Semiorthogonal Decomposition of Derived Categories — Introductory School on Derived Categories

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Lecture 1. — University of Warwick.

A triangulated category $\mathcal{T}$ is an additive category (k-linear). This comes with

- Shift functors: $[t] : \mathcal{T} \to \mathcal{T}$ for each $t \in \mathbb{Z}$ (Note $[t_1] \cdot [t_2] = [t_1 + t_2]$).
- Class of distinguished triangles

$$T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} T_3 \xrightarrow{f_3} T_1[1].$$

better to have equality and not $\simeq$. Can always pass to an equiv. category to make this equality.

The reader is referred to Gelfand-Manin for all axioms of triangulated categories. The most important axiom is:

For any $f : T_1 \to T_2$, $\exists$ a distinguished triangle $T_1 \xrightarrow{f} T_2 \longrightarrow T_3 \longrightarrow T_1[1]$

$T_3 =: \mathrm{Cone}(f)$.

The isomorphism class of $\mathrm{Cone}(f)$ is well defined but is not functorial.

Some remarks about

$$\begin{array}{ccccccc} T_1' & \longrightarrow & T_2' & \longrightarrow & T_3' & \longrightarrow & T_1'[1] \\ \downarrow & & \downarrow & & \wr\downarrow & & \\ T_1 & \xrightarrow{f} & T_2 & \longrightarrow & T_3 & \longrightarrow & T_1[1] \end{array}$$

Some notation:

- $D(\mathcal{C}) :=$ the derived category of the abelian category $\mathcal{C}$.
- $D^b(\mathcal{C}) =$ Bounded derived category.
- $\mathrm{Ext}^i(T, T') = \mathrm{Hom}^i(T, T') = \mathrm{Hom}(T, T'[i])$.
- $\mathrm{Hom}^\bullet(T, T') = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i(T, T')[-i] \in \mathrm{grmod}(k) = D(k)$.
- $V \in \mathrm{mod}(k)$; $T \in \mathcal{T}$ then $V \otimes_k T = T^{\oplus \dim V}$

  This is not a good definition (non-functorial)

  So you define it by the functor it represents

$$\left.\begin{array}{l} \mathrm{Hom}(V \otimes T, T') = \mathrm{Hom}(V^*, \mathrm{Hom}(T, T')) \\ \mathrm{Hom}(T', V \otimes T) = V \otimes \mathrm{Hom}(T', T). \end{array}\right\} (*)$$

- $V^\bullet \in \mathrm{grmod}(k)$; $V^\bullet = \bigoplus_{i \in \mathbb{Z}} V_i[-i]$ (Replace Hom with $\mathrm{Hom}^\bullet$

  $V^\bullet \otimes T = \bigoplus_{i \in \mathbb{Z}} V_i \otimes T[-i]$. and V with $V^\bullet$ in the above conditions $(*)$)

The most important notion for these lectures is that of:

Semiorthogonal Decomposition.

Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be strictly full triangulated subcategories

is closed under shifts and cones.

Def: $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition if

Can also take $\mathrm{Hom}^\bullet$

1) $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$ (ie $\forall A \in \mathcal{A}$, $B \in \mathcal{B}$ we have $\mathrm{Hom}(B, A) = 0$).

2) For any $T \in \mathcal{T}$ there is a distinguished triangle $T_\mathcal{B} \to T \to T_\mathcal{A} \to T_\mathcal{B}[1]$, with $T_\mathcal{A} \in \mathcal{A}$ and $T_\mathcal{B} \in \mathcal{B}$.

Properties:

1) Functoriality: For any $f: T \to T'$ there exist unique $f_\mathcal{A}$ and $f_\mathcal{B}$ such that the following diagram commutes:

in proof ... map below is just composition with this arrow

$$\begin{tikzcd}
T_\mathcal{B} \arrow[r] \arrow[d, dashed, "f_\mathcal{B}"'] & T \arrow[r] \arrow[d, "f"] & T_\mathcal{A} \arrow[r] \arrow[d, dashed, "f_\mathcal{A}"] & T_\mathcal{B}[1] \arrow[d, dashed, "{f_\mathcal{B}[1]}"] \\
T'_\mathcal{B} \arrow[r] & T' \arrow[r] & T'_\mathcal{A} \arrow[r] & T'_\mathcal{B}[1]
\end{tikzcd}$$

comments in proof

Proof: Apply $\mathrm{Hom}(-, T'_\mathcal{A})$. Get long exact sequence)

$$\cdots \to \underset{\overset{\parallel}{0}}{\mathrm{Hom}(T_\mathcal{B}[1], T'_\mathcal{A})} \to \underset{\overset{\ni}{f_\mathcal{A}}}{\mathrm{Hom}(T_\mathcal{A}, T'_\mathcal{A})} \xrightarrow{\cong} \mathrm{Hom}(T, T'_\mathcal{A}) \to \underset{\overset{\parallel}{0}}{\mathrm{Hom}(T_\mathcal{B}, T'_\mathcal{A})} \to \cdots$$

$\Rightarrow \exists!\, f_\mathcal{A}$ s.t the middle square commutes.

Exercise: $\exists!\, f_\mathcal{B}$ s.t the left square commutes. $\square$

The exercise shows that the left square commutes.

Corollary:

$\left.\begin{array}{l} T \mapsto T_\mathcal{A} \\ f \mapsto f_\mathcal{A} \end{array}\right\}$ is a functor and so is $\begin{array}{l} T \mapsto T_\mathcal{B} \\ f \mapsto f_\mathcal{B} \end{array}$.

These are both functors from $\mathcal{T} \to \mathcal{A}$ and $\mathcal{T} \to \mathcal{B}$.

Let the adjoint functors be:

$\mathcal{A} \xrightarrow{\alpha} \mathcal{T}$ and $\mathcal{B} \xrightarrow{\beta} \mathcal{T}$.

Then $T \mapsto T_\mathcal{A}$ is left adjoint to α (denoted α^*) and $T \mapsto T_\mathcal{B}$ is right adjoint to β (denoted $\beta^!$)

$$\mathrm{Hom}(T, \alpha A) \ni f \quad \sim \quad f_{\mathcal{A}} \in \mathrm{Hom}_{\mathcal{J}}(T_{\mathcal{A}}, \alpha A) = \mathrm{Hom}_{\mathcal{A}}(T_{\mathcal{A}}, A)$$

this association gives an isomorphism as shown

$$\begin{tikzcd}
T_{\mathcal{B}} \arrow[r] \arrow[d, dotted] & T \arrow[r] \arrow[d, "f"] & T_{\mathcal{A}} \arrow[r] \arrow[d, dotted, "f_{\mathcal{A}}"] & T_{\mathcal{B}}[1] \arrow[d, dotted] \\
0 \arrow[r] & \alpha A \arrow[r, "\mathrm{id}"'] & \alpha A \arrow[r] & 0
\end{tikzcd}$$

$T_{\mathcal{A}} = \alpha^* T$

$f_{\mathcal{A}} = \alpha^* f$

Exercise: Prove the above $\cong$ for $T_{\mathcal{B}}$.

Definition: A full subcategory $\mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{J}$ is left admissable if $\exists\ \alpha^* : \mathcal{J} \to \mathcal{A}$.
right admissable if $\exists\ \alpha^! : \mathcal{J} \to \mathcal{A}$.

