcal{C}$ to a DG functor $F^{\text{pre-tr}}:\mathcal{A} := (\mathbf{I}'_2)^{\text{pre-tr}} \to \mathcal{C}$. Then $F^{\text{pre-tr}}$ sends the unique object of $\mathcal{B}$ to a contractible object $Y \in \mathcal{C}^{\text{pre-tr}}$. A choice of a homotopy between id_Y and 0 defines a DG functor $\mathcal{A}/\mathcal{B} \to \mathcal{C}^{\text{pre-tr}}$. By Lemma 3.7.2, $\mathrm{Ho}^{\cdot}(\mathcal{R}') = \mathbf{I}'_2$.

(ii) Notation: given a DG category $\mathcal{A}$ and $a \in \mathrm{Ob}\,\mathcal{A}$ one defines $\mathcal{A}/a$ to be the fiber product in the Cartesian square

$$\begin{array}{ccc} \mathcal{A}/a & \longrightarrow & \mathcal{M}or\,\mathcal{A} \\ \downarrow & & \downarrow{\scriptstyle t} \\ * & \xrightarrow{i_a} & \mathcal{A} \end{array} ,$$

where $\mathcal{M}or\,\mathcal{A}$ is the DG category from 2.9, t sends an $\mathcal{A}$-morphism to its target, $*$ is the DG category with one object whose endomorphism algebra equals k, and $i_a:* \to \mathcal{A}$ maps the object of $*$ to a. Decompose $F:\mathbf{I}'_2 \to \mathcal{C}$ as $F = s\overline{F}$, where $s:\mathcal{C}/F(2) \to \mathcal{C}$ sends a $\mathcal{C}$-morphism to its source and $\overline{F}:\mathbf{I}'_2 \to \mathcal{C}/F(2)$ is the composition of the DG functor $\mathbf{I}'_2 \to \mathbf{I}_2/2$ that sends $i \in \{1,2\}$ to the unique $\mathbf{J}'_2$-morphism $e_{i2}:i \to 2$ and the DG functor $\mathbf{I}_2/2 \to \mathcal{C}/F(2)$ corresponding to $F:\mathbf{I}_2 \to \mathcal{C}$ (here $\mathbf{I}_2$ is considered as a DG category). Now define $\mathcal{R}'$ from (D.1) as follows: $\mathrm{Ob}\,\mathcal{R}' := \mathrm{Ob}\,\mathbf{I}'_2 = \{1,2\}$, $\mathrm{Hom}(j_1, j_2) = \mathrm{Hom}(\overline{F}(j_1), \overline{F}(j_2))$ for $j_1 = j_2 \in \mathrm{Ob}\,\mathcal{R}' := \mathrm{Ob}\,\mathbf{I}'_2$, and composition in $\mathcal{R}'$ comes from composition in $\mathcal{C}/F(2)$. We have a canonical decomposition of $\overline{F}$ as $\mathbf{I}'_2 \to \mathcal{R}' \to \mathcal{C}/F(2)$, and to get (D.1) one uses $s:\mathcal{C}/F(2) \to \mathcal{C}$. To show that $\mathrm{Ho}^{\cdot}(\mathcal{R}') = \mathbf{I}_2$ use that $F(e_{i2})$ is a homotopy equivalence. □

Appendix E. The 2-category of DG categories

In E.1–E.4 we recall the definition of the 2-category of DG categories used by Keller in [22], and in E.7.1–E.7.4 we mention a different approach used by Kontsevich. We prefer to work with the weak notion of 2-category due to Bénabou. The definition and basic examples of 2-categories can be found in [3] or [39, Chapter XII], where they are called "bicategories." Let us just recall that we have to associate to each two DG categories $\mathcal{A}_1, \mathcal{A}_2$ a category $T(\mathcal{A}_1, \mathcal{A}_2)$ and to define the composition functors $T(\mathcal{A}_1, \mathcal{A}_2) \times T(\mathcal{A}_2, \mathcal{A}_3) \to T(\mathcal{A}_1, \mathcal{A}_3)$. The 2-category axioms say that composition should be weakly associative and for every DG category $\mathcal{A}$ there is a weak unit object

in $T(\mathcal{A},\mathcal{A})$. The meaning of "weak" is clear from the following example: a 2-category with one object is the same as a monoidal category.

