Derived categories of Fano varieties

Semiorthogonal decompositions

Let $\mathcal{T} = D^b(X)$, the bounded derived category of a smooth projective variety, or, more generally, nice (e.g. smooth and proper enhanced) triangulated category

Def A semiorthogonal decomposition (SOD)

$$\mathcal{T} = \langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n \rangle$$

is a collection of full triang. subcats $\mathcal{T}_i \subset \mathcal{T}$ s.t.

① $\mathrm{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0$ for $i > j$

② $\forall T \in \mathcal{T}$ $\exists$ $0 = T_n \to T_{n-1} \to \dots \to T_1 \to T_0 = T$ with $\boxed{\mathrm{Cone}\,(T_i \to T_{i-1}) \in \mathcal{T}_i}$

E.g. $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ if for any T $\exists$ $\boxed{T_{\mathcal{B}} \to T \to T_{\mathcal{A}}}$ (and $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$)

Ⓛ If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ then $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ in are unique and $T \mapsto T_{\mathcal{A}}$, $T \mapsto T_{\mathcal{B}}$ are functors (e.g. $\varphi : T \to T' \Rightarrow \varphi_{\mathcal{A}} : T_{\mathcal{A}} \to T'_{\mathcal{A}}, \varphi_{\mathcal{B}} : T_{\mathcal{B}} \to T'_{\mathcal{B}}$)

(Pf)

$$\begin{tikzcd} T_{\mathcal{B}} \arrow[r] \arrow[d, dashed, "\varphi_{\mathcal{B}}"'] & T \arrow[r] \arrow[d, "\varphi"] & T_{\mathcal{A}} \arrow[d, dashed, "\varphi_{\mathcal{A}}"] \\ T'_{\mathcal{B}} \arrow[r] & T' \arrow[r] & T'_{\mathcal{A}} \end{tikzcd}$$

The composition $T_{\mathcal{B}} \to T \xrightarrow{\varphi} T' \to T_{\mathcal{A}}$ is zero because $\boxed{\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0} \Rightarrow \exists \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}$. Moreover, they are unique $\Rightarrow$ functorial

~~This~~ In particular, taking $T' = T$, $\varphi = id$, and using this uniqueness we conclude that $T'_{\mathcal{A}} \cong T_{\mathcal{A}}$, $T'_{\mathcal{B}} \cong T_{\mathcal{B}}$ ∎

Ⓛ The functor $T \mapsto T_{\mathcal{A}}$ is left adjoint to $\mathcal{A} \hookrightarrow \mathcal{T}$
$T \mapsto T_{\mathcal{B}}$ is right adjoint to $\mathcal{B} \hookrightarrow \mathcal{T}$

(Pf) Apply the above to $T' \in \mathcal{A}$ and $T \in \mathcal{B}$ ∎

Rem If a functor $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ has left adjoint Φ^* or right adjoint $\Phi^!$ then Φ is fully faithful $\Longleftrightarrow$ $\Phi^*\Phi \xrightarrow{\sim} id$ or $id \xrightarrow{\sim} \Phi^!\Phi$.

Pf $\Phi^*\Phi F \to F \to F' \Rightarrow$

$$\begin{tikzcd} \mathrm{Hom}(F', G) \arrow[r] & \mathrm{Hom}(F, G) \arrow[r, "\sim"] \arrow[dr, "\Phi"'] & \mathrm{Hom}(\Phi^*\Phi F, G) \arrow[d, phantom, "\wr\wr"] \\ & & \mathrm{Hom}(\Phi F, \Phi G) \end{tikzcd}$$

$\Rightarrow \mathrm{Hom}(F', G) = 0$ $\forall G$ $\Rightarrow F' = 0$
$\Rightarrow \Phi^*\Phi \xrightarrow{\sim} id$; inverting the argument we get ($\Leftarrow$). ∎

(Cor) If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ the embedding functors $\alpha : \mathcal{A} \hookrightarrow \mathcal{T}$ and $\beta : \mathcal{B} \hookrightarrow \mathcal{T}$ have left adjoint α^*, right adjoint $\beta^!$; moreover $\alpha^*\alpha \xrightarrow{\sim} id_{\mathcal{A}}$; $id_{\mathcal{B}} \xrightarrow{\sim} \beta^!\beta$ and $\boxed{\beta\beta^! T \to T \to \alpha\alpha^* T}$ is the dec. triang

Def $\mathcal{A}$ is left admissible if $\exists \alpha^*$, right admissible if $\exists \alpha^!$
admissible if $\exists \alpha^*$ and $\alpha^!$

Prop If $\mathcal{A} \subset \mathcal{T}$ is l.adm then $\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$
If $\mathcal{A} \subset \mathcal{T}$ is r.adm then $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$

here $${}^{\perp}\mathcal{A} = \{ T \in \mathcal{T} \mid \mathrm{Hom}(T, \mathcal{A}) = 0 \}$$ $$\mathcal{A}^{\perp} = \{ T \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{A}, T) = 0 \}$$

Pf $$T' \to T \to \alpha\alpha^* T. \Rightarrow \mathrm{Hom}(\alpha\alpha^* T, \alpha A) \to \mathrm{Hom}(T, \alpha A) \to \mathrm{Hom}(T', \alpha A)$$ $$\mathrm{Hom}(\alpha\alpha^*T, \alpha A) \cong \mathrm{Hom}(\alpha^* T, A) \cong \mathrm{Hom}(T, \alpha A) \xrightarrow{\sim} \quad \text{hence } T' \in {}^{\perp}\mathcal{A} \ \blacksquare$$

Prop If $\mathcal{T}$ is smooth & proper (e.g. $\mathcal{T} = D^b(X)$, X sm & prop)
then $\mathcal{A} \subset \mathcal{T}$ is l.adm $\iff$ $\mathcal{A}$ is r.adm $\iff$ $\mathcal{A}$ is adm $\iff$ $\mathcal{A}$ is sm. & prop

Def $E \in \mathcal{T}$ is _exceptional_ if $\boxed{\mathrm{Ext}^\bullet(E,E) \cong \Bbbk}$

L E is exc. $\iff$ $\Phi_E : D^b(\Bbbk) \to \mathcal{T}$, $V^\bullet \mapsto V^\bullet \otimes E$ is fully faithful.
The image $\Phi_E(D^b(\Bbbk)) =: \langle E \rangle \subset \mathcal{T}$ for exc. E is admissible.

Pf ~~Ext~~$^\bullet(V^\bullet \otimes E, \mathcal{F}) \cong$ ~~Ext~~$^\bullet(V^\bullet,$ ~~Ext~~$^\bullet(E, \mathcal{F})) \Rightarrow \Phi_E^!(\mathcal{F}) := \mathrm{Ext}^\bullet(E, \mathcal{F})$ is r.adj.
Similarly, $\Phi_E^*(\mathcal{F}) = \mathrm{Ext}^\bullet(\mathcal{F}, E)^\vee$ is l.adj.
Now $\Phi_E^!(\Phi_E(V^\bullet)) =$ ~~Ext~~$^\bullet(E, V^\bullet \otimes E) \cong \mathrm{Ext}^\bullet(E,E) \otimes V^\bullet$;
this is $\cong V^\bullet \iff \mathrm{Ext}^\bullet(E,E) \cong \Bbbk$. $\blacksquare$

Ex If X is a _Fano variety_ (sm & proj, $-K_X$ ample)
then any _line bundle_ on X is exceptional

Pf $$\mathrm{Ext}^\bullet(\mathcal{L}, \mathcal{L}) \cong H^\bullet(X, \mathcal{L}^\vee \otimes \mathcal{L}) \cong H^\bullet(X, \mathcal{O}_X).$$
$H^0(X, \mathcal{O}_X) = \Bbbk$ because X is connected and reduced and proper.
$H^{>0}(X, \mathcal{O}_X) = H^{>0}(X, \mathcal{O}_X(\underbrace{-K_X}_{\text{ample}} + K_X)) = 0$ by _Kodaira vanishing_ $\blacksquare$

Cor X Fano $\Rightarrow D^b(X) = \langle \mathcal{O}_X, {}^{\perp}\mathcal{O}_X \rangle = \langle \mathcal{O}_X^{\perp}, \mathcal{O}_X \rangle$.