If $\mathcal{J} = \langle \mathcal{A}, \mathcal{B} \rangle$ then:

Exercise: $\begin{cases} \mathcal{B} = {}^{\perp}\mathcal{A} := \{ T \in \mathcal{J} \mid \mathrm{Hom}(T, \mathcal{A}) = 0 \} = \mathrm{Ker}\, \alpha^* \\ \mathcal{A} = \mathcal{B}^{\perp} := \{ T \in \mathcal{J} \mid \mathrm{Hom}(\mathcal{B}, T) = 0 \} = \mathrm{Ker}\, \beta^! \end{cases}$

Proposition: If $\mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{J}$ is left admissable $\Rightarrow \mathcal{J} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$
If $\mathcal{B} \hookrightarrow \mathcal{J}$ is right admissable $\Rightarrow \mathcal{J} = \langle \mathcal{B}^{\perp}, \mathcal{B} \rangle$

Proof: $\alpha^* : \mathcal{J} \to \mathcal{A}$ $\quad T \in \mathcal{J}, \quad \alpha^* T \in \mathcal{A}, \quad \alpha\alpha^* T \in \mathcal{J}$.

aim to show this

$$\underset{\underset{{}^{\perp}\mathcal{A}}{\cap}}{T'} \longrightarrow T \overset{(*)}{\longrightarrow} \underbrace{\alpha\alpha^* T}_{\underset{\mathcal{A}}{\cap}} \longrightarrow T'[1]$$

$$\begin{tikzcd}
\alpha^* T' \arrow[r] \arrow[d, phantom, "\parallel"] & \alpha^* T \arrow[r] \arrow[dr, bend right, "\mathrm{id}"'] & \alpha^* \alpha \alpha^* T \arrow[r] \arrow[d, phantom, "\parallel"] & \alpha^* T'[1] \\
0 & & \alpha^* T &
\end{tikzcd}$$

from axioms of triangulated categories.

becaus (*) came from adjunction

Definition: Suppose that $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \in \mathcal{T}$ (strictly full triangulated subcategories)

We say $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ is a semi-orthogonal decomposition if:

1) $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$.

2) For all $T \in \mathcal{T}$ there exists $0 = T_n \to T_{n-1} \to \dots \to T_1 \to T_0 = T$, such that $\mathrm{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$

Remark: Set $n = 2$. Then

$$0 = T_2 \to \underbrace{T_1 \to T_0}_{\text{you just take this part of the data to get the required triangle.}} = T \qquad \begin{array}{l} \mathrm{Cone}(\overset{0}{\overset{\|}{T_2}} \to \overset{T_1}{\overset{\|}{T_1}}) \in \mathcal{A}_2 \\ \mathrm{Cone}(T_1 \to T) \in \mathcal{A}_1 \end{array}$$

$$\underset{\mathcal{A}_2}{\overset{T_1}{\cap}} \to T \to \underset{\mathcal{A}}{\overset{\mathrm{Cone}(T_1 \to T)}{\cap}} \to T_1[1]$$

So this gives the triangle for the $n=2$ definition.

Exercise:

1) Prove functoriality of $T \mapsto \mathrm{Cone}(T_i \to T_j)$ $\forall\, i > j$.

2) $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle \Rightarrow \mathcal{T} = \langle \mathcal{A}_{\le i}, \mathcal{A}_{\ge i+1} \rangle$ where $\mathcal{A}_{\le i} = \langle \mathcal{A}_1, \dots, \mathcal{A}_i \rangle$ and $\mathcal{A}_{\ge i+1} = \langle \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle$.

3) $\mathcal{A}_i = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle$

4) If you have a semi-orthog. collection of triangulated subcategories, $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{T}$ and $\mathcal{A}_1, \dots, \mathcal{A}_i$ are left admissable and $\mathcal{A}_{i+1}, \dots, \mathcal{A}_n$ are right admissable.

Then $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_i, \underset{\mathcal{A}_{\le i}^{\perp} \cap {}^{\perp}\mathcal{A}_{\ge i+1}}{\underset{\|}{\mathcal{B}}}, \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle$.

Example: Take the Kronecker quiver with 2 verts and n arrows:

$$\underbrace{\bullet \;\; \begin{matrix} \longrightarrow \\ \longrightarrow \\ \vdots \\ \longrightarrow \end{matrix} \;\; \bullet}_{n \text{ arrows}} = Q_n$$

Then $D(\mathrm{Rep}\, Q_n) = \langle S_1, S_2 \rangle$. (more on this tomorrow)

(end.)

Lecture 2.

How to check that a subcategory is admissable?

Saturatedness.

To define this we need the notion of covariant cohomological functors.

A covariant cohomological functor $\mathcal{T} \to \text{mod-}k$ is a functor $H : \mathcal{T} \to \text{mod-}k$ such that for any distinguished triangle $T_1 \to T_2 \to T_3 \to T_1[1]$ in $\mathcal{T}$, we have that

$$\cdots \to H(T_1) \to H(T_2) \to H(T_3) \to H(T_1[1]) \to \cdots$$

is a long exact sequence.

Example: For each $T \in \mathcal{T}$ we have representable functors:

$$h_T : \mathcal{T} \to \text{mod-}k \quad ; \quad h_T(-) = \text{Hom}(T, -)$$

$$h^T : \mathcal{T}^{op} \to \text{mod-}k \quad ; \quad h^T(-) = \text{Hom}(-, T)$$

Definition: A category $\mathcal{T}$ (of finite type) is right (left) saturated if each contravariant (covariant) cohomology functor is representable.

Theorem:
- X a smooth proj. variety $\Rightarrow D^b(X)$ is saturated.
- A is a finite dimensional algebra of finite global dimension then $D^b(\text{mod-}A)$ is saturated.
- If $\mathcal{T}$ is saturated, and $\mathcal{A} \subseteq \mathcal{T}$ is left (right) admissable. Then $\mathcal{A}$ is saturated.
- $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$ with all $\mathcal{A}_i$ are saturated. Then $\mathcal{T}$ is saturated.
- If $\mathcal{T}$ has a strong generator then $\mathcal{T}$ is saturated.

This implies all the above.

vect. spaces

Exercise: Prove by hand (not using the above Theorem) that $D^b(\text{mod-}k)$ is saturated.

Proposition: Let $\mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{T}$ be a full triangulated subcategory.
If $\mathcal{A}$ is left (resp. right) saturated $\Rightarrow$ $\mathcal{A}$ is left (right) admissable.

