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Introduction to Homological Algebra

Lecture 3: $\boxed{08/09/2007}$

$\boxed{\S 3.1.}$ Let us recall the Verdier quotient construction discussed at the end of last time. Let $\mathcal{D}$ be any triangulated category.

We have reflection operations: given a diagram of the form

$$\begin{tikzcd} & L \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M & & N \end{tikzcd} \qquad (*)$$

we say that a diagram

$$\begin{tikzcd} M \arrow[dr, "\gamma"'] & & N \arrow[dl, "\delta"] \\ & K & \end{tikzcd} \qquad (**)$$

is a <u>refection</u> of $(*)$ if

$$L \xrightarrow{(\alpha,\beta)} M \oplus N \xrightarrow{\gamma-\delta} K$$

is part of an exact triangle.

This determines $(**)$ up to a non-unique isomorphism.

$\boxed{\S 3.2.}$ Exercise. Given a reflection diagram

$$\begin{tikzcd} & L \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ M \arrow[dr, "\gamma"'] & & N \arrow[dl, "\delta"] \\ & K & \end{tikzcd} \qquad (1)$$

the induced morphism $\mathrm{Cone}(\alpha) \to \mathrm{Cone}(\delta)$ is an isomorphism.

$\boxed{\S 3.3.}$ Corollary. Let $\mathcal{V} \subset \mathcal{D}$ be a strictly full triangulated subcategory. Given a reflection diagram (3.2.1), the arrow α is a $\mathcal{V}$-quasi-isomorphism if and only if δ is a $\mathcal{V}$-quasi-isomorphism.

$\boxed{\S 3.4.}$ The reflection operations allow us to define composition and addition of morphisms in $\mathcal{D}/\mathcal{V}$.

$\boxed{\S 3.5.}$ Admissible pairs.

Suppose we have two strictly full triangulated subcategories

$$\mathcal{U} \subset \mathcal{D} \supset \mathcal{V}$$

Def: The pair $(\mathcal{U}, \mathcal{V})$ is said to be <u>admissible</u> if the following conditions hold:

(1) $\forall\, U \in \mathcal{U},\ V \in \mathcal{V}$,
$$\mathrm{Hom}_{\mathcal{D}}(U, V) = 0$$

(2) $\forall\, M \in \mathcal{D}$, there is a distinguished triangle
$$U \longrightarrow M \longrightarrow V \longrightarrow U[1]$$
with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

§3.6. <u>Lemma</u>: (a) The distinguished triangle in the definition (3.5.2) is unique up to a unique isomorphism, and depends functorially on M.

[Proof: it is completely straightforward to check that any diagram of the form
$$\begin{tikzcd} U \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, dashed] & U[1] \\ U' \arrow[r] & M' \arrow[r] & V' \arrow[r] & U'[1] \end{tikzcd}$$
can be <u>uniquely</u> completed to a commutative diagram by the dashed arrows as shown.]

(b) The functor $M \longmapsto \mathcal{U}_M$ obtained in (a) is a triangulated functor $\mathcal{D} \longrightarrow \mathcal{U}$ which is right adjoint to $\mathcal{U} \hookrightarrow \mathcal{D}$

[Proof: by an exercise from the previous lecture, it's enough to verify the adjunction property, which is quite obvious from the definition of admissible pairs and the long exact sequence of Hom's.]

(c) Dually, the functor $M \longmapsto \mathcal{V}_M$ is a triangulated functor $\mathcal{D} \longrightarrow \mathcal{V}$ which is left adjoint to the inclusion $\mathcal{V} \hookrightarrow \mathcal{D}$.

(d) Each of the categories $\mathcal{U}, \mathcal{V}$ determines the other one uniquely:

$$\mathcal{U} = {}^{\perp}\mathcal{V}, \qquad \mathcal{V} = \mathcal{U}^{\perp}$$

right orthogonal complement to $\mathcal{U}$

left orthogonal complement to $\mathcal{V}$

$$\| $$

$$\{X \in \mathcal{D} \mid \mathrm{Hom}(X, V) = 0 \quad \forall V \in \mathcal{V}\}$$

In particular, every direct summand of an object of $\mathcal{D}$ that lies in $\mathcal{U}$ (resp., $\mathcal{V}$) also lies in $\mathcal{U}$ (resp., $\mathcal{V}$). In other words, $\mathcal{U}$ & $\mathcal{V}$ are <u>thick</u> <u>subcategories</u>.