Def $E_1, \dots, E_n$ is an _exc. collection_ if E_i are exc, and $\mathrm{Ext}^\bullet(E_i, E_j) = 0$ $\boxed{\forall i > j}$

L $E_1, \dots, E_n$ is an exc.coll $\Rightarrow \mathcal{T} = \langle E_1, \dots, E_n, {}^{\perp}\langle E_1, \dots, E_n \rangle \rangle$
$$\dots = \langle E_1, \dots, E_i, {}^{\perp}\langle E_1, \dots, E_i \rangle \cap \langle E_{i+1}, \dots, E_n \rangle^{\perp}, E_{i+1}, \dots, E_n \rangle$$
$$\dots = \langle \langle E_1, \dots, E_n \rangle^{\perp}, E_1, \dots, E_n \rangle.$$

Ex If X is Fano and $\boxed{-K_X = mH}$ ($m :=$ _Fano index_)
then $(\mathcal{L}, \mathcal{L}(H), \dots, \mathcal{L}((m-1)H))$ is an exc. coll. $\forall$ line bdle $\mathcal{L}$

Pf Indeed, $\mathrm{Ext}^\bullet(\mathcal{L}(iH), \mathcal{L}(jH)) \cong H^p(X, \mathcal{O}_X((j-i)H)$
$$= H^p(X, \mathcal{O}_X((m+j-i)H + K_X) \Rightarrow H^{>0} = 0$$
(Serre duality!) $= H^{n-p}(X, \mathcal{O}_X((i-j)H + K_X)^\vee \Rightarrow H^{<n} = 0 \ \blacksquare$

Ex $X = \mathbb{P}^n \Rightarrow K_X = -(n+1)H \Rightarrow (\mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X(nH))$ is an exc. coll
(Beilinson's collection)

Def Exc. coll. $E_1, \dots, E_k$ is full if $\mathcal{T} = \langle E_1, \dots, E_k \rangle$
(i.e. any orthogonal is zero)

Th Beilinson's collection is full

Pf $0 \to \mathcal{O}(-n) \to \mathcal{O}(1-n)^{\oplus n} \to \dots \to \mathcal{O}(-2)^{\oplus \binom{n}{2}} \to \mathcal{O}(-1)^{\oplus n} \to \mathcal{O} \to \mathcal{O}_x \to 0 \quad \forall x \in \mathbb{P}^n$ ■

Ex $X = Q^n \Rightarrow K_X = -nH \Rightarrow (\mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X((n-1)H))$
This collection is not full.

Th [Kapranov] $D(Q^n) = \begin{cases} \langle S, \mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X((n-1)H) \rangle & \text{if } n \text{ is odd} \\ \langle S_+, S_-, \mathcal{O}_X, \mathcal{O}_X(H), \dots, \mathcal{O}_X((n-1)H) \rangle & \text{if } n \text{ is even} \end{cases}$

Here $S, S_\pm$ are spinor bundles ([Ottaviani])

e.g. $n = 1 \Rightarrow X \simeq \mathbb{P}^1 \Rightarrow S \simeq \mathcal{O}(-1)$
$n = 2 \Rightarrow X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow S_+ \simeq \mathcal{O}(-1,0)$, $S_- \simeq \mathcal{O}(0,-1)$
$n = 3 \Rightarrow X \simeq IGr(2,4) \Rightarrow S \simeq \mathcal{U}$.
$n = 4 \Rightarrow X \simeq Gr(2,4) \Rightarrow S_+ \simeq \mathcal{U}$, $S_- \simeq \mathcal{U}^\perp \left(:= (V/\mathcal{U})^\vee \right)$

Th [Kapranov]
$D(Gr(k,n)) = \langle \Sigma^\alpha \mathcal{U}^\vee \mid n-k \geq \alpha_1 \geq \dots \geq \alpha_k \geq 0 \rangle$

Conj G reductive, $P \subset G$ parabolic $\Rightarrow D(G/P)$ has a f.e.c.

Conj X sm proj cellular $\Rightarrow D(X)$ has a f.e.c

Relative situation $f : X \to S$ proper morphism (and flat)

~~Def~~ **Prop** 1) If $\forall s \in S$ $E|_{X_s}$ is exc. $\Rightarrow \Phi_E := f^*(-) \otimes E$ is ff & adm.
2) If $\forall s \in S$ $(E_1|_{X_s}, \dots, E_n|_{X_s})$ is exc. $\Rightarrow (\Phi_{E_1}(D(S)), \dots, \Phi_{E_n}(D(S)))$ is SO
3) If $\forall s \in S$ —"— is full $\Rightarrow D(X) = \langle \dots \rangle$.

Pf $\mathrm{Ext}^\bullet(f^*\mathcal{F} \otimes E, f^*\mathcal{G} \otimes E) \simeq \mathrm{Ext}^\bullet(f^*\mathcal{F}, f^*\mathcal{G} \otimes E \otimes E^\vee)$
$\simeq \mathrm{Ext}^\bullet(\mathcal{F}, f_*(f^*\mathcal{G} \otimes E \otimes E^\vee))$

$f_*(f^*\mathcal{G} \otimes (E \otimes E^\vee)) \simeq \mathcal{G} \otimes f_*(E \otimes E^\vee)$, so need $f_*(E \otimes E^\vee) \simeq \mathcal{O}_S$

$f_*(E \otimes E^\vee)|_s \simeq H^\bullet(X_s, (E \otimes E^\vee)|_{X_s}) \simeq \mathrm{Ext}^\bullet(E|_{X_s}, E|_{X_s}) \simeq \mathbb{k}$.
$\Rightarrow f_*(E \otimes E^\vee)$ is a line bundle. Moreover

$f^*\mathcal{O}_S = \mathcal{O}_X \longrightarrow E \otimes E^\vee \rightsquigarrow \mathcal{O}_S \to f_*(E \otimes E^\vee) \Rightarrow f_*(E \otimes E^\vee) \simeq \mathcal{O}_S$

⋮

■

Ex Orlov's projective bundle formula

If $\mathcal{V}$ is a v.b. of rank r then

$$D(\mathbb{P}_S(\mathcal{V})) = \langle f^*D(S)\otimes\mathcal{O},\ f^*D(S)\otimes\mathcal{O}(H), \dots, f^*D(S)\otimes\mathcal{O}((r-1)H)\rangle$$

where H is a relative hyperplane class.

Fourier-Mukai functors

$$\mathcal{K}\in D(X\times Y) \Rightarrow \boxed{\Phi_{\mathcal{K}}(\mathcal{F}) := p_{Y*}(p_X^*\mathcal{F}\otimes\mathcal{K})}$$

There ~~is a~~ a criterion to check that $\Phi_{\mathcal{K}}$ is fully faithful.
(Bondal-Orlov + Bridgeland)

Rem One can~~t~~ take any $Z \overset{f_X}{\underset{f_Y}{\rightrightarrows}} \begin{matrix} X \\ Y\end{matrix}$ and $\mathcal{K}\in D(Z)$ and define $\Phi_{\mathcal{K}}(\mathcal{F}) = f_{Y*}(f_X^*\mathcal{F}\otimes\mathcal{K})$

But $\exists f: Z\to X\times Y$ s.t. $f_X = p_X\circ f$, $f_Y = p_Y\circ f \Rightarrow \boxed{\Phi_{\mathcal{K}} \cong \Phi_{f_*\mathcal{K}}}$

Ex Orlov's blowup formula

$$\begin{tikzcd} X = \mathrm{Bl}_Z(Y) \arrow[d, "\pi"'] & E \arrow[l, hook', "i"'] \arrow[d, "p"] \cong \mathbb{P}_Z(\mathcal{N}_{Z/Y}) \\ Y & Z \arrow[l, hook'] \end{tikzcd}$$

codim $Z = c$

$$D(X) = \langle \pi^*D(Y),\ \underbrace{i_*p^*D(Z), \dots, i_*(p^*D(Z)\otimes\mathcal{O}_E((2-c)E))}_{c-1 \text{ copy of } D(Z)}\rangle$$

Conj C curve of $g\geq 2$, $r\geq 2$, $(d,r)=1$, $\deg\mathcal{L}=d$

$M = M_C(r,\mathcal{L})$ $[\dim = (r^2-1)(g-1)$, Fano$]$

$\mathcal{U}$ = universal bundle on $C\times M$

Then $\Phi_{\mathcal{U}}: D(C)\longrightarrow D(M)$ is fully faithful

Known: $r=g=2$. [BO]
: $r=2$, C hyperelliptic [FK]
: $r=2$, C any [Narasimhan]
: $g\geq r+3$, $d=1$ [Belmans-Mukhopadhyay]
: $g\geq r+3$, $(r,d)=1$ [Lee-Moon].