Proof: $T \in \mathcal{T}$, $\quad T_{\mathcal{B}} \to T \to T_{\mathcal{A}} \to T_{\mathcal{B}}[1]$. where $\mathcal{B} = {}^{\perp}\mathcal{A} \ni T'$

$$\mathrm{Hom}(T', \alpha(-)) = 0$$
$$h_{T'} \circ \alpha = 0.$$

$h_T \circ \alpha : \mathcal{A} \to \text{mod-}k$

$\overset{\shortparallel}{\mathrm{Hom}}(T, \alpha(-)) \cong h_A$ for some $A \in \mathcal{A}$ (by saturatedness)

remark: can also prove by showing $\alpha^*(T) = A$

$\mathrm{Hom}(T, \alpha A) \cong h_A(A) = \mathrm{Hom}(A, A) \ni 1_A$

$$T' \to T \to \alpha A \to T'[1]$$

$$h_{T'} \circ \alpha \longrightarrow \underset{\underset{h_A}{\shortparallel}}{h_T \circ \alpha} \xrightarrow{\cong} h_{\alpha A} \circ \alpha \Rightarrow h_{T'} \circ \alpha = 0$$

$$h_{\alpha A} \circ \alpha = \mathrm{Hom}(\alpha A, \alpha(-)) = \mathrm{Hom}(A, -) = h_A(-).$$

$\square$

We now consider the question:

How can one construct a fully faithful functor $D^b(\text{mod-}k) \longrightarrow \mathcal{T}$?

$$D^b(\text{mod-}k) \cong \text{gr-mod-}k$$
$$\bigoplus_{i \in \mathbb{Z}} k^{n_i}[i]$$
$$k \longmapsto E.$$

In other words, let me take

$$E \in \mathcal{T} \rightsquigarrow \varphi_E : D^b(\text{mod-}k) \to \mathcal{T}$$
$$V^\bullet \longmapsto V^\bullet \otimes_k E.$$

To check whether a functor is fully faithful, it is useful to write down an adjoint functor.

$\mathrm{Hom}(\varphi_E(V^\bullet), F) = \mathrm{Hom}(V^\bullet \otimes E, F) = \mathrm{Hom}(V^\bullet, \mathrm{Hom}^\bullet(E, F))$

$\varphi_E^!(F) = \mathrm{Hom}^\bullet(E, F)$

$\varphi_E^*(F) = \mathrm{Hom}^\bullet(F, E)^\vee$

$\varphi_E^! \circ \varphi_E(V^\bullet) = \mathrm{Hom}^\bullet(E, V^\bullet \otimes E) = \mathrm{Hom}^\bullet(E, E) \otimes V^\bullet$

φ_E is fully faithful $\Leftrightarrow$ $\mathrm{Hom}^\bullet(E, E) = k$ ie $\mathrm{Hom}(E, E) = k$, $\mathrm{Ext}^i(E, E) = 0$ for $i \neq 0$.

Objects with this useful property have a name (next page):

Definition: We say an object $E \in \mathcal{T}$ is exceptional if $\mathrm{Hom}^\bullet(E, E) = k$.

Proposition: If $E \in \mathcal{T}$ is exceptional then

- $\mathcal{T} = \langle \Phi_E(D^b(\text{mod-}k)), {}^\perp E \rangle$
- $\mathcal{T} = \langle E^\perp, \Phi_E(D^b(\text{mod-}k)) \rangle$

[as an abuse of notation we sometimes write $\mathcal{T} = \langle E, {}^\perp E \rangle$ and $\mathcal{T} = \langle E^\perp, E \rangle$].

Definition: An exceptional collection in $\mathcal{T}$ is a collection $E_1, E_2, \dots, E_n$ of exceptional objects which is semi-orthogonal i.e $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ for $i > j$.

Proposition: For any exceptional collection $E_1, \dots, E_n$, there is a semi-orthogonal decomposition:

$$\mathcal{T} = \langle E_1, \dots, E_i, {}^\perp\langle E_1, \dots, E_i \rangle \cap \langle E_{i+1}, \dots, E_n \rangle^\perp, E_{i+1}, \dots, E_n \rangle.$$

Definition: An exceptional collection $E_1, \dots, E_n$ is <u>full</u> if it generates $\mathcal{T}$

i.e $\mathcal{T} = \langle E_1, E_2, \dots, E_n \rangle$.

Example: $D^b(\mathbb{P}^1)$. We claim that $(\mathcal{O}(-1), \mathcal{O})$ is a full exceptional collection.

Proof: Step 1.

$$\begin{aligned}
\mathrm{Hom}^\bullet(\mathcal{O}(k), \mathcal{O}(l)) &= H^\bullet(\mathbb{P}^1, R\mathcal{H}om(\mathcal{O}(k), \mathcal{O}(l))) \\
&= H^\bullet(\mathbb{P}^1, \mathcal{O}(k)^\vee \overset{\mathbb{L}}{\otimes} \mathcal{O}(l)) \\
&= H^\bullet(\mathbb{P}^1, \mathcal{O}(-k) \overset{\mathbb{L}}{\otimes} \mathcal{O}(l)) \\
&= H^\bullet(\mathbb{P}^1, \mathcal{O}(l-k))
\end{aligned}$$

You can try the same calculation on an arbitrary variety

(*)

For $l = k$, $\mathrm{Hom}(\mathcal{O}(k), \mathcal{O}(k)) = k \Rightarrow$ any line bundle on $\mathbb{P}^1$ is exceptional

(The same argument works on any connected X s.t $h^{0,q}(X) = 0$ for $q \geq 1$).

In particular any line bundle on a Fano variety is exceptional

(Cont. →)

Have to assume $\mathrm{char}(k) = 0$ to use Kodaira vanishing.

Step 2:

$$\mathrm{Hom}^\bullet(\mathcal{O}, \mathcal{O}(-1)) = H^\bullet(\mathbb{P}^1, \mathcal{O}(-1)) = 0.$$

If X is a Fano variety then

$$-K_X = r \cdot H \quad \text{for } r \in \mathbb{Z}_{>0} \text{ with } H \text{ ample.}$$

Then

$$H^\bullet(X, \mathcal{O}_X(-iH)) = 0 \quad \text{for } 1 \leq i \leq r-1.$$

$\Rightarrow (\mathcal{O}_X((1-r)H), \ldots, \mathcal{O}_X(-H), \mathcal{O}_X)$ is an exceptional collection in $D^b(X)$.

Step 3: Checking fullness (this is the most involved step typically)

It is enough to check that $\langle \mathcal{O}(-1), \mathcal{O} \rangle^\perp = 0$.

We give 3 proofs of this fact.

(a) (Very specific to $\mathbb{P}^1$)

It is enough to check that there are no indecomposables in $\langle \mathcal{O}(-1), \mathcal{O} \rangle^\perp = 0$.

$\mathcal{O}(k)$, $\mathcal{O}_{nx}$.

Exercise: Check that these are not in $\langle \mathcal{O}(-1), \mathcal{O} \rangle^\perp$.

(b) Method of Resolution of the diagonal.