The 2-category of DG categories is only the tip of the "iceberg" of DG categories. In E.8 we make some obvious remarks regarding the whole iceberg, but its detailed description is left to the experts (see 1.8).

E.1. Flat case

First, let us construct the 2-category *FlatDGcat* of flat DG categories ("flat" is a shorthand for "homotopically flat over k," see 3.3). Define $T(\mathcal{A}_1,\mathcal{A}_2)\subset D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ to be the full subcategory of quasi-functors in the sense of [22, §7] (see also [26]). According to [22], a *quasi-functor* from $\mathcal{A}_1$ to $\mathcal{A}_2$ is an object $\Phi\in D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ such that for every $X\in\mathcal{A}_1$ the object $\Phi(X)\in D(\mathcal{A}_2)$ belongs to the essential image of the Yoneda embedding $\mathrm{Ho}(\mathcal{A}_2)\to D(\mathcal{A}_2)$ (here $\Phi(X)$ is the restriction of $\Phi:\mathcal{A}_1\otimes\mathcal{A}_2^\circ\to k$-DGmod to $\{X\}\otimes\mathcal{A}_2=\mathcal{A}_2$). In other words, an object of $D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ is a quasi-functor if it comes from a DG functor from $\mathcal{A}_1$ to the full subcategory of quasi-representable DG $\mathcal{A}_2^\circ$-modules ("quasi-representable" means "quasi-isomorphic to a representable DG $\mathcal{A}_2^\circ$-module," see C.16). The composition of $\Phi\in D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ and $\Psi\in D(\mathcal{A}_2^\circ\otimes\mathcal{A}_3)$ is defined to be $\Phi\overset{L}{\otimes}_{\mathcal{A}_2}\Psi$, and the associativity isomorphism is the obvious one.

$D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ is a graded k-category (the morphisms $\Phi_1\to\Phi_2$ of degree n are the elements of $\mathrm{Ext}^n(\Phi_1,\Phi_2)$). This structure induces a structure of graded k-category on $T(\mathcal{A}_1,\mathcal{A}_2)$.

E.2. Remark. If $\mathcal{A}_2$ is pretriangulated in the sense of 2.4, then the subcategory $T(\mathcal{A}_1,\mathcal{A}_2)\subset D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ is triangulated.

E.3. General case

It suffices to define for every DG category $\mathcal{A}$ a 2-functor $\beth: S_{\mathcal{A}}\to$ *FlatDGcat*, where *FlatDGcat* is the 2-category of flat DG categories and $S_{\mathcal{A}}$ is a non-empty 2-category such that for every $s_1,s_2\in\mathrm{Ob}\,S_{\mathcal{A}}$ the category of 1-morphisms $s_1\to s_2$ has one object and one morphism ("$\beth$" is the Hebrew letter Dalet). We define $\mathrm{Ob}\,S_{\mathcal{A}}$ to be the class of all flat resolutions of $\mathcal{S}$ (by Lemma B.5, $\mathrm{Ob}\,S_{\mathcal{A}}\neq\emptyset$). $\beth$ sends each $\widetilde{\mathcal{A}}\in\mathrm{Ob}\,S_{\mathcal{A}}$ to itself considered as an object of *FlatDGcat*. The unique 1-morphism from $\widetilde{\mathcal{A}}_1\in\mathrm{Ob}\,S_{\mathcal{A}}$ to $\widetilde{\mathcal{A}}_2\in\mathrm{Ob}\,S_{\mathcal{A}}$ is mapped by $\beth$ to $Hom_{\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_2}\in T(\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_2)\subset D(\widetilde{\mathcal{A}}_1^\circ\otimes\widetilde{\mathcal{A}}_2)$, where the DG $\widetilde{\mathcal{A}}_1\otimes\widetilde{\mathcal{A}}_2^\circ$-module $Hom_{\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_2}$ is defined by

$$(X_1,X_2)\mapsto\mathrm{Hom}\big(\pi_2(X_2),\pi_1(X_1)\big),\quad X_i\in\widetilde{\mathcal{A}}_i \tag{E.1}$$

and π_i is the DG functor $\widetilde{\mathcal{A}}_i\to\mathcal{A}$. To define $\beth$, one also has to specify a quasi-isomorphism

$$Hom_{\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_2}\overset{L}{\otimes}_{\mathcal{A}_2}Hom_{\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}_3}\to Hom_{\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_3} \tag{E.2}$$

for every three resolutions $\widetilde{\mathcal{A}}_i \to \mathcal{A}$. It comes from the composition morphism

$$Hom_{\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_2} \otimes_{\mathcal{A}_2} Hom_{\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}_3} \to Hom_{\widetilde{\mathcal{A}}_1,\widetilde{\mathcal{A}}_3}.$$

