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Introduction to Homological Algebra

Lecture 1: $\boxed{08/07/2007}$

$\boxed{\S 1.1.}$ Homological algebra is more of a very convenient language, rather than a mathematical theory. Note also that homological algebra is still not fully developed: for example, the notion of a triangulated category that many people use these days is not the "correct one".

$\boxed{\S 1.2.}$ Today we will give a review of the basic structures of homological algebra.

$\mathcal{A}b$ = the category of abelian groups

One has the notion of a complex of abelian groups. There are two standard constructions with complexes.

1) Shifts: if $M = (M^{\cdot}, d)$ is a complex, then for any $a \in \mathbb{Z}$, we define a new complex, $M[a]$, by $M[a]^n = M^{a+n}$
$d_{M[a]} = (-1)^a d_M$.

2) Cones: Consider a morphism of complexes $M \xrightarrow{f} N$. We can define a complex $\mathrm{Cone}(f)$ as follows:

$$\mathrm{Cone}(f)^n = N^n \oplus M^{n+1}$$

$$d_{\mathrm{cone}(f)} = \begin{pmatrix} d_N & f \\ 0 & -d_M \end{pmatrix}$$

It is trivial to check that

$$d_N \circ f = f \circ d_M \implies d^2_{\mathrm{cone}(f)} = 0.$$

Another way to write this definition: consider the total complex of the double complex

$$\begin{tikzcd}
& 0 & 0 & \\
\cdots \arrow[r] & N^n \arrow[u] \arrow[r] & N^{n+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & M^n \arrow[u, "f"] \arrow[r] & M^{n+1} \arrow[u, "f"] \arrow[r] & \cdots \\
& 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$\boxed{\S 1.3.}$ Exercise. Consider an embedding of complexes $N \hookrightarrow C$ which has termwise splittings, not necessarily compatible with the differentials.

Note that, in any case, C/N is also a complex. Set $M = (C/N)[-1]$.

Choose splittings

$$C^n \cong N^n \oplus (C/N)^n \qquad (*)$$

and write the differential on C in terms of this splitting:

$$d_C = \begin{pmatrix} d_N & f \\ 0 & d_{M[1]} \end{pmatrix}$$

We can consider f as a morphism of graded abelian groups $M \longrightarrow N$.

Exercise: Show that f is a morphism of complexes, and the direct sum decomposition $(*)$ identifies C with the cone of the morphism f.

This essentially tautological exercise provides a slightly different way of thinking about cones.

§1.4. Additive categories

An additive category is a category where all the Hom-sets are equipped with abelian group structures such that composition of morphisms is biadditive, and which has finite products.

Remark: In particular, looking at the product of an empty collection of objects, we see that there is a final object.

Exercises. Check that the final object of an additive category is in fact a zero object (i.e., it is also initial). Check that finite products are the same as finite coproducts. Use this to show that being additive is a property of a category (i.e., any given category has at most one additive structure).

§1.5. Image of an idempotent.

Let $\mathcal{A}$ be an additive category. Suppose we have $M \in \mathcal{A}$, and $e \in \mathrm{End}(M)$ is an idempotent, i.e., $e^2 = e$.

Def: We say that e has an image if there is a decomposition $M = M_1 \oplus M_2$ with respect to which the matrix of e equals $\begin{pmatrix} id_{M_1} & 0 \\ 0 & 0 \end{pmatrix}$. One can check that in this case M_1 is unique, and we write $M_1 = \mathrm{Im}(e)$.

Remark: If $e \in \mathrm{End}(M)$ is an idempotent, so is $id_M - e$. If e has an image, then, with the notation above, $\mathrm{Im}(id_M - e) = M_2$.

Def: We say that an additive category $\mathcal{A}$ is idempotently complete, or Karoubian, if every idempotent endomorphism of every object of $\mathcal{A}$ has an image.