Derived categories of Fano varieties II

Del Pezzo varieties

II.1

Yesterday we discussed exceptional collections which give a way to construct SOD's with simplest components $D(\mathbb{C})$

How can one construct more complicated admissible subcats?

Let
$$\begin{tikzcd} & X\times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$
, $\mathcal{E} \in D(X\times Y) \rightsquigarrow \boxed{\begin{array}{l}\Phi_{\mathcal{E}} : D(X) \to D(Y) \\ \mathcal{F} \mapsto q_*(p^*\mathcal{F}\otimes\mathcal{E})\end{array}}$

(all functors are derived).

A bit more general
$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$
. $\mathcal{E}\in D(Z) \rightsquigarrow \Phi_{\mathcal{E}}(\mathcal{F}) = g_*(f^*\mathcal{F}\otimes\mathcal{E})$

This reduces to the previous: $\exists\, \varphi : Z \to X\times Y$ s.t. $p\circ\varphi = f,\ q\circ\varphi = g$

$\Rightarrow \boxed{\Phi_{Z,\mathcal{E}} \cong \Phi_{X\times Y,\varphi_*\mathcal{E}}}$, but sometimes ~~the~~ the second is more convenient.

These are Fourier-Mukai functors

An FM functor $\Phi_{\mathcal{E}}$ has both adjoints:

$$\Phi^!_{\mathcal{E}} \cong \Phi_{\mathcal{E}^\vee\otimes\omega_X[\dim X]}, \quad \Phi^*_{\mathcal{E}} \cong \Phi_{\mathcal{E}^\vee\otimes\omega_Y[\dim Y]},$$

so one can compute the compositions and check full faithfulness of $\Phi_{\mathcal{E}}$, hence admissibility of $\Phi_{\mathcal{E}}(D(X)) \subset D(Y)$.

Alternatively, one can use the following result.

Thm (Bondal - Orlov, ~ 00)

$\Phi_{\mathcal{E}}$ is fully faithful $\iff \forall\, x_1, x_2 \in X\ \ \mathrm{Ext}^\bullet(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) \xrightarrow{\sim} \mathrm{Ext}^\bullet(\mathcal{E}_{x_1}, \mathcal{E}_{x_2})$

i.e. ① $\forall\, x_1 \neq x_2$ $\quad \mathrm{Ext}^\bullet(\mathcal{E}_{x_1}, \mathcal{E}_{x_2}) = 0$

② $\forall\, x$ $\quad \mathrm{Ext}^\bullet(\mathcal{E}_x, \mathcal{E}_x) \cong \Lambda^\bullet(T_xX[-1])$

Ex (Orlov) Let $Y = \mathbb{P}_X(\mathcal{V})$, $r = \mathrm{rk}(\mathcal{V})$,
$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "id"] & \\ X & & Y \end{tikzcd}$$
. $\mathcal{E} = \mathcal{O}_{Y/X}(i) \Rightarrow \Phi_{\mathcal{E}}$ is f.f.

and $D(Y) = \langle \Phi_{\mathcal{O}(i)}(D(X)), \Phi_{\mathcal{O}(i+1)}(D(X)), \dots, \Phi_{\mathcal{O}(i+r-1)}(D(X)) \rangle$

Projective bundle formula

Ex Orlov Let $Y = Bl_Z(X)$, $E = \mathbb{P}_Z(\mathcal{N})$, $c = \mathrm{codim}\, Z$
$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "id"] & \\ X & \mathcal{O}_Y & Y \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr, "i"] & \\ Z & \mathcal{O}_{E/Z}(k) & Y \end{tikzcd} \Rightarrow$$

$$D(Y) = \langle \underset{\pi^*(D(X))}{\underset{\shortparallel}{\Phi_{\mathcal{O}_Y}(D(X))}}, \Phi_{\mathcal{O}_E}(D(Z)), \dots, \Phi_{\mathcal{O}_E(c-2)}(D(Z)) \rangle$$

Blowup formula

$\boxed{II.2}$

What can we say about derived categories of Fano varieties?

Let us go from small dimensions up

① dim = 1 $\Rightarrow X \cong \mathbb{P}^1 \Rightarrow D(X) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$.

② dim = 2 $\Rightarrow X$ - del Pezzo surface

10 types

- $X = \mathbb{P}^2 \Rightarrow D(X) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$
- $X = Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \Rightarrow D(X) = \langle \mathcal{S}_+, \mathcal{S}_-, \mathcal{O}, \mathcal{O}(1) \rangle$
- $X = Bl_{z_1,\dots,z_r}(\mathbb{P}^2)$, $1 \le r \le 8$, z_i in general position

$\Rightarrow D(X) = \langle \mathcal{O}, \mathcal{O}(h), \mathcal{O}(2h), \mathcal{O}_{E_1}, \dots, \mathcal{O}_{E_r} \rangle$

What about dim = 3? Fano 3-folds have been classified [Fano-Iskovskikh-Mori-Mukai]

Altogether, there are 105 deformation families, see $\boxed{\text{fanography.info}}$

So, it is hard (and doesn't make much sense) to discuss all of them, so we will concentrate on some.

~~Def A del Pezzo 3-fold is a Fano 3-fold such that $-K_X = 2H$ for some~~

Def The Fano index of X is the maximal $m \geq 1$ such that $-K_X = mH$, H ample.

Thm (Kobayashi-Ochiai) $m(X) \le \dim(X) + 1$. Moreover

- $m(X) = \dim(X) + 1 \Rightarrow X \cong \mathbb{P}^4$
- $m(X) = \dim(X) \Rightarrow X \cong Q^4$

In both cases we know that $D(X)$ has a f.e.c.

So, in dim = 3 we are left with 2 cases: **Rem** $\boxed{X \cap H = \text{del Pezzo surface } \text{~~of deg~~}}$

- $\boxed{m(X) = 2 \quad = \text{del Pezzo 3-folds}}$
- $m(X) = 1 \quad = \text{prime Fano 3-folds and more.}$

From classification: If $-K_Y = 2H$, define $d(Y) := H^3$

Then $1 \le d \le 7$ with 1 deformation family of each degree (except 6) and 2 deformation families of deg 6.

The easiest is $Y_3 \subset \mathbb{P}^4$ = cubic 3-fold

and $Y_4 \subset \mathbb{P}^5$ = intersection of two quadrics

Then $Y_5 = Gr(2,5) \cap \mathbb{P}^6$

$Y_6' = Fl(1,2;3) \cong (\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^7$

$Y_6'' = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$Y_7 = Bl_{pt}(\mathbb{P}^3)$

and $(Y_2 \xrightarrow{2:1} \mathbb{P}^3 \supset Z_4) =$ quartic in $\mathbb{P}(1,1,1,1,2)$

$Y_1 =$ sextic in $\mathbb{P}(1,1,1,2,3)$

For any $Y = Y_d$ we have $D(Y_d) = \langle \mathcal{B}_{Y_d}, \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle$
We will say that $\mathcal{B}_{Y_d}$ is the <u>nontrivial part</u> of $D(Y_d)$
What can we say about it?

$\boxed{d=4}$ <u>Theorem</u> (Bondal-Orlov) $\boxed{\mathcal{B}_{Y_4} \cong D(C_2)}$

<u>Sketch of proof</u>

Need a Fourier-Mukai functor $D(C_2) \longrightarrow D(Y_d)$,
so need a family of bundles on Y_d parameterized by C_2.

Recall $Y = Q_1 \cap Q_2 \subset \mathbb{P}^5 \Rightarrow$ pencil of quadrics $\{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$.
Among them 6 quadratic cones, other smooth.

- For each smooth Q_λ we have $S_{\lambda,+}|_Y$, $S_{\lambda,-}|_Y$ – two vector bundles of rank 2 on Y
- For each singular $Q_\lambda = C(\overline{Q}_\lambda)$ we have $\overline{S}_\lambda|_Y$ – one vector bundle of rank 2

Together they form a family of rank 2 bundles on Y parameterized by a double cover of $\mathbb{P}^1$ branched at 6 pts, i.e. by a (hyperelliptic) curve C of genus 2, i.e. a rank 2 bundle $\mathcal{E}$ on $C \times Y$

One can prove (e.g. by BO-criterion) that $\Phi_{\mathcal{E}}$ is f.f, it lands in ~~$\mathcal{B}$~~ $\mathcal{B}_Y$, and is essentially surjective ∎

<u>Generalizations</u>

- $D(Q_1 \cap Q_2) = \langle D(\Gamma), \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-2) \rangle$
  where Γ is a hyperelliptic curve or a root stack on $\mathbb{P}^1$
- $D(Q_1 \cap Q_2 \cap \ldots \cap Q_k)$: HPD for double Veronese
- $D(M_C(2,L))$ – the most fascinating subject, see [Tevelev-Torres]

Geometric consequences

II.4

① $F_1(Y) \cong \mathrm{Jac}(C)$

Pf $0 \to I_L \to \mathcal{O}_Y \to \mathcal{O}_L \to 0$, $I_L \in \mathcal{B}_Y$.

and under the equivalence $D(C) \cong \mathcal{B}_Y$
they correspond to line bundles (of ~~E~~ degree 0) on C.