E.4. Each $T(\mathcal{A}_1, \mathcal{A}_2)$ is equipped with a graded k-category structure, and if $\mathcal{A}_2$ is pretriangulated then $T(\mathcal{A}_1, \mathcal{A}_2)$ is equipped with a triangulated structure. We already know this if $\mathcal{A}_1$ and $\mathcal{A}_2$ are flat (see E.1, Remark E.2), and in the general case we get it by transport of structure via the equivalence $T(\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2) \to T(\mathcal{A}_1, \mathcal{A}_2)$ corresponding to flat resolutions $\widetilde{\mathcal{A}}_1 \to \mathcal{A}_1$ and $\widetilde{\mathcal{A}}_1 \to \mathcal{A}_2$.

E.5. Remarks.

(i) $T(\mathcal{A}_1, \mathcal{A}_2)$ is a full subcategory of the following triangulated category $D(\mathcal{A}_1^\circ \overset{L}{\otimes} \mathcal{A}_2)$ equipped with a triangulated functor $R : D(\mathcal{A}_1^\circ \otimes \mathcal{A}_2) \to D(\mathcal{A}_1^\circ \overset{L}{\otimes} \mathcal{A}_2)$, which is an equivalence if $\mathcal{A}_1$ or $\mathcal{A}_2$ is flat. The objects of $D(\mathcal{A}_1^\circ \overset{L}{\otimes} \mathcal{A}_2)$ are triples $(\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, M)$, where $\widetilde{\mathcal{A}}_i$ is a flat resolution of $\mathcal{A}_i$ and $M \in D(\widetilde{\mathcal{A}}_1^\circ \otimes \widetilde{\mathcal{A}}_2)$. Morphisms of degree n from $(\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, M)$ to $(\widetilde{\mathcal{A}}'_1, \widetilde{\mathcal{A}}'_2, M')$ are elements of

$$\mathrm{Ext}^n_{\widetilde{\mathcal{A}}'_1 \otimes (\widetilde{\mathcal{A}}'_2)^\circ}\big(\big(Hom_{\widetilde{\mathcal{A}}'_1,\widetilde{\mathcal{A}}_1} \otimes Hom_{\widetilde{\mathcal{A}}_2,\widetilde{\mathcal{A}}'_2}\big) \otimes_{\widetilde{\mathcal{A}}_1 \otimes \widetilde{\mathcal{A}}_2^\circ} M, M'\big).$$

One defines composition in $D(\mathcal{A}_1^\circ \overset{L}{\otimes} \mathcal{A}_2)$ and $R : D(\mathcal{A}_1^\circ \otimes \mathcal{A}_2) \to D(\mathcal{A}_1^\circ \overset{L}{\otimes} \mathcal{A}_2)$ in the obvious way.

(ii) $D(\mathcal{A}^\circ \overset{L}{\otimes} \mathcal{A})$ equipped with the functor $\overset{L}{\otimes}_{\mathcal{A}}$ is a monoidal category. $Hom_{\mathcal{A}} := Hom_{\mathcal{A},\mathcal{A}}$ viewed as an object of $D(\mathcal{A}^\circ \overset{L}{\otimes} \mathcal{A})$ is a unit object.

E.6. Ind-version and duality

We are going to define an involution $\circ$ of the 2-category *DGcat* which preserves the composition of 1-morphisms, reverses that of 2-morphisms, and sends each $\mathcal{A} \in DGcat$ to $\mathcal{A}^\circ$.