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \\ & \mathbb{P}^1 \arrow[uu, "\text{diag.}"] \arrow[ul, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

he completes the bottom later

Speck

$$0 \to \mathcal{O}(-1,-1) \to \mathcal{O} \to \Delta_* \mathcal{O} \to 0$$

rotate the triangle:

$$\mathcal{O} \to \Delta_* \mathcal{O} \to \mathcal{O}(-1,-1)[1].$$

Derived pushforward → Speck followed by derived pullback ↗ Speck

$K \in D^b(\mathbb{P}^1 \times \mathbb{P}^1)$

"Fourier-Mukai functor"

$$\Phi_K : D^b(\mathbb{P}^1) \longrightarrow D^b(\mathbb{P}^1)$$
$$F \longmapsto Rq_*(K \overset{L}{\otimes} Lp^* F)$$

Apply this to get:

$$\Phi_{\mathcal{O}}(F) \longrightarrow \Phi_{\Delta_*\mathcal{O}}(F) \longrightarrow \Phi_{\mathcal{O}(-1,-1)[1]}(F) \quad \text{a distinguished triangle in } D^b(\mathbb{P}^1).$$

(later we conclude that

$\Phi_{\mathcal{O}}(F) \in \mathrm{Im}\, \varphi_{\mathcal{O}}$ $\quad \Phi_{\Delta_*\mathcal{O}}(F) = F$ and last term $\in \mathrm{Im}\, \varphi_{\mathcal{O}(-1)}$.)

$$\Phi_{\mathcal{O}}(F) = Rq_*(\mathcal{O} \overset{L}{\otimes} Lp^*F) = Rq_* Lp^* F = H^{\cdot}(\mathbb{P}^1, F) \otimes \mathcal{O}_{\mathbb{P}^1}.$$

$$\Phi_{\mathcal{O}(-1,-1)[1]}(F) = H^{\cdot}(\mathbb{P}^1, F(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)[1].$$

$$\Phi_{\Delta_*\mathcal{O}}(F) = Rq_*(R\Delta_*\mathcal{O} \overset{L}{\otimes} Lp^*F) = Rq_* R\Delta_*(\mathcal{O} \overset{L}{\otimes} L\Delta^* Lp^* F)$$
$$= R(q\circ\Delta)_* L(p\circ\Delta)^*F = R1_* L1^* F = F.$$

This completes the proof. (third method next lecture)

(end)

Lecture 3.

We have already discussed two proofs of fullness

(a) Classification of indecomposables.

(b) Resolution of the diagonal. This technique extends to $\mathbb{P}^n$ and to $Gr(k,n)$.

(c) In the third approach we will use the Euler sequence:

$$\mathcal{O}(-2) \xrightarrow{\binom{y}{-x}} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{(x,y)} \mathcal{O}$$

this shows that $\mathcal{O}(-2) \in \langle \mathcal{O}(-1), \mathcal{O} \rangle$. Then look at

$$\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)$$

Therefore $\mathcal{O}(-3) \in \langle \mathcal{O}(-2), \mathcal{O}(-1) \rangle \subseteq \langle \mathcal{O}(-1), \mathcal{O} \rangle$

By induction we see that $\mathcal{O}(-t) \in \langle \mathcal{O}(-1), \mathcal{O} \rangle$ for all $t \geq 0$. (*)

So it is enough to check that $\langle \mathcal{O}(-1), \mathcal{O} \rangle^{\perp} = 0$

$$\begin{array}{c} \langle \mathcal{O}(-1), \mathcal{O} \rangle^{\perp} \\ \cap\!\!\mid \text{ by } * \\ \mathcal{O}(-t)^{\perp} \end{array}$$

So it is enough to show that $\bigcap \mathcal{O}(-t)^{\perp} = 0$.

Now $\mathcal{H}$ means sheaf cohomology ie $F^{\bullet}$ - complex of coherent sheaves

$$\mathrm{Hom}^{\bullet}(\mathcal{O}(-t), F) = H^{0}(\mathcal{H}^{\bullet}(F)(t)) \quad \text{for } t \gg 0.$$

then $\mathcal{H}^{i}(F) = \dfrac{\mathrm{Ker}(F^{i} \to F^{i+1})}{\mathrm{Im}(F^{i-1} \to F^{i})}$.

If LHS $= 0 \Rightarrow \mathcal{H}^{\bullet}(F) = 0 \Rightarrow F = 0$, so we are done

Another version of (c):

(c)′ $0 \to \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_p \to 0$.

$$\mathcal{O}_p \in \langle \mathcal{O}(-1), \mathcal{O} \rangle \Rightarrow \langle \mathcal{O}(-1), \mathcal{O} \rangle^{\perp} \subseteq \mathcal{O}_p^{\perp}.$$

$\mathcal{O}_p^{\perp} = \{ F \mid \operatorname{sup} \mathcal{H}^{\bullet}(F) \not\ni p \}$.

$\therefore \bigcap \mathcal{O}_p^{\perp} = \{ F \mid \operatorname{supp} \mathcal{H}^{\bullet}(F) = \phi \} = 0$ So we are done.

Examples of full exceptional collections.

1) $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$, the Belinson exceptional collection.

Exercise: Prove this using methods (b), (c) and (c)'

2) $\underset{\substack{\cap \\ \mathbb{P}^{n+1}}}{D^b(\overset{\text{quadric}}{Q^n})} = \langle \mathcal{A}, \mathcal{O}(1-n), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$

In the case when k is algebraically closed of char $k = 0$,

then $\mathcal{A} = \begin{cases} \langle S \rangle & \text{if } n \text{ is odd} \quad (S \text{ is called a spinor bundle}) \\ \langle S^+, S^- \rangle & \text{if } n \text{ is even} \quad (S^+, S^- \text{ are spinor bundles}). \end{cases}$

For instance:

$n = 1$, $S = \mathcal{O}(-1)$

$n = 2$, $S^+, S^- = \mathcal{O}(-1,-2), \mathcal{O}(-2,-1)$.

$n = 4$, $S^{\pm}$ – twists of tautological bundles.

3) $D^b(Gr(k,n))$.

Let $\mathcal{U} \subseteq \mathcal{O}^{\oplus n}$ be the tautological bundle. $r(\mathcal{U}) = k$.

$\mathcal{U} \longrightarrow \mathcal{G}$ – $GL(k)$-bundle.

For any sequence of integers $\underbrace{(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)}_{\lambda}$ we have $\Sigma^\lambda(\mathcal{U})$ (Schur functor)

eg. $\lambda = (a, 0, \dots, 0) \Rightarrow \Sigma^\lambda(\mathcal{U}) = S^a\mathcal{U}$. ; $\Sigma^{(0,\dots,0)}\mathcal{U} = \mathcal{O}$

$\lambda = (\underbrace{1, \dots, 1}_{a}, \underbrace{0, \dots, 0}_{k-a}) \Rightarrow \Sigma^\lambda\mathcal{U} = \Lambda^a\mathcal{U}$; $\Sigma^{(1,\dots,1)}\mathcal{U} = \Lambda^k\mathcal{U} = \mathcal{O}(-1)$.

We have

$D^b(Gr(k,n)) = \langle \Sigma^\lambda(\mathcal{U}) \rangle_{n-k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0}$ ← this exceptional collection is indexed by Young diagrams.

the order is opposite to "C"

$n-k$

k

Let k be alg. closed of char $= 0$.

Conjecture: For any semisimple G and any parabolic $P \subset G$, there is a full exceptional collection in $D^b(G/P)$.

Known cases:

1) G is of type A

2) types B, C, D it is known for some G/P. Also known for flag varieties

3) G_2 – known

4) E_6, F_4 known for some G/P.

E_7, E_8 nothing is known.

Exercise: If $E_1, E_2, \dots, E_n$ is a full exceptional collection in an arbitrary triangulated category $\mathcal{T}$ then $\{[E_1], [E_2], \dots, [E_n]\}$ is a $\mathbb{Z}$-basis in $K_0(\mathcal{T})$

(So we get an obstruction to the existence of a full exceptional collection).

Exercise: Show that any full exceptional collection in $D^b(\mathbb{P}^1)$ is:

$$(\mathcal{O}(k)[s], \mathcal{O}(k+1)[t]) \quad \text{for some } k, s, t \in \mathbb{Z}.$$

Mutation of Exceptional Collections.