② $F_2(Y) \cong \mathbb{P}^3_C$

Pf $\mathrm{Ext}^1(I_C, \mathcal{O}_Y(-1)) \cong \mathbb{C} \Rightarrow 0 \to \mathcal{O}_Y(-1) \to \mathcal{F}_C \to I_C \to 0$

$\mathcal{F}_C \in \mathcal{B}_Y$; in fact $\mathcal{F}_C \cong \Phi_{\mathcal{E}}(\mathcal{O}_x)$, $x \in C$.

Thus $F_2(Y) = \{(x, \varphi) \mid [\varphi] \in \mathrm{Hom}(\mathcal{O}_Y(-1), \Phi_{\mathcal{E}}(\mathcal{O}_x)) \cong \mathbb{C}^4\}$
up to rescaling

Exer Describe $F_3(Y)$ geometrically and categorically

$\boxed{d=5}$ $D(Y_5) = \langle \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}, \mathcal{O}(1) \rangle$
Thm (Orlov) hence $\mathcal{B}_Y = \langle \mathcal{U}, \mathcal{U}^\perp \rangle \cong D(\cdot \substack{\to\\\to\\\to} \cdot)$

Geometric consequences

① $F_1(Y) \cong \mathbb{P}^2$: $0 \to \mathcal{U} \to \mathcal{U}^\perp \to I_L \to 0$

② $F_2(Y) \cong \mathbb{P}^4$: $0 \to \mathcal{O}_Y(-1) \to \mathcal{F}_C \to I_C \to 0$,

$\mathcal{F}_C \cong \mathcal{U} \Rightarrow F_2(Y) \cong \mathbb{P}(\mathrm{Hom}(\mathcal{O}(-1), \mathcal{U})) \cong \mathbb{P}^4$.

Exer Describe $F_3(Y)$ ...

$\boxed{d \geq 6}$ $\mathcal{B}_Y$ has a full exc. collection

Exerci Prove this and describe F_1, F_2, F_3

$\boxed{d \leq 3}$ $\mathcal{B}_Y$ is a fractional Calabi-Yau category, (probably indecomposable)

$\mathbb{S}^3_{\mathcal{B}_{Y_3}} \cong [5]$ Sdim $= 5/3$

$\mathbb{S}^2_{\mathcal{B}_{Y_2}} \cong [4]$ (but $\mathbb{S} \not\cong [2]$) Sdim $= 2$ $(= 6/3)$

$\mathbb{S}^3_{\mathcal{B}_{Y_1}} \cong [7]$ Sdim $= 7/3$

Derived categories of Fano varieties III

Yesterday we discussed $Y_4 = Q_1 \cap Q_2 \subset \mathbb{P}^5$:

$D(Y_4) = \langle D(\Gamma_2), \mathcal{O}, \mathcal{O}(H) \rangle$ (in particular, $\mathcal{B}_Y \simeq D(\Gamma_2)$)

$F_1(Y) = \mathrm{Jac}(\Gamma_2)$: $I_L \in \mathcal{B}_Y \longleftrightarrow$ line bundles

$F_2(Y) \xrightarrow{\mathbb{P}^3} \Gamma_2$ $\quad 0 \to \mathcal{O}_Y(-1) \to F_C \to I_C \to 0$, $F_C \in \mathcal{B}_Y \longleftrightarrow \mathcal{O}_x$.

$F_3(Y) \xrightarrow{Gr(2,4)} \mathrm{Jac}(\Gamma_2)$

What about ~~the~~ other types of del Pezzo 3-folds

d=5 $\quad \underline{Y_5 = Gr(2,5) \cap \mathbb{P}^6}$

Thm (Orlov) $\quad D(Y) = \langle \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}, \mathcal{O}(H) \rangle$

In particular, $\mathcal{B}_Y \simeq \langle \mathcal{U}, \mathcal{U}^\perp \rangle$.

Moreover, $\mathrm{Ext}^1(\mathcal{U}, \mathcal{U}^\perp) \simeq \mathbb{C}^3$, so $\boxed{\mathcal{B}_Y \simeq D(\bullet \overset{\to}{\underset{\to}{\to}} \bullet)}$

Pf HPD for $Gr(2,5)$ gives

$D(Y) = \langle \mathcal{U}, \mathcal{O}, \mathcal{U}^*, \mathcal{O}(H) \rangle \Rightarrow D(Y) = \langle \mathcal{U}, \underset{\mathcal{U}^\perp}{\underset{\parallel}{\mathbb{L}_{\mathcal{O}}(\mathcal{U}^\vee)}}, \mathcal{O}, \mathcal{O}(H) \rangle$

■

Geometric consequences

① $F_1(Y) \simeq \mathbb{P}^2$: $\quad I_L \in \mathcal{B}_Y$, $\quad 0 \to \mathcal{U} \to \mathcal{U}^\perp \to I_L \to 0$

$\Rightarrow F_1(Y) \simeq \mathbb{P}(\mathrm{Hom}(\mathcal{U}, \mathcal{U}^\perp)) \simeq \mathbb{P}^2$

② $F_2(Y) \simeq \mathbb{P}^4$: $\quad 0 \to \mathcal{O}(-1) \to F_C \to I_C \to 0$, $\quad F_C \in \mathcal{B}_Y$,

$F_C \simeq \mathcal{U} \Rightarrow F_2(Y) \simeq \mathbb{P}(\underset{V^*}{\underset{\wr\wr}{\mathrm{Hom}(\mathcal{O}(-1), \mathcal{U})}}) \simeq \mathbb{P}^4$.

Exer Describe $F_3(Y)$

d ≥ 6 $\quad \overset{g=2}{Y_6' = Fl(1,2;3)}$, $\overset{g=3}{Y_6'' = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$, $\overset{g=2}{Y_7 = Bl_z \mathbb{P}^3}$

Exer • Prove that $\mathcal{B}_Y$ is generated by $2g+2$ exc. objects.

• Describe $F_1(Y)$, $F_2(Y)$ geometrically and in terms of these exc. collections

d ≤ 3 $\quad Y_3 \subset \mathbb{P}^4$; $Y_2 \underset{④}{\subset} \mathbb{P}(11112)$; $Y_1 \underset{⑥}{\subset} \mathbb{P}(11123)$

Thm If $Y \underset{d}{\subset} \mathbb{P}(w_0 w_1, \dots w_n)$ then (set $w = \sum w_i$, $c = \gcd(d, w)$)

$\mathbb{S}_{\mathcal{B}_Y}^{d/c} \simeq \left[\frac{(n+1)d - 2w}{c} \right] \Rightarrow \mathrm{Sdim} = n + 1 - 2\frac{w}{d} = \begin{cases} 5/3 & \text{for } Y_3 \\ 4/2 & \text{for } Y_2 \\ 7/3 & \text{for } Y_1 \end{cases}$

Not much can be said aside of this.

Exer $\quad F_2(Y_2) \xrightarrow{\mathbb{P}^2} F_1(Y_3)$

Prime Fano 3-folds

Def X is a prime Fano variety if $\mathrm{Pic}(X) = \mathbb{Z}K_X$

Classification: $(-K_X)^3 = 2g(X) - 2$ ⟵ anticanonical degree
$g(X)$ ← genus

- $2 \le g(X) \le 12$, $g(X) \neq 11$
- $g(X) \geq 4 \Rightarrow -K_X$ is very ample, $H^0 = g+2$
- $g(X) \geq 5 \Rightarrow X \subset \mathbb{P}^{g+1}$ is an intersection of quadrics
  $g(X) \leq 5 \Rightarrow$ complete intersection in a weighted projective space

Mukai Ket $g(X) \geq 4$; assume $\boxed{g = r \cdot s}$ $\boxed{r, s \geq 2}$

Then there is a globally generated stable vec. bun. $\mathcal{E}$ on X with $r(\mathcal{E}) = r$, $c_1(\mathcal{E}) = -K_X$, $h^0(\mathcal{E}) = r+s$.