To define it at the level of 1-morphisms and 2-morphisms consider the 2-category $DGcat_{\mathrm{ind}}$ whose objects are DG categories, as before, but the category $\underrightarrow{T}(\mathcal{A}, \mathcal{K})$ of 1-morphisms from a DG category $\mathcal{A}$ to a DG category $\mathcal{K}$ equals $D(\mathcal{A}^\circ \overset{L}{\otimes} \mathcal{K})$ (1-morphisms are composed in the obvious way). Clearly, $DGcat \subset DGcat_{\mathrm{ind}}$. The DG category $DGcat_{\mathrm{ind}}$ has a canonical involution $\bullet$ which reverses the composition of 1-morphisms and preserves that of 2-morphisms: at the level of objects one has $\mathcal{A}^\bullet := \mathcal{A}^\circ$, and to define $\bullet$ at the level of 1-morphisms and 2-morphisms, one uses the obvious equivalence between $\underrightarrow{T}(\mathcal{A}, \mathcal{K})$ and $\underrightarrow{T}(\mathcal{K}^\circ, \mathcal{A}^\circ)$.

Now it is easy to see that each $F \in T(\mathcal{A}, \mathcal{K}) \subset \underrightarrow{T}(\mathcal{A}, \mathcal{K})$ has a right adjoint $F^* \in \underrightarrow{T}(\mathcal{K}, \mathcal{A})$ and $(F^*)^\bullet \in T(\mathcal{A}^\circ, \mathcal{K}^\circ) \subset \underrightarrow{T}(\mathcal{A}^\circ, \mathcal{K}^\circ)$. So putting $F^\circ := (F^*)^\bullet$, one gets the promised involution of *DGcat*.

Remarks.

(i) It is easy to show that if $\mathcal{K} \in DGcat$ is pretriangulated and $\mathrm{Ho}(\mathcal{K})$ is Karoubian then $F \in \underrightarrow{T}(\mathcal{A}, \mathcal{K})$ has a right adjoint if and *only* if $F \in T(\mathcal{A}, \mathcal{K})$.

(ii) At the 2-category level the definitions of the right derived DG functor from 5.2 and 7.3 amount to the following one. Suppose that in the situation of Theorem 1.6.2, we are given $F \in T(\mathcal{A}, \mathcal{A}')$. Then $RF \in \underrightarrow{T}(\mathcal{C}, \mathcal{A}')$ is the composition of $F \in T(\mathcal{A}, \mathcal{A}') \subset \underrightarrow{T}(\mathcal{C}, \mathcal{A}')$ and the right adjoint $\xi^* \in \underrightarrow{T}(\mathcal{C}, \mathcal{A})$ of $\xi \in T(\mathcal{A}, \mathcal{C})$.

E.7. Relation with Kontsevich's approach

E.7.1. Let $\mathcal{A}, \mathcal{K}$ be DG categories and suppose that $\mathcal{A}$ is flat. Given a DG functor $F : \mathcal{A} \to \mathcal{K}$ denote by Φ_F the DG $\mathcal{A} \otimes \mathcal{K}^\circ$-module $(X, Y) \mapsto \mathrm{Hom}(Y, F(X))$. Clearly, $\Phi_F \in D(\mathcal{A}^\circ \otimes \mathcal{K})$ belongs to $T(\mathcal{A}, \mathcal{K})$. Let us describe the full subcategory of $T(\mathcal{A}, \mathcal{K})$ formed by the DG $\mathcal{A} \otimes \mathcal{K}^\circ$-modules Φ_F. One has $\Phi_F = \mathrm{Ind}_{\mathrm{id}_\mathcal{A} \otimes F^\circ}(\mathit{Hom}_\mathcal{A})$, where F° is the DG functor $\mathcal{A}^\circ \to \mathcal{K}^\circ$ corresponding to $F : \mathcal{A} \to \mathcal{K}$ and $\mathit{Hom}_\mathcal{A}$ is the $\mathcal{A}^\circ \otimes \mathcal{A}$-module $(X, Y) \mapsto \mathrm{Hom}(X, Y)$. As $\mathcal{A}$ is homotopically flat over k the morphism $L\,\mathrm{Ind}_{\mathrm{id}_\mathcal{A} \otimes F^\circ}(\mathit{Hom}_\mathcal{A}) \to \mathrm{Ind}_{\mathrm{id}_\mathcal{A} \otimes F^\circ}(\mathit{Hom}_\mathcal{A})$ is a quasi-isomorphism. Therefore, the adjunction between derived induction and restriction yields a canonical isomorphism

$$\mathrm{Ext}^n(\Phi_F, \Phi_G) = \mathrm{Ext}^n\big(L\,\mathrm{Ind}_{\mathrm{id}_\mathcal{A} \otimes F^\circ}(\mathit{Hom}_\mathcal{A}), \Phi_G\big) \xrightarrow{\sim} \mathrm{Ext}^n(F, G), \tag{E.3}$$