Suppose $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ then we can write down also $\mathcal{T} = \langle \mathcal{B}, \mathcal{C} \rangle$

where $\underset{\text{you should think of Hom as a type of pairing}}{\mathcal{A} = \mathcal{B}^\perp}$ $\qquad \mathcal{C} = {}^\perp\mathcal{B}$.

Let us write down an explicit equivalence. $\qquad \alpha, \beta, \gamma$

$\alpha^*, \beta^*, \beta^!, \gamma^!$

Proposition: $\mathcal{A} \cong \mathcal{C}$. More precisely $\gamma^!\alpha$ and $\alpha^*\gamma$ are mutually inverse

Proof: Start with an arbitrary object $T \in \mathcal{T}$. Then the composition triangle for T looks like:

$$\gamma\gamma^! T \longrightarrow T \longrightarrow \beta\beta^* T \qquad \text{where the morphisms are canonical.}$$

Set $T = \alpha A$

$$\gamma\gamma^! \alpha A \longrightarrow \alpha A \longrightarrow \beta\beta^*\alpha A \qquad\qquad \alpha^*\beta = 0 \text{ by semi-orthogonality}$$

$$\alpha^*\gamma\gamma^!\alpha A \longrightarrow \underset{\substack{\| \\ A}}{\alpha^*\alpha A} \longrightarrow \underset{\substack{\| \\ 0}}{\underbrace{\alpha^*\beta}}\beta^*\alpha A \quad \text{because } \mathcal{B} = \ker \alpha^*$$

therefore this $= A$ completing the proof. $\square$

Corollary:

a) $\alpha\alpha^* : \mathcal{T} \to \mathcal{T}$

1) vanishes on $\mathcal{B}$

2) induces an equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{A}$

b) $\gamma\gamma^! : \mathcal{T} \to \mathcal{T}$

1) vanishes on $\mathcal{B}$

2) induces an equivalence $\mathcal{A} \to \mathcal{C}$.

Definition: $\mathbb{L}_{\mathcal{B}} := \alpha\alpha^*$ is called the left mutation functor

$\mathbb{R}_{\mathcal{B}} := \gamma\gamma^!$ is called the right mutation functor.

Proposition: If $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_i, \dots, \mathcal{A}_n \rangle$ is a semi-orthogonal decomposition then for any i, the following are semi-orthogonal decompositions:

$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i, \mathbb{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1}), \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle$.

$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathbb{L}_{\mathcal{A}_i}(\mathcal{A}_{i+1}), \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle$.

⌈ In general we have the following triangle for right mutation functors

$\mathbb{R}_{\mathcal{B}} \to id \to \beta\beta^*$ (1)

and for left mutation functors

$\beta\beta^! \to id \to \mathbb{L}_{\mathcal{B}}$

So $\mathbb{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1}) \subseteq \langle \mathcal{A}_{i-1}, \mathcal{A}_i \rangle$

The semiorthogonality of $\mathcal{A}_i$, $\mathbb{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1})$ follows from the corollary ⌋

One category lies in $\mathbb{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1})$ and the other in $\mathcal{A}_i$, arguing using (1) ⌋.

Exercise:

1) Compute the mutations $\mathbb{L}_{\mathcal{O}(k)}\mathcal{O}(k+1)$ and $\mathbb{R}_{\mathcal{O}(k+1)}\mathcal{O}(k)$ on $\mathbb{P}^1$

2) Do all compositions of at most 3 mutations to $(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$ on $\mathbb{P}^2$.

Lecture 4.

Last lecture we discussed left and right mutations.

The left mutation is the cone of the morphism $\beta\beta^! \longrightarrow id_T$

ie $\quad \beta\beta^! \longrightarrow id_{\mathcal{T}} \longrightarrow \mathbb{L}_{\mathcal{B}}$

and similarly the right mutation is the fiber:

$$\mathbb{R}_{\mathcal{B}} \longrightarrow id_{\mathcal{T}} \longrightarrow \beta\beta^*$$

This gives a braid group action.

To see this, we will check that the braid group relations hold:

Assume WLOG $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle$

We will show that: $\quad R_1 R_2 R_1 = R_2 R_1 R_2.$

Proof:

$$\begin{array}{l} \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle \xrightarrow{R_1} \langle \mathcal{A}_2, \mathbb{R}_{A_1}(\mathcal{A}_2), \mathcal{A}_3 \rangle \xrightarrow{R_2} \langle \mathcal{A}_2, \mathcal{A}_3, \mathbb{R}_{\mathcal{A}_3}(\mathbb{R}_{\mathcal{A}_2}(\mathcal{A}_1)) \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow R_1 \\ \qquad\qquad\qquad\qquad\qquad \langle \mathcal{A}_3, \mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_2), \mathbb{R}_{\mathcal{A}_3}(\mathbb{R}_{\mathcal{A}_2}(\mathcal{A}_1)) \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ① \end{array}$$

also

$$\begin{array}{l} \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle \xrightarrow{R_2} \langle \mathcal{A}_1, \mathcal{A}_3, \mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_2) \rangle \xrightarrow{R_1} \langle \mathcal{A}_3, \mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_1), \mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_2) \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow R_2 \\ \qquad\qquad\qquad\qquad\qquad \langle \mathcal{A}_3, \mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_2), \mathbb{R}_{\mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_2)}(\mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_1)) \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ② \end{array}$$

and these coincide. – either by a direct argument

or using the following exercise:

Exercise: $\mathcal{B} = \langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ Then

$$\mathbb{R}_{\mathcal{B}} = \mathbb{R}_{\mathcal{B}_2} \mathbb{R}_{\mathcal{B}_1} \quad \text{and} \quad \mathbb{L}_{\mathcal{B}} = \mathbb{L}_{\mathcal{B}_1} \circ \mathbb{L}_{\mathcal{B}_2}.$$

Using this exercise:

$$① = \mathbb{R}_{\langle \mathcal{A}_2, \mathcal{A}_3 \rangle}(\mathcal{A}_1) \quad \text{and} \quad ② = \mathbb{R}_{\langle \mathcal{A}_3, \mathbb{R}_{\mathcal{A}_3}(\mathcal{A}_2) \rangle}(\mathcal{A}_1).$$

Serre Functors.

These are categorical interpretations of Serre Duality.

Definition: A Serre functor for $\mathcal{T}$ is an autoequivalence $S: \mathcal{T} \to \mathcal{T}$ such that

$$\mathrm{Hom}^\bullet(T_1, T_2)^\vee \cong \mathrm{Hom}^\bullet(T_2, ST_1).$$

Example: If $\mathcal{T} = D^b(X)$ with X smooth, projective then $S = - \otimes \omega_X[\dim X]$ is a Serre functor.

For instance: if $D^b(\text{mod-}k) = D^b(\operatorname{Spec} k) \rightsquigarrow S = \mathrm{id}$.

Proposition: If a Serre functor exists, then it is unique.

Proof: $S_1, S_2 : \mathcal{T} \to \mathcal{T}$. Then

$$\mathrm{Hom}(S_1T, S_2T) = \mathrm{Hom}(T, S_1T)^\vee = \mathrm{Hom}(T, T)^{\vee\vee} = \mathrm{Hom}(T, T) \ni \mathrm{id}_T. \qquad \square.$$

Proposition: If $\mathcal{T}$ is saturated then $S_{\mathcal{T}}$ exists.