$\boxed{\S 1.6.}$ Exercise. Every additive category $\mathcal{A}$ admits a unique (in the appropriate sense) fully faithful additive embedding into a Karoubian category $\mathcal{A} \hookrightarrow \mathcal{A}^{Kar}$ such that every object of $\mathcal{A}^{Kar}$ is the image of an idempotent in $\mathcal{A}$.

Remark: It is not always possible to turn an additive category into an abelian one "by brute force".

$\boxed{\S 1.7.}$ Graded additive categories.

Def: A graded additive category is an additive category $\mathcal{A}$ equipped with gradings on the Hom-groups which are compatible with compositions.

Def: A differential graded (DG) structure on an additive category $\mathcal{A}$ is a grading in the sense above together with differentials of degree 1 on the Hom-groups satisfying:

$$d(f\circ g) = (df)\circ g + (-1)^{\deg(f)}\cdot(f\circ(dg)),$$

whenever f is homogeneous and $f\circ g$ is defined.

Remark. If $\mathcal{A}$ is a DG additive category and $M\in\mathcal{A}$, then $\mathrm{End}_{\mathcal{A}}(M)$ is a d.g.a.

differential graded algebra

$\boxed{\S 1.8.}$ If $\mathcal{A}$ is a DG category, the homotopy category $Ho(\mathcal{A})$ is the additive category defined by

$$\mathrm{Objects}(Ho(\mathcal{A})) = \mathrm{Objects}(\mathcal{A})$$

$$\mathrm{Hom}_{Ho(\mathcal{A})}(M,N) = H^0(\mathrm{Hom}_{\mathcal{A}}(M,N)).$$

$\boxed{\S 1.9.}$ Example of a DG category.

Fix an additive category $\mathcal{B}$, and let $C(\mathcal{B})$ be the category of complexes in $\mathcal{B}$.

If we only consider morphisms of complexes of degree 0, then $C(\mathcal{B})$ becomes a plain additive category. However, it is better to define $C(\mathcal{B})$ as a DG category.

<u>Namely</u>: If $M, N \in C(\mathcal{B})$, we define a complex of abelian groups $\mathrm{Hom}^\bullet(M,N)$ by

$$\mathrm{Hom}^n(M,N) = \prod_{a \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(M^a, N^{a+n}).$$

The differential

$$d : \mathrm{Hom}^n(M,N) \longrightarrow \mathrm{Hom}^{n+1}(M,N)$$

is given by

$$d(f) = d_N \circ f + (-1)^n f \circ d_M .$$

It is easy to check that this makes $C(\mathcal{B})$ into a DG category.

<u>Remark</u>: $Z^0(\mathrm{Hom}^\bullet(M,N)) = \{$usual (degree 0) morphisms of complexes $M \longrightarrow N\}$

and $H^0(\mathrm{Hom}^\bullet(M,N)) = \{$homotopy classes of degree 0 morphisms of complexes $M \longrightarrow N\}$.

$\boxed{\S 1.10.}$ As usual, we have full DG subcategories $C^+(\mathcal{B}), C^-(\mathcal{B}), C^b(\mathcal{B}) \subset C(\mathcal{B})$.

$\boxed{\S 1.11.}$ <u>Remark</u>. Translations and cones can be defined abstractly in any DG category (but they do not always exist):

$\mathcal{C}$ = any DG category

1) $M \in \mathcal{C} \Rightarrow M[a]$ (if it exists) is determined by $\mathrm{Hom}(L, M[a]) = \mathrm{Hom}(L, M)[a]$

2) $f : M \longrightarrow N$ a <u>DG morphism</u> in $\mathcal{C}$

by definition, this means an element of $Z^0 \mathrm{Hom}(M, N)$

$\Rightarrow \mathrm{Cone}(f)$ (if it exists) is determined by $\mathrm{Hom}(L, \mathrm{Cone}(f)) = \mathrm{Cone}\big(\mathrm{Hom}(L, M) \longrightarrow \mathrm{Hom}(L, N)\big)$.