where $\mathrm{Ext}^n(F, G) := \mathrm{Ext}^n_{\mathcal{A} \otimes \mathcal{A}^\circ}(\mathit{Hom}_\mathcal{A}, \mathcal{H}om(F, G))$ and $\mathcal{H}om(F, G) := \mathrm{Res}_{\mathrm{id}_\mathcal{A} \otimes F^\circ}(\Phi_G)$, i.e., $\mathcal{H}om(F, G)$ is the DG $\mathcal{A} \otimes \mathcal{A}^\circ$-module $(X, Y) \mapsto \mathrm{Hom}(F(Y), G(X))$, $X, Y \in \mathcal{A}$. The morphism $\mathrm{Ext}^m(F_2, F_3) \otimes \mathrm{Ext}^n(F_1, F_2) \to \mathrm{Ext}^{m+n}(F_1, F_3)$ coming from (E.3) is, in fact, induced by the morphism $\mathcal{H}om(F_2, F_3) \otimes \mathcal{H}om(F_1, F_2) \to \mathcal{H}om(F_1, F_3)$ and the quasi-isomorphism $(\mathit{Hom}_\mathcal{A}) \otimes_\mathcal{A} (\mathit{Hom}_\mathcal{A}) \to \mathit{Hom}_\mathcal{A}$. So we have described the full subcategory of $T(\mathcal{A}, \mathcal{K})$ formed by the DG $\mathcal{A} \otimes \mathcal{K}^\circ$-modules Φ_F. The next statement shows that it essentially equals $T(\mathcal{A}, \mathcal{K})$ if $\mathcal{A}$ is semi-free.

E.7.2. Proposition. *If $\mathcal{A}$ is semi-free over k then every object of $T(\mathcal{A}, \mathcal{K})$ is isomorphic to Φ_F for some $F : \mathcal{A} \to \mathcal{K}$.*

Proof. An object $\Phi \in T(\mathcal{A}, \mathcal{K})$ is a DG $\mathcal{A} \otimes \mathcal{K}^\circ$-module. Consider Φ as a DG functor $\mathcal{A} \to \mathcal{K}' \subset \mathcal{K}^\circ$-DGmod, where $\mathcal{K}'$ is the full DG subcategory of quasi-representable DG modules. We have the DG functors $\mathcal{K} \leftarrow \mathcal{K}'' \xrightarrow{\pi} \mathcal{K}'$, where $\mathcal{K}''$ is the DG category whose objects are triples consisting of an object $Y \in \mathcal{K}$, a DG $\mathcal{A} \otimes \mathcal{K}^\circ$-module Ψ, and a quasi-isomorphism $f : h_Y \to \Psi$ (see C.16.3 for a precise definition of $\mathcal{K}''$). We also have a canonical DG functor $\mathrm{Cone} : \mathcal{K}'' \to \mathcal{K}^\circ$-DGmod, which sends (Y, Ψ, f) to $\mathrm{Cone}(f)$ (the definition of the Cone functor on morphisms is clear from 2.9). $\mathcal{A}$ is semi-free and π is a surjective quasi-equivalence, so by Lemma B.6 our DG functor $\mathcal{A} \to \mathcal{K}'$ lifts to a DG functor $\mathcal{A} \to \mathcal{K}''$. Let $F : \mathcal{A} \to \mathcal{K}$ be the composition $\mathcal{A} \to \mathcal{K}'' \to \mathcal{K}$. One has an exact sequence of DG $(\mathcal{A} \otimes \mathcal{K}^\circ)$-modules $0 \to \Phi \to M \to \Phi_F[1] \to 0$, where M corresponds to the composition $\mathcal{A} \to \mathcal{K}' \xrightarrow{\mathrm{Cone}} \mathcal{K}^\circ$-DGmod. As M is acyclic we get a $T(\mathcal{A}, \mathcal{K})$-isomorphism $\Phi_F \xrightarrow{\sim} \Phi$. □