Moreover, if $g(X) \geq 5$, such $\mathcal{E}$ is unique and exceptional and the pair $(\mathcal{O}_X, \mathcal{E})$ is exceptional

It defines $\boxed{X \to Gr(r, r+s)}$

Explicitly ① $g = 4 = 2 \cdot 2 \Rightarrow X \to Gr(2,4) = Q^4$
- either $X = Gr(2,4) \cap \{\text{cubic}\}$
- or $X = CQ^3 \cap \{\text{cubic}\}$

② $g = 6 = 2 \cdot 3 \Rightarrow X \longrightarrow Gr(2,5)$
- either $X = Gr(2,5) \cap \mathbb{P}^7 \cap Q$ ordinary GM 3-fold
- or $X = C(Gr(2,5) \cap \mathbb{P}^6) \cap Q$ special

③ $g = 8 = 2 \cdot 4 \Rightarrow X \longrightarrow Gr(2,6)$
$X = Gr(2,6) \cap \mathbb{P}^9$

④ $g = 9 = 3 \cdot 3 \Rightarrow X \longrightarrow Gr(3,6)$
$X = LGr(3,6) \cap \mathbb{P}^{10}$

⑤ $g = 10 = 2 \cdot 5 \Rightarrow X \longrightarrow Gr(2,7)$
$X = G_2Gr(2,7) \cap \mathbb{P}^{11}$

⑥ $g = 12 = 2 \cdot 6 = 3 \cdot 4 \Rightarrow X \longrightarrow Gr(2,8)$,
$X \longrightarrow Gr(3,7)$

$$\boxed{X = I_3Gr(3,7)}$$

⑦ $g = 7$ ⤳ replace $-K_X$ by $-2K_X$: $(-2K_X)^2 = 48$
⤳ $2g' - 2 = 48 \Rightarrow g' = 25 = 5 \cdot 5$
⤳ super-Mukai bundle, $X \to Gr(5,10)$,
$X = OGr_+(5,10) \cap \mathbb{P}^8$.

What can we say about derived categories?

Whenever there is a Mukai bundle ($g \in \{4,6,7,8,9,10,12\}$)

we have $D(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}^\vee \rangle$

What can we say about $\mathcal{A}_X$? $\overset{\shortparallel}{\mathcal{E}}$ - the Mukai bundle

$\boxed{g = 9}$ $X = LGr(3,6) \cap \mathbb{P}^{10}$.

$LGr(3,6) \subset \mathbb{P}^{13}$, $\mathbb{P}^{10} \subset \mathbb{P}^{13}$

$\Downarrow$

quartic hypersurface $= [LGr(3,6)]^\vee \subset \check{\mathbb{P}}^{13}$ $\mathbb{P}^2 \subset \check{\mathbb{P}}^{13}$

$\Rightarrow \Gamma_X := [LGr(3,6)]^\vee \cap \mathbb{P}^2$ = plane quartic curve, smooth! $\boxed{g(\Gamma_X) = 3}$

[Note that the construction of Γ_X is similar to the construction of C_2 from Y_4.]

Homological projective duality: $\boxed{\mathcal{A}_X \cong D(\Gamma_X)}$

$\|$ for $LGr(3,6)$ $\|$

Geometric consequences

- $F_1(X)$ is hard (curve of large genus) we skip it.
- $F_2(X) \xrightarrow{\mathbb{P}^1} \Gamma_X$
- $F_3(X) \cong Jac(\Gamma_X)$

Pf Very similar to Y_4: $D(X) = \langle \overset{D(\Gamma_X)}{\overset{\cong}{\mathcal{A}_X}}, \mathcal{U}, \mathcal{O} \rangle$,

① $0 \to F_{C_2} \to \mathcal{U} \to \mathcal{O}_X \to \mathcal{O}_{C_2} \to 0$

$F_{C_2} \in \mathcal{A}_X \longleftrightarrow \mathcal{O}_y \in \Gamma_X$. $\Rightarrow F_2(X) \xrightarrow{\varphi} \Gamma_X$,

$\varphi^{-1}(y) = \mathbb{P}(\underset{\mathbb{C}^2}{\underset{\cong}{\mathrm{Hom}(\mathcal{E}_y, \mathcal{U})}}) \cong \mathbb{P}^1$

② $I_{C_3} \in \mathcal{A}_X \longleftrightarrow$ line bundle

$\Rightarrow F_3(X) \cong Jac(\Gamma_X)$

∎

Rem In this case I don't know any argument for this without using derived cats.

$\boxed{g=10}$ $X = G_2Gr(2,7) \cap \mathbb{P}^{11}$

$G_2Gr(2,7) \subset \mathbb{P}^{13}$ $\qquad \mathbb{P}^{11} \subset \mathbb{P}^{13}$

$[G_2Gr(2,7)]^\vee \subset \check{\mathbb{P}}^{13}$ $\qquad \mathbb{P}^1 \subset \check{\mathbb{P}}^{13}$

$\shortparallel$ sextic hypersurface $\qquad \mathbb{P}^1 \cap [G_2Gr(2,7)]^\vee = 6$ points

$\rightsquigarrow$ $\Gamma_X \xrightarrow{2:1} \mathbb{P}^1$ branched at these 6 pts.

$g(\Gamma_X) = 2$

HPD for $G_2Gr(2,7)$: $\mathcal{A}_X \simeq D(\Gamma_X)$

Geometric consequences

- $F_1(X)$ is hard
- $F_2(X) \simeq \mathrm{Jac}(\Gamma_X)$
- $F_3(X) \xrightarrow{\mathbb{P}^2} \Gamma_X$

$\boxed{\text{Pf}}$ ① $I_{C_2} \in \mathcal{A}_X \longleftrightarrow$ line bundle $\Rightarrow F_2(X) \simeq \mathrm{Jac}(\Gamma_X)$

② $0 \to \mathcal{U} \to \mathcal{F}_{C_3} \to I_{C_3} \to 0$ $\qquad \mathcal{F}_{C_3} \in \mathcal{A}_X \longleftrightarrow \mathcal{O}_y$

$\mathrm{Hom}(\mathcal{U}, \mathcal{F}_{C_3}) \simeq \mathbb{C}^3 \Rightarrow \mathbb{P}^2$-bundle.

$\square$

$\boxed{g=7}$ $X = OGr_+(5,10) \cap \mathbb{P}^8$

$OGr_+(5,10) \subset \mathbb{P}^{15}$, $\qquad \mathbb{P}^8 \subset \mathbb{P}^{15}$

$OGr_-(5,10) \subset \check{\mathbb{P}}^{15}$ $\qquad \mathbb{P}^6 \subset \check{\mathbb{P}}^{15}$

$\rightsquigarrow$ $\Gamma_X = OGr_-(5,10) \cap \mathbb{P}^6$, $g(\Gamma_X) = 7$

HPD for $OGr_+(5,10)$: $\mathcal{A}_X \simeq D(\Gamma_X)$

Consequence : $F_2(X) \simeq S^2 \Gamma_X$

$0 \to \mathcal{E}_{y_1} \oplus \mathcal{E}_{y_2} \to \mathcal{U} \to \mathcal{O} \to \mathcal{O}_{C_2} \to 0$

$\boxed{\text{Exer}^*}$ $F_3(X) = ?$

Derived categories of Fano varieties

Prime Fano 3-folds with even g

IV.1

Yesterday: • $2 \leq g(x) \leq 12$, $g(x) \neq 11$

• $g(x) = r \cdot s \Rightarrow \boxed{D(x) = \langle \mathcal{A}_x, \mathcal{O}_x, \underset{\mathcal{U}_r^\vee}{\underset{\shortparallel}{\mathcal{E}_r}} \rangle}$ if $g(x) \geq 6$ or $g(x) = 4$ and general

• Three cases where $\mathcal{A}_x \cong D(\Gamma_x)$:

- $g(x) = 10 \Rightarrow g(\Gamma_x) = 2$
- $g(x) = 9 \Rightarrow g(\Gamma_X) = 3$
- $g(X) = 7$ $(\Rightarrow g' = 25) \Rightarrow g(\Gamma_x) = 7$.