(e) The Verdier quotients $\mathcal{D}/\mathcal{U}$ and $\mathcal{D}/\mathcal{V}$ exist. (Recall that the general construction of the Verdier quotients that we gave may sometimes fail to work due to set-theoretical difficulties.)

Moreover, the natural compositions

$$\mathcal{U} \hookrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{V} \text{ and}$$

$$\mathcal{V} \hookrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{U}$$

are equivalences of triangulated categories.

[Proof: check that the functors $M \longmapsto U_M$ and $M \longmapsto V_M$ satisfy the universal properties of the Verdier quotients by $\mathcal{V}$ and $\mathcal{U}$, respectively.]

(f) <u>Converse</u>: suppose $\mathcal{D}$ is a triangulated category and $\mathcal{U} \subset \mathcal{D}$ is a thick strictly full triangulated subcategory. TFAE:

(i) $\mathcal{U} \hookrightarrow \mathcal{D}$ has a left adjoint

(ii) $\mathcal{D}/\mathcal{U}$ exists and $\mathcal{D} \to \mathcal{D}/\mathcal{U}$ has a right adjoint

(iii) $(\mathcal{U}, \mathcal{U}^{\perp})$ is an admissible pair.

§3.7. Example.

X = topological space

$Z \subset X$ closed subspace, $U = X \setminus Z$

First consider the abelian setting:

$Sh(X)$ = category of sheaves of abelian groups on X

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

$$\rightsquigarrow \quad Sh(Z) \xrightarrow{i_*} Sh(X) \underset{j_!}{\overset{j^*}{\rightleftarrows}} Sh(U)$$

The functor $j^* = j^!$ identifies $Sh(U)$ with the Serre quotient $Sh(X)/Sh(Z)$.

However, this situation is not as symmetric as in the triangulated setting.

Passing to the derived categories, we obtain

$$D(Sh(Z)) \xrightarrow{i_*} D(Sh(X)) \xleftarrow{Rj_*} D(Sh(U)),$$

and these form an admissible pair of subcategories of $D(Sh(X))$.

This example is important!

$\boxed{\S 3.8.}$ Derived categories

$\mathcal{A}$ = abelian category

As discussed in the first lecture, we have a DG category structure on $C(\mathcal{A})$, the category of complexes of objects of $\mathcal{A}$.

We put $K(\mathcal{A}) = Ho(C(\mathcal{A}))$, which is a triangulated category.

Let $K(\mathcal{A})^{acycl} \subset K(\mathcal{A})$ denote the full subcategory consisting of acyclic complexes. The <u>derived category</u> of $\mathcal{A}$ is defined as the Verdier quotient

$$D(\mathcal{A}) = K(\mathcal{A}) / K(\mathcal{A})^{acycl}$$

$\boxed{\S 3.9.}$ It is natural to ask whether one can understand $D(\mathcal{A})$ in terms of the approach to the Verdier quotients via admissible pairs.

In other words, when are the left and right orthogonal complements to $K(\mathcal{A})^{acycl}$ in $K(\mathcal{A})$ "sufficiently large", so as to produce admissible pairs in $K(\mathcal{A})$.

$\boxed{\S 3.10.}$ <u>Definition</u>.

$${}^{\perp}K(\mathcal{A})^{acycl} = \{\text{homotopically projective complexes in } K(\mathcal{A})\}$$

$$\left(K(\mathcal{A})^{acycl}\right)^{\perp} = \{\text{homotopically injective complexes in } K(\mathcal{A})\}$$

The question in §3.9 can be rephrased as follows: does every complex M over $\mathcal{A}$ admit a quasi-isomorphism $P \longrightarrow M$ (resp., $M \longrightarrow I$) where P is homotopically projective (resp., I is homotopically injective). Such a quasi-isomorphism is called a <u>homotopically projective resolution</u> of M (resp., a <u>homotopically injective resolution</u> of M).