E.7.3. The standard resolution

Consider the category DGalg of (non-unital) associative DG algebras and the category DGcoalg of (non-counital) cocomplete coassociative coalgebras (a coalgebra U is *cocomplete* if for every $u \in U$ there exists $n \in \mathbb{N}$ such that u is annihilated by the n-fold coproduct $\Delta_n : U \to U^{\otimes n}$). If $U \in \mathrm{DGcoalg}$ and $A \in \mathrm{DGalg}$ then $\mathrm{Hom}(U, A) \in \mathrm{DGalg}$ (the product of $f : U \to A$ and $g : U \to A$ is defined to be the composition of the coproduct $U \to U \otimes U$, the map $f \otimes g : U \otimes U \to A \otimes A$, and the product $m : A \otimes A \to A$). Define the *Maurer–Cartan functor* $\mathrm{MC} : \mathrm{DGcoalg}^\circ \times \mathrm{DGalg} \to \mathrm{Sets}$ as follows: $\mathrm{MC}(U, A)$ is the set of elements $\omega \in \mathrm{Hom}(U, A)$ of degree 1 such that $\mathrm{d}\omega + \omega^2 = 0$. There exist functors $B : \mathrm{DGalg} \to \mathrm{DGcoalg}$ and $\Omega : \mathrm{DGcoalg} \to \mathrm{DGalg}$ such that $\mathrm{MC}(U, A) = \mathrm{Mor}(U, BA) = \mathrm{Mor}(\Omega U, A)$ (they are called "bar construction" and "cobar construction"). As Ω is left adjoint to B, we have the adjunction morphisms $\Omega BA \to A$ and $U \to B\Omega U$. In fact, they are quasi-isomorphisms. The above statements are classical (references will be given in E.9).

Caution: while B sends quasi-isomorphisms to quasi-isomorphisms this is *not* true for Ω. Indeed, consider the morphism $\varphi : 0 \to k$, where k is equipped with the obvious DG algebra structure. Then $B(\varphi)$ is a quasi-isomorphism but $\Omega B(\varphi)$ is not.

It is easy to see that if A is a semi-free DG k-module then ΩBA is a semi-free DG algebra (in the non-unital sense), so ΩBA is a semi-free resolution of A. ΩBA is non-unital even if A is unital. The DG algebra one gets by adding the unit to a DG algebra B will be denoted by $u(B)$. If A is unital then $u(A)$ is the Cartesian product of DG algebras A and k, so we get a quasi-isomorphism $u(\Omega BA) \to u(A) = A \times k$. Let us call it the *standard resolution* of $A \times k$. It is semi-free (in the unital sense) if A is a semi-free DG k-module.

As explained in [24,31,33,36], there is a similar construction in the more general setting of DG categories. Given a DG category $\mathcal{A}$, let $\mathcal{A}_{\mathrm{discr}}$ denote the DG category with $\mathrm{Ob}\,\mathcal{A}_{\mathrm{discr}} = \mathrm{Ob}\,\mathcal{A}$ such that the endomorphism DG algebra of each object of $\mathcal{A}_{\mathrm{discr}}$ equals k and $\mathrm{Hom}_{\mathcal{A}_{\mathrm{discr}}}(X, Y) = 0$ if X, Y are different objects of $\mathcal{A}_{\mathrm{discr}}$. Let $u(\mathcal{A}) \subset \mathcal{A} \times \mathcal{A}_{\mathrm{discr}}$ be the full DG subcategory formed by objects (a, a), $a \in \mathrm{Ob}\,\mathcal{A} = \mathrm{Ob}\,\mathcal{A}_{\mathrm{discr}}$. There is a *standard resolution* $\mathrm{Stand}(\mathcal{A}) \to u(\mathcal{A})$. If all Hom complexes of $\mathcal{A}$ are semi-free over k then $\mathrm{Stand}(\mathcal{A})$ is semi-free.