Proof: $h_{T_1}^\vee \overset{?}{=} h^{ST_1}$ $\square$.

Proposition: $\langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$. Then

$$L_{\mathcal{A}_1} L_{\mathcal{A}_2} \dots L_{\mathcal{A}_{n-1}}(\mathcal{A}_n) = S_{\mathcal{T}}(\mathcal{A}_n) \text{ and}$$

$$R_{\mathcal{A}_n} R_{\mathcal{A}_{n-1}} \dots R_{\mathcal{A}_2}(\mathcal{A}_1) = S_{\mathcal{T}}^{-1}(\mathcal{A}_1).$$

Proof: $L_{\mathcal{A}_1} \dots L_{\mathcal{A}_{n-1}}(\mathcal{A}_n) = \langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle^\perp$

$$\mathcal{A}_n = {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle$$

$\mathcal{B} \subseteq \mathcal{T}$

$S_{\mathcal{T}}({}^\perp\mathcal{B}) = \mathcal{B}^\perp$ and

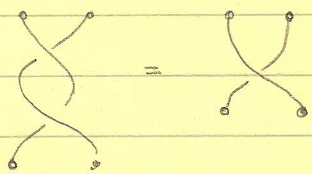
$S_{\mathcal{T}}^{-1}(\mathcal{B}^\perp) = {}^\perp\mathcal{B}.$ $\square$.

Now it is well known that $Z(Br_n) \cong \mathbb{Z}$. The generator is a distinguished central element of the braid group $(L_1 L_2 \cdots L_{n-1})^n$ and it just acts as the Serre functor S_J because:

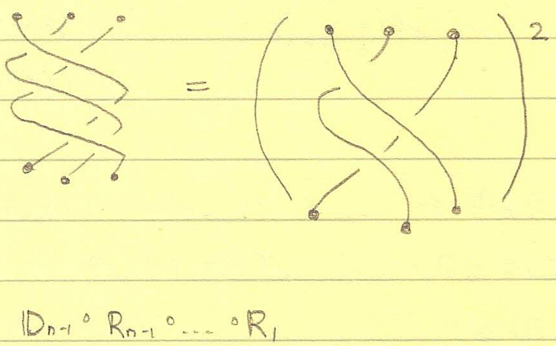
$$
\begin{array}{c}
J = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \\
\downarrow L_1 \cdots L_{n-1} \\
\langle S_J(\mathcal{A}_n), \mathcal{A}_1, \ldots, \mathcal{A}_{n-1} \rangle \\
\downarrow L_1 \cdots L_{n-1} \\
\langle S_J(\mathcal{A}_{n-1}), S_J(\mathcal{A}_n), \mathcal{A}_1, \ldots, \mathcal{A}_{n-2} \rangle \\
\downarrow \\
\vdots \\
\downarrow \\
\langle S_J(\mathcal{A}_1), S_J(\mathcal{A}_2), \ldots, S_J(\mathcal{A}_n) \rangle
\end{array}
$$

Similarly, $(R_{n-1}, \ldots, R_2 R_1)^n = S_J^{-1}$

Br_2:



Br_3:



Br_n

$\rotatebox[origin=c]{90}{$\in$}$

$\mathbb{D}_n = \mathbb{D}_{n-1} \circ R_{n-1} \circ \cdots \circ R_1$

$\mathbb{D}_n^2 = (\mathbb{R}_{n-1} \cdots \mathbb{R}_1)^n$

Exercise: $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle \xrightarrow{\mathbb{D}_n} \underbrace{\langle \mathcal{B}_n, \dots, \mathcal{B}_1 \rangle}_{}$ Right Dual Semi-Orthogonal Collection.

$$\mathcal{B}_i = {}^{\perp}\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle.$$

ie. $\mathrm{Hom}(\mathcal{B}_i, \mathcal{A}_j) = 0$ for $i \neq j$.

If $\langle E_1, \dots, E_n \rangle \xrightarrow{\mathbb{D}_n} \underbrace{\langle F_n, \dots, F_1 \rangle}_{\text{Right dual exceptional collection}}$

$$\mathrm{Hom}(F_i, E_j) = \begin{cases} 0 & \text{if } i \neq j. \\ k & \text{if } i = j. \end{cases}$$

This can be used to construct:

Resolution of the Diagonal.

X is smooth projective. Consider $D^b(X)$.

$D^b(X) = \langle E_1, E_2, \dots, E_n \rangle$

$D^b(X) = \langle F_n, F_{n-1}, \dots, F_1 \rangle$.

$X \times X \xhookleftarrow{\Delta} X$

Theorem: There is a chain of maps:

$$0 = D_n \to D_{n-1} \to \dots \to D_1 \to D_0 = \Delta_* \mathcal{O}_X, \text{ such that } \mathrm{Cone}(D_i \to D_{i-1}) = E_i \boxtimes F_i^\vee.$$

"sum of rank 1 vectors"

$K \in D^b(X \times X)$ $\qquad \Phi^K_{X \leftarrow X} = \Phi_K$.

$T \in D^b(X)$

$$0 = \Phi_{D_n}(T) \to \Phi_{D_{n-1}}(T) \to \dots \to \Phi_{D_1}(T) \to \Phi_{D_0}(T) = T.$$

$$\Phi_{D_i}(T) \to \Phi_{D_{i-1}}(T) \to \underset{\mathrm{Hom}^\bullet(F_i, T) \otimes E_i.}{\overset{}{\Phi_{E_i \boxtimes F_i^\vee}(T)}}$$

$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad \mathrm{Hom}^\bullet(F_i, T) \otimes E_i$.

Exercise: Compute the RDEC for:

1) $\langle \mathcal{O}(-n), \dots, \mathcal{O} \rangle$ on $\mathbb{P}^n$.

2) $\langle \Sigma^\lambda \mathcal{U} \rangle$ on $Gr(k, n)$.

Lecture 5.

Recall that last time we discussed Resolution of the Diagonal, which stated the following:

If $D^b(X) = \langle E_1, E_2, \dots, E_n \rangle$, then

$$0 = D_n \to D_{n-1} \to \dots \to D_1 \to D_0 = \Delta_* \mathcal{O}_X \in D^b(X \times X)$$

such that $\mathrm{Cone}(D_i \to D_{i-1}) = E_i \boxtimes F_i^\vee \quad (= R\mathcal{H}om(Lp_2^* F_i, Lp_1^* E_i))$

to prove this, we need the following bit of theory:

<u>Base Change for Semi-Orthogonal Decompositions.</u>

$$\begin{tikzcd} X \arrow[d, "p"'] & \ni x \arrow[d] \\ S & s \end{tikzcd} \qquad D^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$$

$$X \xrightarrow{f} S \qquad t \mapsto s$$

Definition: We say that $\mathcal{A} \subseteq D^b(X)$ is <u>S-linear</u> if $\mathcal{A} \overset{L}{\otimes} Lp^* F \subset \mathcal{A}$ for any $F \in D^b(S)$.

$$\begin{tikzcd} X_T \arrow[r, "\tilde{f}"] \arrow[d, "p_T"'] & X \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

<u>Theorem</u>: Assume that $D^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ – S-linear semi-orthogonal decomposition.
Let f be a base change such that T and X are <u>Tor-independent</u> over S.
means $\mathrm{Tor}_{>0}^{\mathcal{O}_{S,s}}(\mathcal{O}_{T,t}, \mathcal{O}_{X,x}) = 0$
Then $D^b(X_T) = \langle \mathcal{A}_{1T}, \mathcal{A}_{2T}, \dots, \mathcal{A}_{nT} \rangle$ is a T-linear semi-orthog. decomposition s.t
$R\tilde{f}_*(\mathcal{A}_{iT}) \subseteq \mathcal{A}_i$; $L\tilde{f}^*(\mathcal{A}_i) \subseteq \mathcal{A}_{iT}$.