Rem In all these cases X is rational and $\mathcal{J}(x) \cong \mathcal{J}ac(\Gamma_x)$

What about the remaining cases $(g \in \{4, 6, 8, \textcircled{10}, 12\})$, i.e., the cases of even genus $\Longrightarrow \textcircled{\mathcal{E}_2}$

(g=12) In this case we have several Mukai bundles.

Thm $\boxed{D(x) = \langle \mathcal{O}_x, \mathcal{E}_4, \mathcal{E}_3, \mathcal{E}_2 \rangle}$ or (simple mutation)

$D(x) = \langle \underset{\mathcal{U}_3}{\underset{\shortparallel}{\mathcal{E}_3^\vee}}, \underset{\mathcal{U}_3^\perp}{\underset{\shortparallel}{\mathcal{E}_4^\vee}}, \mathcal{O}_x, \mathcal{E}_2 \rangle$. $\boxed{\mathrm{Ext}^\circ(\mathcal{U}_3, \mathcal{U}_3^\perp) = \mathbb{C}^3}$

In particular, $\mathcal{A}_X \cong D(\circ \equiv\!> \circ)$ same quiver as in Y_5!

(Rem) X varies in a 6-dim moduli; Y is unique | $\mathcal{J}(X) = \mathcal{J}(Y) = 0$

(g=10) As we already know, $\mathcal{A}_x \cong D(\Gamma_x)$, $g(\Gamma_x) = 2$. | X, Y rat'l | $F_2(x) \cong \mathbb{P}^2 \cong F_1(Y$

Same category as Y_4!

(Rem) X with $\Gamma_x = \Gamma$ varies in a 7-dim moduli, Y with $\Gamma_Y = \Gamma$ is uniqe

$F_2(X) \cong$ | $\mathcal{J}(X) \cong \mathcal{J}ac(\Gamma) \cong \mathcal{J}(Y) \cong F_1(Y)$

X, Y rat'l

(g=8) $X = Gr(2,6) \cap \mathbb{P}^9$

(HPD) $Gr(2,6) \subset \mathbb{P}^{14}$ $\quad \mathbb{P}^9 \subset \mathbb{P}^{14}$

$\underset{\text{(Pfaffian) cubic hypersurface Pf}}{\underset{\shortparallel}{[Gr(2,6)]^\vee}} \subset \check{\mathbb{P}}^{14}$ $\quad \mathbb{P}^4 \subset \check{\mathbb{P}}^{14} \rightsquigarrow \underset{\text{cubic 3-fold, smooth}}{\underset{\uparrow}{Y_X}} := Pf \cap \mathbb{P}^4$

Thm $\boxed{\mathcal{A}_X \cong \mathcal{B}_{Y_X}}$

(Rem) X with given Y_X varies in 5-dim moduli
$\mathcal{J}(X) \cong \mathcal{J}(Y_X)$, $X \overset{bir}{\sim} Y_X$ nonrat'l
$F_2(X) \cong F_1(Y_X)$

To summarize: for $d \in \{3,4,5\}$, $\boxed{g = 2d + 2} \in \{8, 10, 12\}$

and any prime Fano X of genus g

$\exists$ unique del Pezzo Y_X of degree d

such that $\boxed{\mathcal{A}_X \cong \mathcal{B}_{Y_X}}$ $+ \mathcal{J}(X) \cong \mathcal{J}(Y)$, $X \overset{bir}{\sim} Y$
$+ F_2(X) \cong F_1(Y)$

Conjecture The same is true for $d \in \{1,2\} \Rightarrow g \in \{4,6\}$

IV.2

Pro: One can compare the numerical invariants:

$$K_0(\mathcal{A}_X)_{num} := K_0(\mathcal{A}_X)/\mathrm{Ker}(\chi) + \chi$$

$$\simeq \mathbb{Z}^2, \begin{pmatrix} 1-g/2 & -g/2 \\ 3-g & 1-g \end{pmatrix}$$

and $K_0(\mathcal{B}_Y)_{num}, \chi \simeq \mathbb{Z}^2, \begin{pmatrix} -1 & -1 \\ 1-d & -d \end{pmatrix}$

Exer $\exists$ an isomorphism $K_0(\mathcal{A}_X)_{num} \cong K_0(\mathcal{B}_Y)_{num}$ compatible with the Euler forms

Contra: If $\boxed{d=1 \text{ and } g=4}$ then $h^{1,2}(Y)=21$ $h^{1,2}(X)=20$

i.e. $J(Y) \not\cong J(X)$, $HH_0(\mathcal{B}_Y) \cong HH_0(\mathcal{A}_X) \Longrightarrow \mathcal{B}_Y \not\cong \mathcal{A}_X$

$\boxed{\text{Need a correction in this case}}$

In the case $\boxed{d=2 \text{ and } g=6}$ it is also false.

- [Bernardara - Tabuada] For general $X \nexists Y$ with $\mathcal{A}_X \cong \mathcal{B}_Y$
  They proved that $\mathcal{A}_X \cong \mathcal{B}_Y \Longrightarrow J(X) \cong J(Y)$ as ppav, and $J(X)$ vary in a 20-dim moduli, while $J(Y)$ — " — 19-dim moduli
- [Perry (+ Bayer?)] For general $Y \nexists X$ with $\mathcal{A}_X \simeq \mathcal{B}_Y$.
  Both categories are endowed with a natural involutive autoequivalence $\boxed{\tau := \mathbb{S} \circ [-2]}$ and one can identify

$$\mathcal{B}_Y/\tau \simeq D(\text{quartic K3})$$

$$\mathcal{A}_X/\tau \simeq \begin{cases} D(\text{GM surface}), & X \text{ is special} \\ \text{K3 of GM 4-fold}, & X \text{ is ordinary} \end{cases}$$

$\mathcal{A}_X \simeq \mathcal{B}_Y \Rightarrow \mathcal{A}_X/\tau \simeq \mathcal{B}_Y/\tau \Rightarrow D(S) \simeq \text{GMK3}$
but the (derived) period map of GM K3 cats can be controlled and it can be checked that the image contains no $D(S)$ for S with $\mathrm{Pic}(S) \cong \mathbb{Z}$.

- [Zhang] $\forall X$ and $\forall Y$ $\mathcal{A}_X \not\cong \mathcal{B}_Y$

Cannot explain yet.

$\boxed{\text{So, in this case we also need a correction}}$

Thm [K + Shinder]

$\mathcal{A}$

Let $d=2$, $g=6$

For any Y there is a family of (smooth & proper) triang. cats parameterized by a smooth pointed curve $(B, 0)$ ~~$\mathcal{A}\mathcal{X}$~~ (i.e. a B-linear cat $\mathcal{A}$ which is sm & prop $/B$) s.t.

① $\mathcal{A}_0 \cong \mathcal{B}_Y$

② $\forall\, b \neq 0$ $\quad \mathcal{A}_b \cong \mathcal{A}_{X_b}$ for some X_b of genus 6.

Pf. Let $L \subset Y$ be a (general) line (comes from a bitangent to $S \subset \mathbb{P}^3$)
$\Rightarrow \exists!\, L' \subset Y$ s.t. $|L \cap L'| = 2$

Consider $\mathrm{Bl}_{L'}(Y)$; then L is a $(-1,-1)$ curve which can be contracted (by the anticanonical map) to an ordinary double point.

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{L'}(Y) =: \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\pi", "|2H-E|"'] & L \arrow[l, hook'] \arrow[dr] & \\
L \arrow[r, hook] & Y & X & \{x\} \arrow[l, hook']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Cl}(X) \arrow[d] & = & \mathrm{Pic}(\tilde{X}) \arrow[d] \\
\mathrm{Cl}(X,x) & \cong & \mathbb{Z}
\end{tikzcd}$$

Then X is a (maximally non-factorial) Fano 3-fold of genus 6 with ODP $x \in X$, i.e. $\exists$ line bundle $\mathcal{L}$ on $\tilde{X}$ s.t. $\mathcal{L}|_L \cong \mathcal{O}_L(-1)$ (we take $\mathcal{L} := \mathcal{O}_{\tilde{X}}(-H)$)

The pair $(\mathcal{L} \otimes I_L, \mathcal{L})$ is exceptional and

$$\begin{tikzcd}
D(\tilde{X}) \arrow[d, "\pi_*"'] & = \langle \mathcal{L} \otimes I_L, \arrow[d] & \mathcal{L}, \arrow[dl, leftrightarrow] & \mathcal{C} \rangle \\
D(X) & = \langle P, & \mathcal{C} \rangle &
\end{tikzcd}$$

where P is a $\mathbb{CP}^\infty$-object.