E.7.4. A_∞-functors

If $\mathcal{A}$ is any DG category and $\widetilde{\mathcal{A}}$ is a semi-free resolution of $\mathcal{A}$ then $T(\mathcal{A}, \mathcal{K}) = T(\widetilde{\mathcal{A}}, \mathcal{K})$, so E.7.1, Proposition E.7.2 give a graded k-category equivalent to $T(\mathcal{A}, \mathcal{K})$ whose objects are DG functors $\widetilde{\mathcal{A}} \to \mathcal{K}$. In particular, if all Hom complexes of $\mathcal{A}$ are semi-free (or, more generally, homotopically projective) over k, we get a category equivalent to $T(u(\mathcal{A}), \mathcal{K})$ whose objects are DG functors $\mathrm{Stand}(\mathcal{A}) \to \mathcal{K}$. Notice that if k is a field (and if you believe in the axiom of choice, which ensures that modules over a field are free) then every DG k-module is semi-free. The functor $T(\mathcal{A}, \mathcal{K}) \to T(u(\mathcal{A}), \mathcal{K})$ corresponding to the canonical projection $u(\mathcal{A}) \to \mathcal{A}$ is fully faithful (this follows from the decomposition $D(u(\mathcal{A})^\circ \otimes \mathcal{K}) = D(\mathcal{A}^\circ \otimes \mathcal{K}) \oplus D(\mathcal{A}^\circ_{\mathrm{discr}} \otimes \mathcal{K})$). DG functors $\mathrm{Stand}(\mathcal{A}) \to \mathcal{K}$ such that the corresponding object of $T(u(\mathcal{A}), \mathcal{K})$ is in $T(\mathcal{A}, \mathcal{K}) \subset T(u(\mathcal{A}), \mathcal{K})$ are called *A_∞-functors*. More precisely, this is *one* of the versions of the notion of A_∞-functor $\mathcal{A} \to \mathcal{K}$. They differ on how an A_∞ analog of the axiom $F(\mathrm{id}) = \mathrm{id}$ in the definition of usual functor is

formulated (the difference is inessential from the homotopy viewpoint). The above notion is as "weak" as possible.

According to Kontsevich, the structure of graded k-category on $T(\mathcal{A},\mathcal{K})$ comes from a canonical DG category A_∞-funct$(\mathcal{A},\mathcal{K})$ whose objects are A_∞-functors $\mathcal{A}\to\mathcal{K}$. Here is its definition if $\mathcal{A}$ and $\mathcal{K}$ have one object (the general case is similar). Let A, K be the endomorphism DG algebras of these objects. Then an A_∞-functor $\mathcal{A}\to\mathcal{K}$ is a DG algebra morphism $\Omega BA\to K$ satisfying a certain condition (see E.7.3). So it remains to construct a DG category whose objects are elements of $\mathrm{Mor}(\Omega BA,K)=\mathrm{MC}(BA,K)$, i.e., elements ω of the DG algebra $R:=\mathrm{Hom}(BA,K)$ such that $\deg\omega=1$ and $\mathrm{d}\omega+\omega^2=0$. Such ω defines a DG R°-module N_ω: it equals R as a graded R°-module, and the differential in N_ω maps r to $\nabla r:=dr+\omega r$. Now put $\mathrm{Hom}(\omega,\omega'):=\mathrm{Hom}(N_\omega,N_{\omega'})$ and define the composition map $\mathrm{Hom}(\omega,\omega')\times\mathrm{Hom}(\omega',\omega'')\to\mathrm{Hom}(\omega,\omega'')$ in the obvious way.

Remark. According to [33,36], in the more general case that $\mathcal{K}$ is an A_∞-category A_∞-functors $\mathcal{A}\to\mathcal{K}$ form an A_∞-category. Kontsevich informed me, that if $\mathcal{K}$ is a DG category then the A_∞-category of A_∞-functors $\mathcal{A}\to\mathcal{K}$ is a DG category. I do not know if this DG category equals the above DG category A_∞-funct$(\mathcal{A},\mathcal{K})$.

E.8. DG models of $T(\mathcal{A}_1,\mathcal{A}_2)$

Kontsevich's model has already been mentioned in E.7.4: if the Hom complexes of $\mathcal{A}_1$ are semi-free (or, more generally, homotopically projective) over k then $T(\mathcal{A}_1,\mathcal{A}_2)$ is the graded homotopy category of the DG category A_∞-funct$(\mathcal{A}_1,\mathcal{A}_2)$.

Keller's model is easier to define. If $\mathcal{A}_1$ or $\mathcal{A}_2$ is flat then $D(\mathcal{A}_1^\circ\overset{L}{\otimes}\mathcal{A}_2)=D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)=\mathrm{Ho}^\cdot(\underset{\to}{\mathcal{R}})$, where $\mathcal{R}:=\mathcal{A}_1^\circ\otimes\mathcal{A}_2$ and $\underset{\to}{\mathcal{R}}$ is the DG category of semi-free DG $\mathcal{R}^\circ$-modules. This identifies $T(\mathcal{A}_1,\mathcal{A}_2)\subset D(\mathcal{A}_1^\circ\otimes\mathcal{A}_2)$ with the graded homotopy category of a certain full DG subcategory $DG(\mathcal{A}_1,\mathcal{A}_2)\subset\underset{\to}{\mathcal{R}}$, which will be called *Keller's model*.