$\boxed{\S 1.12.}$ <u>Definition</u>. A DG category $\mathcal{C}$ is said to be <u>strongly</u> <u>pre-triangulated</u> if all objects of $\mathcal{C}$ have shifts, and all DG morphisms in $\mathcal{C}$ have cones.

(Roughly speaking, just "pre-triangulated" means that the functors defining shifts and cones are representable at the level of the homotopy category.)

$\boxed{\S 1.13}$ Distinguished triangles.

Fix a strongly pre-triangulated DG category $\mathcal{C}$. By taking all the cohomology of the Hom's in $\mathcal{C}$, we obtain a graded additive category, which we will also denote by $Ho(\mathcal{C})$.

Alternatively, we can define $Ho(\mathcal{C})$ in the old way, and consider

$$\mathrm{Hom}^n_{Ho(\mathcal{C})}(M,N) = \mathrm{Hom}_{Ho(\mathcal{C})}(M,N)$$

in the old sense

Def: A triangle in $Ho(\mathcal{C})$ is a diagram of morphisms of degree 0:

$$M \xrightarrow{u} N \xrightarrow{v} C \xrightarrow{w} M[1]$$

Given such a triangle, we expand it to a long sequence (infinite on both ends) as follows:

$$\ldots \to C[-1] \xrightarrow{-w[-1]} M \xrightarrow{u} N \xrightarrow{v} C \xrightarrow{w} M[1] \xrightarrow{-u[1]} N[1] \xrightarrow{-v[1]} C[1] \xrightarrow{-w[1]} M[2] \xrightarrow{u[2]} N[2] \xrightarrow{v[2]} C[2] \to \ldots$$

Remark. When we shift a DG morphism of objects: $f \mapsto f[a]$, there is no sign change!

<u>Def:</u> A triangle in $Ho(\mathcal{C})$ is said to be <u>distinguished</u> if it is isomorphic to a triangle of the form

$$M \xrightarrow{u} N \longrightarrow \mathrm{Cone}(u) \longrightarrow M[1]$$

the canonical embedding and projection

§1.14. <u>Definition</u>. A <u>triangulated category</u> is a graded additive category $\mathcal{D}$ such that all shifts in $\mathcal{D}$ exist (so we have an actual shift functor $T: \mathcal{D} \longrightarrow \mathcal{D}$) together with a chosen collection of triangles in $\mathcal{D}$ (in the sense of the definition given above), called the <u>distinguished triangles</u>, satisfying the following list of axioms.

(0) Every triangle isomorphic to a distinguished one is distinguished

(1) Every triangle of the form

$$M \xrightarrow{id_M} M \longrightarrow 0 \longrightarrow M[1]$$

is distinguished.

(2) "Rotation invariance" of distinguished triangles: a triangle

$$M \xrightarrow{u} N \xrightarrow{v} C \xrightarrow{w} M[1]$$

is distinguished $\Longleftrightarrow$ the triangle

$$N \xrightarrow{v} C \xrightarrow{w} M[1] \xrightarrow{-u[1]} N[1]$$ is distinguished

(3) Every morphism $M \xrightarrow{f} N$ in $\mathcal{D}$ can be included in a distinguished triangle

$$M \xrightarrow{f} N \longrightarrow C \longrightarrow M[1],$$

and every morphism between morphisms, i.e., a commutative diagram

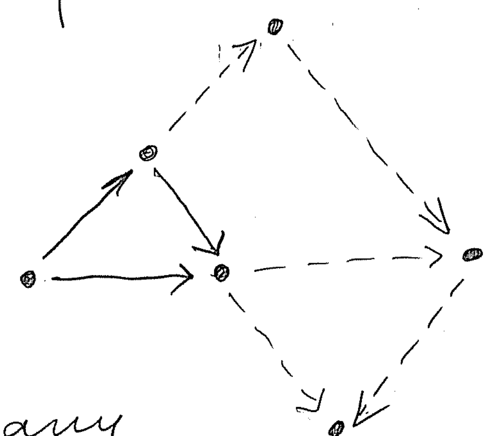
$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[r] \arrow[d] & C \arrow[r] & M[1] \\ M' \arrow[r, "f'"] & N' \arrow[r] & C' \arrow[r] & M'[1] \end{tikzcd}$$

these have already been chosen

can be included in a commutative diagram of distinguished triangles

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & M[1] \arrow[d] \\ M' \arrow[r, "f'"] & N' \arrow[r] & C' \arrow[r] & M'[1] \end{tikzcd}$$

(4) <u>Octahedron Axiom</u>. (The last one; the most complicated and in some sense the most important one.)