$\boxed{\S 3.11.}$ <u>Existence of homotopically injective/projective resolutions</u>

<u>Lemma</u>. Every bounded above complex of projective objects of $\mathcal{A}$ is homotopically projective. Dually, every bounded below complex of injective objects of $\mathcal{A}$ is homotopically injective.

§3.12. Proposition. (a) Suppose $\mathcal{A}$ has enough projectives. Then every bounded above complex of objects of $\mathcal{A}$ admits a homotopically projective resolution.

(b) Suppose $\mathcal{A}$ has enough projectives and countable direct sums. Then every complex of objects of $\mathcal{A}$ admits a homotopically projective resolution.

Proof. Statement (a) is very standard and very old. Statement (b) is much more recent (maybe 15–20 years old).

Let us sketch a proof of (b). Suppose $M^\bullet$ is a complex which admits a filtration

$$0 = M_0^\bullet \subset M_1^\bullet \subset M_2^\bullet \subset \dots \subset M^\bullet$$

such that:

(i) $d(M_i^\bullet) \subset M_{i-1}^\bullet$ for all $i \geq 1$

(in particular, each of the $M_i^\bullet$'s is a subcomplex of $M^\bullet$, and $M_1^\bullet$ has identically zero differential)

(ii) each M_i^n / M_{i-1}^n ($i \geq 1$, $n \in \mathbb{Z}$) is a projective object of $\mathcal{A}$;

(iii) $M^\bullet = \bigcup_{i \geq 1} M_i^\bullet$

We claim that M° is then homotopically projective. This is easy to check. If the filtration on M° is actually finite, use induction. In general, pass to projective limits ...

Exercise: Use truncations to complete the proof of (b) using (a).

$\boxed{\S 3.13.}$ Grothendieck group of a triangulated category

Let $\mathcal{D}$ be an essentially small triangulated category. We define

$K_0(\mathcal{D})$ = abelian group with generators being the isomorphism classes of objects in $\mathcal{D}$, and relations:

$\forall$ distinguished triangle

$$L \longrightarrow M \longrightarrow N \longrightarrow L[1],$$

the relation is $[M] = [L] + [N]$

Consequence: $[M[a]] = (-1)^a [M] \quad \forall a \in \mathbb{Z}$.

Exercise. (a) If $\mathcal{A}$ is an additive category, then $K_0(K^b(\mathcal{A})) = K_0(\mathcal{A})$

(b) If $\mathcal{A}$ is abelian, $K_0(D^b(\mathcal{A})) = K_0(\mathcal{A})$.

<u>Remark:</u> If $\mathcal{A}$ is an abelian category, then in general (unless $\mathcal{A}$ is semisimple), $K_0(\mathcal{A})$ depends on whether we consider $\mathcal{A}$ as an abelian category or only as an additive category!

$\boxed{\S 3.14.}$ What is the meaning of K_0?

Philosophically speaking, triangulated categories are "animations" of abelian groups.

<u>Theorem</u> (Thomason): Let $\mathcal{D}$ be a triangulated category and $\mathcal{P} \subset \mathcal{D}$ a strictly full triangulated subcategory such that every object of $\mathcal{D}$ is a direct summand of an object of $\mathcal{P}$.

Claim: all such subcategories $\mathcal{P} \subset \mathcal{D}$ are in one-to-one correspondence with subgroups of $K_0(\mathcal{D})$.

Explicitly, the bijections are as follows:

$$(\mathcal{D} \supset \mathcal{P}) \longmapsto (\text{image of } K_0(\mathcal{P}) \longrightarrow K_0(\mathcal{D}))$$

in fact, this map is injective

and

$$(K_0(\mathcal{D}) \supset K) \longmapsto \mathcal{P}_K = \{ M \in \mathcal{D} \mid [M] \in K \}$$

<u>Exercise</u>. Prove this theorem.

It is not very difficult.

Example: we must check that $\forall M \in \mathcal{D}$, the object $M \oplus M[1]$ lies in every $\mathcal{P}$ as above. To check this, pick $M' \in \mathcal{D}$ with $M \oplus M' \in \mathcal{P}$. Then

$$\mathrm{cone}(\mathrm{id}_M \oplus 0 : M \oplus M' \longrightarrow M \oplus M') =$$

$$= M \oplus M[1] \in \mathcal{P}.$$

This completes the abstract part of this introduction to derived categories.