One also has the *dual Keller model* $(DG(\mathcal{A}_1^\circ,\mathcal{A}_2^\circ))^\circ$: its graded homotopy category is $T(\mathcal{A}_1^\circ,\mathcal{A}_2^\circ)^\circ=T(\mathcal{A}_1,\mathcal{A}_2)$. The equality $T(\mathcal{A}_1,\mathcal{A}_2)=T(\mathcal{A}_1^\circ,\mathcal{A}_2^\circ)^\circ$ identifies $T(\mathcal{A}_1,\mathcal{A}_2)$ with the graded homotopy category of the DG category $(DG(\mathcal{A}_1^\circ,\mathcal{A}_2^\circ))^\circ$, which is a full DG subcategory of the DG category $\underset{\leftarrow}{\mathcal{R}}:=\{$the dual of the DG category of semi-free DG $\mathcal{R}$-modules$\}$.

If the Hom complexes of $\mathcal{A}_1$ are homotopically projective over k there is a canonical quasi-equivalence A_∞-funct$(\mathcal{A}_1,\mathcal{A}_2)\to DG(\mathcal{A}_1,\mathcal{A}_2)$, which is not discussed here.

Remark. Let $\mathcal{A},\mathcal{C}_1,\mathcal{C}_2$ be DG categories and suppose that $\mathcal{C}_1,\mathcal{C}_2$ are flat. Then $DG(\mathcal{A},\mathcal{C}_1)$, $DG(\mathcal{C}_1,\mathcal{C}_2)$, and $DG(\mathcal{A},\mathcal{C}_2)$ are defined, but in general (if $\mathcal{C}_1$ is not semi-free) the image of

$$\bigotimes_{\mathcal{C}_1}: DG(\mathcal{A},\mathcal{C}_1)\otimes DG(\mathcal{C}_1,\mathcal{C}_2)\to(\mathcal{A}\otimes\mathcal{C}_2^\circ)\text{-DGmod}$$

is not contained in $DG(\mathcal{A},\mathcal{C}_2)$ or even in $\underset{\rightarrow}{\mathcal{R}}$, where $\mathcal{R}:=\mathcal{A}^\circ\otimes\mathcal{C}_2$. So we do not get a composition DG functor $DG(\mathcal{A},\mathcal{C}_1)\otimes DG(\mathcal{C}_1,\mathcal{C}_2)\to DG(\mathcal{A},\mathcal{C}_2)$ but rather a DG functor

$$\Psi: DG(\mathcal{A},\mathcal{C}_1)\times DG(\mathcal{C}_1,\mathcal{C}_2)\times DG(\mathcal{A},\mathcal{C}_2)^\circ\to k\text{-DGmod}, \tag{E.4}$$

which lifts the graded functor

$$T(\mathcal{A},\mathcal{C}_1)\times T(\mathcal{C}_1,\mathcal{C}_2)\times T(\mathcal{A},\mathcal{C}_2)^\circ\to\{\text{Graded } k\text{-modules}\}$$

defined by $(F_1,G,F_2)\mapsto\bigoplus_n \operatorname{Ext}^n(F_2,GF_1)$. One defines (E.4) by

$$(M_1,N,M_2)\mapsto \operatorname{Hom}(M_2,M_1\otimes_{\mathcal{C}_1}N).$$

E.9. Some historical remarks

As explained in [44], the functors B and Ω from E.7.3 go back to Eilenberg–MacLane and J.F. Adams. It was E.H. Brown [7] who introduced $MC(U,A)$; he called its elements "twisting cochains." The fact that the morphism $\Omega BA\to A$ is a quasi-isomorphism appears as [43, Theorem 6.2, pp. 7–28]. All the properties of B and Ω from E.7.3 were formulated in [44] and proved in [21]; their analogs for Lie algebras and commutative coalgebras were proved in [48, Appendix B, §7]. In these works DG algebras and DG coalgebras were assumed to satisfy certain boundedness conditions. The general case was treated in [20,36].

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