Idea of Proof:

$$\langle L\tilde{f}^*(\mathcal{A}_i) \otimes Lp_T^* D^b(T) \rangle \cong \mathcal{A}_{iT}.$$

Remark: $\mathcal{A}_{iT}$ does not depend on the choice of embedding $\mathcal{A}_i \subseteq D^b(X)$.

eg. If $\mathcal{A}_i \cong D^b(Y)$ for some Y/S, then $\mathcal{A}_{iT} \subseteq D^b(Y_T) \mid \overset{\cap}{D^b(Y)}$

(such that Y and T are tor-independent over S.)

Proof of Resolution of the diagonal:

$D^b(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$

$$\begin{tikzcd} X \times X \arrow[rr] \arrow[dd] & & X \arrow[dd] & \\ & x \arrow[rr] \arrow[dl] & & \operatorname{Spec} k \arrow[dl] \\ X \arrow[rr] & & \operatorname{Spec} k & \end{tikzcd}$$

$D^b(X \times X) = \langle \mathcal{A}_{1X}, \dots, \mathcal{A}_{nX} \rangle$
$\quad \rotatebox{90}{$\in$}$
$\Delta_* \mathcal{O}_X \qquad \mathrm{Cone}(D_i \to D_{i-1}) = E_i \boxtimes F_i'$ (before $\mathcal{A}_i$ crossed to give this)

bit added later ---

$D^b(\operatorname{Spec} k) \cong \mathcal{A}_i = \langle E_i \rangle$

$D^b(X) \cong \mathcal{A}_{iX} = \{ E_i \boxtimes F \}$; $F \in D^b(X)$.

Exercise : Apply the resolution of the diagonal to check that $F_i' \cong F_i^{\vee}$.

Let $X \xrightarrow{p} S \ni s$ be a flat morphism.

Let $E_1, E_2, \dots, E_n \in D^b(X)$ be such that :

$E_{1s}, E_{2s}, \dots, E_{ns} \in D^b(X_s)$ is an exceptional collection for any $s \in S$.

Then :

1) $\varphi_{E_i} : D^b(S) \longrightarrow D^b(X)$, $F \longmapsto E_i \overset{L}{\otimes} Lp^*(F)$ is fully faithful, $\varphi_{E_i}(D^b(S))$ is admissable and S-linear.

2) $\varphi_{E_1}(D^b(S)), \dots, \varphi_{E_n}(D^b(S))$ is semi-orthogonal.

3) If $E_{1s}, E_{2s}, \dots, E_{ns}$ is full for all $s \in S$ then ($\Rightarrow$ then each X_s is smooth)

$$D^b(X) = \langle \varphi_{E_1}(D^b(S)), \dots, \varphi_{E_n}(D^b(S)) \rangle.$$

Proof : (See next page).

Proof:

1) $\varphi^!_{E_i}(F) = Rp_* R\mathcal{H}om(E_i, F) = Rp_*(E_i^\vee \overset{L}{\otimes} F)$

$$\varphi^!_{E_i} \circ \varphi_{E_i}(F) = Rp_*\left(E_i^\vee \overset{L}{\otimes} \left(E_i \overset{L}{\otimes} Lp^*(F)\right)\right)$$
$$= Rp_*\left(\left(E_i^\vee \overset{L}{\otimes} E_i\right) \overset{L}{\otimes} Lp^*(F)\right)$$
$$= Rp_*(E_i^\vee \otimes E_i) \overset{L}{\otimes} F \quad \text{by projection formula}$$

$$Lj_s^* Rp_*(E_i^\vee \otimes E_i) = Rp_{s*} Li_s^*(E_i^\vee \otimes E_i) = Rp_{s*}(E_{is}^\vee \otimes E_{is})$$
$$= H^\bullet(X_s, E_{is}^\vee \otimes E_{is}) = \mathrm{Hom}^\bullet(E_{is}, E_{is}) = \Bbbk$$

$$\begin{tikzcd} X \arrow[d, "p"'] & X_s \arrow[l, "i_s"'] \arrow[d, "p_s"] \\ S & s \arrow[l, "j_s"] \end{tikzcd}$$

take the cone of the first morphism

$$\mathcal{O}_S \xrightarrow{\cong} Rp_*(E_i^\vee \otimes E_i) \longrightarrow C$$

$$\mathcal{O}_X = Lp^* \mathcal{O}_S \longrightarrow E_i^\vee \otimes E_i$$

we have this canonical morphism which induces a canonical morphism $\mathcal{O}_S \to$ in the sequence above

$$\begin{tikzcd} Lj_s^* \mathcal{O}_S \arrow[r] \arrow[d, phantom, "\shortparallel"] & Lj_s^* Rp_*(E_i^\vee \otimes E_i) \arrow[r] \arrow[d, phantom, "\shortparallel"] & Lj_s^* C \arrow[d, phantom, "\shortparallel"] \\ \Bbbk \arrow[r, "id"] & \Bbbk & 0 \end{tikzcd}$$

$\Rightarrow C = 0$.

$\Rightarrow \varphi_{E_i}$ is fully faithful and right admissable.

This proves (1).

2) $\mathrm{Hom}(\varphi_{E_i}(F'), \varphi_{E_j}(F'')) = \mathrm{Hom}(F', \overbrace{\varphi^!_{E_i} \varphi_{E_j}}^{\text{want this composition} = 0}(F''))$

$$\varphi^!_{E_i} \circ \varphi_{E_j}(F) = Rp_*(E_i^\vee \otimes E_j) \otimes F$$

$$Lj_s^* Rp_*(E_i^\vee \otimes E_j) = \mathrm{Hom}^\bullet(E_{is}, E_{js}) = 0 \quad \text{for } i > j.$$

3) $D^b(X) = \langle \mathcal{A}, \varphi_{E_1}(D^b(S)), \ldots, \varphi_{E_n}(D^b(S)) \rangle$

$$F \in \mathcal{A} = \langle \cdots \rangle^\perp = \ker \varphi^!_{E_1} \cap \ldots \cap \mathrm{Ker}\, \varphi^!_{E_n}$$

$$0 = \varphi^!_{E_i}(F) = Rp_*(E_i^\vee \otimes F) \Rightarrow 0 = Lj_s^* Rp_*(E_i^\vee \otimes F) = \mathrm{Hom}^\bullet(E_{is}, F_s)$$
$$\Rightarrow \forall\, s \in S,\ F_s = 0 \Rightarrow F = 0.$$

This establishes (3).