Look up the definition in any textbook on derived categories.

<u>Remark</u>. In fact, one should view the octahedron axiom as the analogue for triangulated categories of the "Second Isomorphism Theorem": if $M \subset N \subset L$ are abelian groups (or objects in any abelian category), then canonically

$$L/N \cong (L/M)/(N/M)$$

$\boxed{\S 1.15.}$ <u>Theorem</u>. Let $\mathcal{C}$ be any strongly pre-triangulated DG category. Then the homotopy category $Ho(\mathcal{C})$, with the graded structure and the collection of distinguished triangles defined in §1.14, is a triangulated category.

<u>Remark</u>: The same theorem is true if $\mathcal{C}$ is assumed to be merely pre-triangulated.

$\boxed{\S 1.16.}$ <u>Key Lemma</u>. Let $\mathcal{C}$ be a strongly pre-triangulated DG category.

(1) Consider a DG morphism $M \xrightarrow{u} N$ which is a split embedding (but the splitting is not necessarily a DG morphism). Then we have $N/M \in \mathcal{C}$, and the natural projection $\mathrm{Cone}(u) \longrightarrow N/M$ is a homotopy equivalence.

(2) Any DG morphism in $\mathcal{C}$ is homotopic to a split embedding as above. That is, if $M \xrightarrow{f} L$ is any DG morphism in $\mathcal{C}$, there is a (naturally constructed) commutative diagram of DG morphisms

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "u"'] & & L \\ & N \arrow[ur, "g"'] & \end{tikzcd}$$

where u is a split embedding and g is a homotopy equivalence (i.e., becomes an isomorphism in $Ho(\mathcal{C})$).

(Hint for (2): take $N = L \oplus \mathrm{Cone}(id_M)$.)

Now we sketch a proof of Theorem 1.15.

$\boxed{\S 1.17.}$ It is trivial to verify properties (0) and (1) in the definition of a triangulated category for the category $Ho(\mathcal{C})$.

What about rotation invariance of distinguished triangles? Use the lemma above to reduce to triangles of the form $M \xrightarrow{u} N \longrightarrow \mathrm{Cone}(u) \longrightarrow M[1]$, where u is a split embedding, and prove property (2) in this case.

For property (3), the first part follows by construction. For the second part, given a commutative diagram in $Ho(\mathcal{C})$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[r] \arrow[d, "h"] & \mathrm{Cone}(f) \arrow[r] & M[1] \\ M' \arrow[r, "f'"] & N' \arrow[r] & \mathrm{Cone}(f') \arrow[r] & N[1] \end{tikzcd}$$

Lift f, f', g, h to honest DG morphisms in $\mathcal{C}$. The resulting square in $\mathcal{C}$ will fail to commute by a coboundary in $\mathrm{Hom}^0_{\mathcal{C}}(M, N')$. If we write this coboundary as the differential of some element of $\mathrm{Hom}^{-1}_{\mathcal{C}}(M, N')$, we get a DG morphism $\mathrm{Cone}(f) \longrightarrow \mathrm{Cone}(f')$ which completes the diagram above.

For the octahedron axiom, start with any commutative triangle

$$\begin{tikzcd} & N \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & L \end{tikzcd}$$

and use the key lemma twice to replace it with a commutative triangle

$$\begin{tikzcd} & N' \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & L'' \end{tikzcd}$$

where all the maps are split embeddings. Then use the second isomorphism theorem (for complexes of abelian groups).