[Namikawa] $\exists$ smoothing of X, i.e.

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d] & \mathcal{X} \arrow[d, "f"] & \mathcal{X}^\circ \arrow[l, hook'] \arrow[d] \\
\{0\} \arrow[r, hook] & B & B^\circ \arrow[l, hook']
\end{tikzcd}$$

s.t. $\mathcal{X}$ is smooth, $\mathcal{X}^\circ$ is smooth $/B^\circ$ and f is proper.

We prove that $i_* P$ is exceptional and define $\mathcal{D} \subset D(\mathcal{X})$ by $D(\mathcal{X}) = \langle i_* P, \mathcal{D} \rangle$

Then $\mathcal{D}_0 \cong \mathcal{C}$, $\mathcal{D}_b \simeq D(\mathcal{X}_b)$ for $b \neq 0$

The last step is to show that $(\mathcal{O}, \mathcal{E}_2)$ is defined in this family.

Consider $\mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{ev} \mathcal{O}_{L'}(1)$, define

$0 \to \mathcal{E}_Y \to \mathcal{O}_Y(H) \oplus \mathcal{O}_Y(H) \to \mathcal{O}_{L'}(2) \to 0$ not locally free!

Then
- $\exists\, \tilde{\mathcal{E}}$ on $Bl_{L'}(Y) = \tilde{X}$ s.t. $\mathcal{E}_Y \cong \rho_* \tilde{\mathcal{E}}$ <u>vector bundle!</u>
- $\tilde{\mathcal{E}}|_L \cong \mathcal{O}_L$ [illegible] $\oplus\, \mathcal{O}_L$ [illegible]
- $\exists\, \mathcal{E}_X$ on X s.t. $\tilde{\mathcal{E}} \cong \pi^* \mathcal{E}_X$
- $(\mathcal{O}_X, \mathcal{E}_X)$ is an exceptional pair.

Deformation ~~famil~~ theory allows ~~t~~ us to extend (after étale base change) this family to $\mathcal{X}$.

$\mathcal{E}_{\mathcal{X}}|_{\mathcal{X}_b}$ has the same parameters as $\mathcal{E}_{\mathcal{X}_b} \Rightarrow$ equal!

Thus $\mathcal{D} = \langle \mathcal{A}, f^*D(B) \otimes \mathcal{O}_{\mathcal{X}}, f^*D(B) \otimes \mathcal{E}_{\mathcal{X}} \rangle$

Then $\mathcal{A}_b = \mathcal{A}_{\mathcal{X}_b}$ by definition

On the central bundle we have $\mathcal{D}_0 = \langle \mathcal{A}_0, \mathcal{O}, \overset{\overset{\mathcal{E}}{\cong}}{\mathcal{E}} \rangle$

$\Rightarrow D(Bl_{L'} Y) = \langle \underline{I_L \otimes \mathcal{I}}, \underline{\mathcal{I}}, \mathcal{A}_0, \underline{\mathcal{O}}, \underline{\tilde{\mathcal{E}}} \rangle$

On the other hand

$D(Bl_{L'} Y) = \langle \mathcal{B}_Y, \underline{\mathcal{O}}, \underline{\mathcal{O}(H)}, \underline{\mathcal{O}_E}, \underline{\mathcal{O}_E(H)} \rangle$

A sequence of mutations identifies $\mathcal{A}_0 \simeq \mathcal{B}_Y$ ■

<u>Rem</u> Same argument works for $d \geq 3$, $g = 2d+2$. just replace L' by a smooth rat'l curve C_{d-1} of degree $d-1$ and take L to be its unique bisecant line.

Moreover, in these cases one can choose $\mathcal{A}$ in such a way that $\mathcal{A}_b \simeq \mathcal{A}_0$ for all $b \in B$

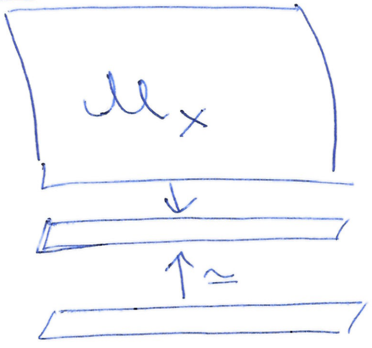
<u>Rem</u> For $d=1$, let $\overline{Y}$ be a 1-nodal dP3-fold, $Y \to \overline{Y}$ its small resolution (algebraic space) and C_0 its exceptional curve. Then the same construction works and produces a family $\mathcal{A} / (B, 0)$ with $\mathcal{A}_0 \simeq \mathcal{B}_Y := \langle \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle^\perp$ and $\mathcal{A}_b \simeq \mathcal{A}_{\mathcal{X}_b}$

The obtained $\mathcal{X}_b$ are all <u>general</u>!

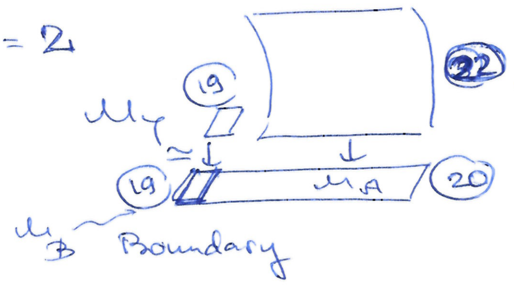
Picture with moduli spaces

$\boxed{d \geq 3}$

dim fibers = $g-3 = 2d-1$

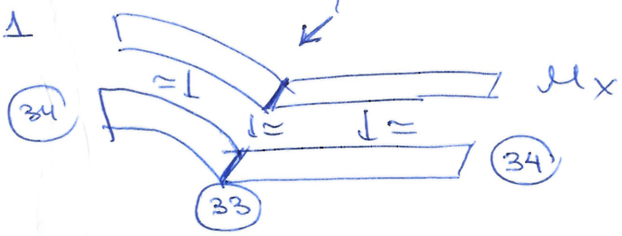


$d=2$



$d=1$

{nodal factorial Y} ⟷ {nodal mnf X}



In all cases $\boxed{\mathcal{M}_{(Y_d, C_{d-1})} \subset \partial \mathcal{M}_{X_g}}$

For $d \geq 3$ this boundary div. is horizontal
For $d \leq 2$ it is vertical

Geometric consequences

For a family $\mathfrak{X}$ as above

$\boxed{d \geq 3}$ $\Rightarrow$ $\exists\ \mathfrak{F} \to B$ s.t. $\mathfrak{F}_0 \cong F_1(Y)$, $\mathfrak{F}_b \cong F_2(\mathfrak{X}_b)$

$\boxed{d=2}$ $\Rightarrow$ $\exists\ \mathfrak{F} \to B$ s.t. $\mathfrak{F}_0 \cong Bl_{[4]}(F_1(Y))$, $\mathfrak{F}_b \cong F_2(\mathfrak{X}_b)$

$\boxed{d=1}$ $\Rightarrow$ something similar but not clear.

Derived categories of Fano varieties

So far, we talked about Fano 3-folds (mostly).
What about higher dimensions?

Starting from dimension 4, no classification is available.
only some lists... But it is clear that a classification
must be really long!

dim	1	2	3	4
# Fano	1	10	105	?
# Fano with $g=1$	1	1	17	?

Higher dimensional ~~[illegible]~~ del Pezzo varieties (coind = 2)

$d=7 \Rightarrow n \leq 3$

$d=6 \Rightarrow n \leq 4$; $\mathbb{P}^2 \times \mathbb{P}^2$, f.e.c. # 9 ; in $\mathcal{B}_Y$ # 6.