Example:

$X = \mathbb{P}_S(\mathcal{E})$, $r(\mathcal{E}) = n+1$

$\mathcal{O}(-n), \mathcal{O}(1-n), \ldots, \mathcal{O}(-1), \mathcal{O}.$

$\Rightarrow D^b(X) = \langle \mathcal{O}(-n) \otimes D^b(S), \mathcal{O}(1-n) \otimes D^b(S), \ldots, \mathcal{O}(-1) \otimes D^b(S), D^b(S) \rangle$

$X \xrightarrow{p} S$ is a $\mathbb{P}^n$-fibration (Severi-Brauer varieties).

$\rightsquigarrow$ Azumaya Algebra (locally a matrix algebra)

In etale topology $\mathcal{U} \to S$

$$X_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{P}^n = \mathbb{P}_{\mathcal{U}}(\mathcal{E}_{\mathcal{U}})$$

$\mathcal{E}_{\mathcal{U}}$ is defined only up to a twist by a line bundle.

$\mathrm{End}(\mathcal{E}_{\mathcal{U}}) = \mathcal{E}_{\mathcal{U}}^{\vee} \otimes \mathcal{E}_{\mathcal{U}}$ is canonically defined and so glues to a sheaf of algebras on S. $\mathcal{R} \to$ azumaya algebra.

$Br(S) \ni \beta \leftarrow\!\!\!\sim \mathcal{R}$ $\qquad\qquad$ $\beta \in H^2_{et}(S, \mathcal{O}_S^{\times})$

$D^b(S, \beta) = D^b(\mathrm{coh}(S, \mathcal{R}))$

β-twisted sheaves

Theorem (Bernardara)

$\mathcal{O}(-i) \in D^b(X, p^*\beta^{-i})$ which is a $p^*\beta^{-i}$-twisted sheaf then

$$D^b(X) = \langle \mathcal{O}(-n) \otimes D^b(S, \beta^n), \mathcal{O}(1-n) \otimes D^b(S, \beta^{n-1}), \ldots, \mathcal{O}(-1) \otimes D^b(S, \beta), D^b(S) \rangle$$

(end).

Lecture 6.

Blowup

Let X be a (not necessarily smooth) variety and let $Z \subseteq X$ be a subvariety.

Let $\tilde{X} = Bl_Z(X)$

We have the blowup diagram:

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] & E \arrow[l, hook', "i"'] \arrow[d, "p"] \\ X & Z \arrow[l, hook'] \end{tikzcd}$$

If $Z \subseteq X$ is a locally complete intersection then: $E \cong \mathbb{P}_Z(\mathcal{N}_{Z/X})$.

Let $c = \operatorname{codim} Z = r(\mathcal{N}_{Z/X})$.

Theorem: (Orlov).

$$D(\tilde{X}) = \langle Ri_*(\mathcal{O}_{E/Z}(1-c) \otimes Lp^* D^b(Z)), \ldots, Ri_*(\mathcal{O}_{E/Z}(-1) \otimes Lp^* D^b(Z)), L\pi^* D^b(X) \rangle.$$

Example:

$$D(Bl_{pt}\mathbb{P}^2) = \langle \mathcal{O}_E(-1), \mathcal{O}(-2H), \mathcal{O}(-H), \mathcal{O} \rangle.$$

Exercise:

1) Mutate $\mathcal{O}_E(-1)$ to the right.

2) Write down the resolution of the diagonal and the associated Belinson spectral sequence.

Other Birational Transformations.

Conjecture:

a) If $X \dashrightarrow X'$ is a flip then $D(X') \hookrightarrow D(X)$.

b) If $X \dashrightarrow X'$ is a flop then $D(X') \cong D(X)$.

(b) Was proved by Bridgeland in the case dim = 3.

There is a more general conjecture:

Let $X \dashrightarrow X'$ be a birational isomorphism.

$$\begin{tikzcd}[column sep=small] X \arrow[rr, dashed, leftrightarrow] & & X' \\ & \tilde{X} \arrow[ul, "\pi"] \arrow[ur, "\pi'"'] & \end{tikzcd}$$

Conjecture: If $K_{\tilde{X}/X} - K_{\tilde{X}/X'} \geqslant 0$ then $D^b(X) \hookrightarrow D^b(X')$

Griffiths Component of $D^b(X)$

Let $D(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \rangle$ be a maximal semi-orthogonal decomposition.

Drop all $\mathcal{A}_i$ which can be embedded (as admissable subcategories) into $D^b(Z)$ with
this actually holds automatically

$$\dim Z \leqslant \dim X - 2 \quad \text{and} \quad Z \text{ smooth}$$

The Griffiths component is the set of components which are left.

Problems: 1) Why should such a maximal semi-orthog. decomposition exist?

2) Is the Griffiths component (as defined) independent of the choice of maximal semi-orthogonal decomposition.

1) should follow from the following:

Conjecture (Noetherian Property).

If X is smooth and projective then any descending chain $D^b(X) \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$ stabilizes.

We also remark that the Jordan-Hölder property is false, by the following example:

$$Q = \left\{ \bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet \overset{c}{\underset{d}{\rightrightarrows}} \bullet \;\middle|\; \begin{matrix} ca = 0 \\ db = 0 \end{matrix} \right\} \qquad D^b(Q) = \langle S_3, S_2, S_1 \rangle$$

$$E = \left\{ \mathbb{C} \overset{1}{\underset{0}{\rightrightarrows}} \mathbb{C} \overset{0}{\underset{1}{\rightrightarrows}} \mathbb{C} \right\} \qquad D^b(Q) = \langle E, {}^{\perp}E \rangle.$$

Other Examples of Semi-Orthogonal Decompositions.

$$S \xleftarrow{p} X \hookrightarrow \mathbb{P}_S(\mathcal{E})$$

$$\mathcal{L} \hookrightarrow S^2\mathcal{E}^\vee \qquad r(\mathcal{E}) = n.$$

Clifford Algebra

$$D^b(X) = \langle D^b(S, \mathcal{C}l_0), \mathcal{O}(3-n)\otimes Lp^*(D^b(S)), \dots, \mathcal{O}(-1)\otimes Lp^*(D^b(S)), Lp^*(D^b(S)) \rangle.$$

Homological Projective Duality.

$$\mathbb{P}(W) \xhookrightarrow{V_2} \mathbb{P}(S^2W) \supseteq H.$$

$$X \subseteq \mathbb{P}(W) \times \mathbb{P}(S^2W^\vee) \qquad \deg X = (2,1)$$

$$\downarrow$$

$$\mathbb{P}(S^2W^\vee)$$

So we get:

$$D^b(X) = \langle D^b(\mathbb{P}^2(S^2W^\vee), \mathcal{C}l_0), \underbrace{D^b(\mathbb{P}(S^2W^\vee)), \dots,}_{(n-2)\text{ times}} D^b(\mathbb{P}(S^2W^\vee)) \rangle.$$

Homological Projective Duality gives us:

$$D^b(X_L) = \langle D^b(L, \mathcal{C}l_0), \mathcal{O}(1+2d-n), \dots, \mathcal{O} \rangle \ ; \quad n \geq 2d$$

where

$$\mathbb{P}^{d-1} = L \subseteq \mathbb{P}(S^2W^\vee)$$

$$X_L = \mathbb{P}(W) \cap L^\perp \subseteq \mathbb{P}(S^2W)$$

A complete intersection of d quadrics in $\mathbb{P}(W)$

(end of course).