$d=5 \Rightarrow n \leq 6$; $Gr(2,5) \cap \mathbb{P}^{n+3}$ f.e.c. # $2n-2$; in $\mathcal{B}_Y$ # $n-1$

infinite series

$d=4 \Rightarrow n \in \mathbb{Z} \Rightarrow \begin{cases} \mathcal{B}_Y \simeq \mathbb{D}(C_{\frac{n+1}{2}}), & n \text{ odd.} \\ \mathcal{B}\text{f.e.c } \# n+5, & n \text{ even} \end{cases}$

$d=3 \Rightarrow n \in \mathbb{Z}$; $S^3 \simeq [n+2]$ (Sdim $= \frac{n+2}{3}$)

$\boxed{\begin{matrix} d=3 \\ n=4 \\ K3 \end{matrix}}$

$d=2 \Rightarrow n \in \mathbb{Z}$; $S^2 \simeq \tau[n+1]$, n even
$S \simeq \tau[2k]$, $n=4k-1$ (Sdim $= \frac{n+1}{2}$)
$S \simeq [2k+1]$, $n=4k+1$

$d=1 \Rightarrow n \in \mathbb{Z}$; S^6dim $= \frac{2n+1}{3}$

Higher dimensional Mukai varieties (coind = 3)

$D(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}_X, \mathcal{O}_X(1), \mathcal{E}_X(1), \ldots, \mathcal{O}_X(n-3), \mathcal{E}_X(n-3) \rangle$

$g=12 \Rightarrow n \leq 3$

$g=10 \Rightarrow n \leq 5$
$n=4 \Rightarrow \mathcal{A}_X =$ fec # 2
$n=5 \Rightarrow \mathcal{A}_X = 0$

$g=9 \Rightarrow n \leq 6$
$n=4 \Rightarrow \mathcal{A}_X =$ fec # 4
$n=5,6 \Rightarrow \mathcal{A}_X = 0$

$g=8 \Rightarrow n \leq 8$
$n=4 \Rightarrow \mathcal{A}_X =$ fec # 8
$n=5 \Rightarrow \mathcal{A}_X \simeq D(\Gamma_1)$
$n=6 \Rightarrow \mathcal{A}_X =$ fec # 4
$n=7 \Rightarrow \mathcal{A}_X =$ fec # 2
$n=8 \Rightarrow \mathcal{A}_X =$ fec # 3

$g=7 \Rightarrow n \leq 10$
$n=4 \Rightarrow \mathcal{A}_X =$ fec # 12
$n \geq 5 \Rightarrow \mathcal{A}_X = 0$

$g=6 \Rightarrow n \leq 6$
$n \in \{3,5\} \Rightarrow \mathcal{A}_X$ is of Enriques type
↗ $n \in \{4,6\} \Rightarrow \mathcal{A}_X$ is of K3 type

Constant in the fibers of the period maps.

The most interesting story that happens in dim 4 is when $\mathcal{B}_Y$ or $\mathcal{A}_X$ is of K3 type,

(i.e. $S \simeq [2]$, $HH_\bullet(\mathcal{A}) = \mathbb{C}[-2] \oplus \mathbb{C}^{22} \oplus \mathbb{C}[2]$)

There are a few examples:

ⓐ Cubic 4-folds

ⓑ GM 4-folds and 6-folds

ⓒ Verra 4-folds $= X \xrightarrow{2:1} \mathbb{P}^2 \times \mathbb{P}^2$ branched at $(2,2)$

ⓓ Debarre-Voisin 20-folds $= Gr(3,10) \cap H$.

ⓐ Cubic 4-folds

$$\mathcal{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$$

There are examples where $\mathcal{A}_X \simeq \mathcal{D}(S)$, S: K3

ⓐ • $\mathbb{P}^2 \sqcup \mathbb{P}^2 \subset X \Rightarrow Bl_{\mathbb{P}^2 \sqcup \mathbb{P}^2}(X) \cong Bl_S(\mathbb{P}^2 \times \mathbb{P}^2)$

$S = (2,1) + (1,2)$

$\Rightarrow X$ rat'l, $\mathcal{A}_X \simeq S$

ⓑ • X - Pfaffian, i.e. $X = [Gr(2,6)]^\vee \cap \mathbb{P}^5 \subset \mathbb{P}^{14}$

$S = Gr(2,6) \cap \mathbb{P}^8$

HPD: $\mathcal{A}_X \simeq \mathcal{D}(S)$; X rat'l

ⓒ • Other ~~exam~~ rat'l examples by [AHTVA], [RS]; in these case one should also have $\mathcal{A}_X \simeq \mathcal{D}(S)$, but no explicit construction of this sort is known

• $\mathbb{P}^2 \subset X \Rightarrow Bl_{\mathbb{P}^2}(X) \longrightarrow \mathbb{P}^2$ - quadric surface bundle with degeneration along a sextic curve

$\Rightarrow$ (quadric bundle formula) $\mathcal{A}_X \cong \mathcal{D}(S, \beta)$,

where $S \xrightarrow{2:1} \mathbb{P}^2$ ramified over the sextic

$\beta \in Br(S)$ - 2-torsion Brauer class

$\mathcal{D}(S,\beta)$ = twisted derived category

Among these there are countably many subfamilies with $\beta = 0$ ($\Rightarrow \mathcal{A}_X \simeq \mathcal{D}(S)$) and rat'l.

• X nodal at $x_0 \Rightarrow Bl_{x_0} X \cong Bl_S \mathbb{P}^4$, $S = (2) \cap (3)$

$\Rightarrow \mathcal{A}_X \simeq \mathcal{D}(S)/\langle K \rangle \Rightarrow$ can be included in a family

• X chordal $\Rightarrow$? limiting family

Conj X is rat'l $\Longleftrightarrow \mathcal{A}_X \simeq \mathcal{D}(S)$

ⓑ+ⓒ ~~Start~~ <u>GM 4-folds</u> : similar story

- ~~there are exam~~

$\mathcal{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}_X, \mathcal{O}_X(1), \mathcal{E}_X(1) \rangle$

- there are cases where $\mathcal{A}_X \simeq \mathcal{D}(S)$ and X rat'l
- there are cases where $\mathcal{A}_X \simeq \mathcal{D}(S, \beta)$
- in the nodal case $\mathcal{A}_X \cong \mathcal{A}_{X_{\text{Verra}}} \simeq \mathcal{D}(S_1, \beta_1) \simeq \mathcal{D}(S_2, \beta_2$

  (because of the two quadric surface bundle structures)

The same rationality conjecture

ⓑ ~~Ex~~ <u>GM sixfolds</u>

$\mathcal{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{E}_X, \dots, \mathcal{O}_X(3), \mathcal{E}_X(3) \rangle$

The same family of K3 cats:

some are $\simeq \mathcal{D}(S)$, some are $\simeq \mathcal{D}(S, \beta)$

But <u>all</u> X are rat'l ?

ⓓ <u>DK 20-folds</u>

Expected the same story, but so far no example of equivalence $\mathcal{A}_X \simeq \mathcal{D}(S)$ or $\mathcal{D}(S, \beta)$ is known (but see [Benedetti-Song] for interesting construction of (S, β) associated with special X)

<u>Are there other examples</u> : [Iliev-Manivel] <u>suggestions</u>

One new example: $(\mathbb{P}^3)^3 \cap \mathbb{P}^{62}$

<u>Küchle 4-folds</u> :

ⓔ ~~$X \simeq (\mathbb{P}^1)^5 \cap \mathbb{P}^{30}$~~ $X = \mathrm{Gr}(5,10) \rightsquigarrow \Lambda^2\mathcal{U}^\vee \oplus \Lambda^2\mathcal{U}^\vee \oplus \mathcal{O}(1)$

ⓕ $X = \mathrm{Gr}(5,8) \rightsquigarrow \Lambda^3\mathcal{U}^\vee \oplus \mathcal{O}(1)$

<u>Thm</u> . $\mathrm{Gr}(5,10) \rightsquigarrow \Lambda^2\mathcal{U}^\vee \oplus \Lambda^2\mathcal{U}^\vee \cong (\mathbb{P}^1)^5$

. $\mathrm{Gr}(5,8) \rightsquigarrow \Lambda^3\mathcal{U}^\vee \simeq \mathrm{Bl}_{v_2(\mathbb{P}^2)}(\mathbb{P}^5)$

<u>Cor</u> . $X_e \simeq (\mathbb{P}^1)^5 \cap \mathbb{P}^{30} \cong \mathrm{Bl}_S(\mathbb{P}^1)^4$, $\mathcal{A}_X \simeq \mathcal{D}(S)$

. $X_f \simeq \mathrm{Bl}_{v_2(\mathbb{P}^2)}(X_a) \Rightarrow \mathcal{A}_{X_f} \simeq \mathcal{A}_{X_a}$

<u>Rem</u> $\mathcal{A}_{X_{IM}} \simeq \mathcal{D}((\mathbb{P}^3 \times \mathbb{P}^3) \cap \mathbb{P}^{11})$

ⓖ $= (c5)$ $\boxed{X = \mathrm{Gr}(3,7) \rightsquigarrow \Lambda^2\mathcal{U}^\vee \oplus \Lambda^4\mathcal{U}^\vee \oplus \mathcal{O}(1)}$

<u>Fatighenti - Mongardi list</u> (23 cases $\exists$, nothing new)