Izvestiya: Mathematics **82**:4 694–751 *Izvestiya RAN: Ser. Mat.* **82**:4 53–114
DOI: https://doi.org/10.1070/IM8756

On linear sections of the spinor tenfold. I

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Abstract. We discuss the geometry of transverse linear sections of the spinor tenfold X, the connected component of the orthogonal Grassmannian of 5-dimensional isotropic subspaces in a 10-dimensional vector space endowed with a non-degenerate quadratic form. In particular, we show that if the dimension of a linear section of X is at least 5, then its integral Chow motive is of Lefschetz type. We discuss the classification of smooth linear sections of X of small codimension. In particular, we check that there is a unique isomorphism class of smooth hyperplane sections and exactly two isomorphism classes of smooth sections of codimension 2. Using this, we define a natural quadratic line complex associated with a linear section of X. We also discuss the Hilbert schemes of linear spaces and quadrics on X and its linear sections.

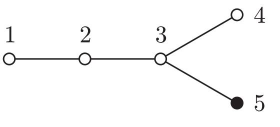
Keywords: spinor variety, linear sections, Chow motives, birational transformations, classification of algebraic varieties, Hilbert schemes.

§ 1. Introduction

1.1. Overview. The *spinor tenfold*

$$X=\mathrm{Spin}(10)/\mathbf{P}_5\subset\mathbb{P}^{15}$$

is one of the most interesting rational homogeneous spaces. Here $\mathrm{Spin}(10)$ is the simply connected covering of the special orthogonal group $\mathrm{SO}(10)$, and $\mathbf{P}_5$ is its parabolic subgroup associated with the last vertex of the Dynkin diagram D_5 (the black vertex in the picture below):



The spinor tenfold is classically represented as a connected component

$$X\cong\mathrm{OGr}_+(5,\mathrm{V})$$

of the isotropic Grassmannian $\mathrm{OGr}(5,\mathrm{V})$ for a non-degenerate quadratic form on a 10-dimensional vector space V. However, we note that the Plücker embedding $\mathrm{OGr}_+(5,\mathrm{V})\subset\mathrm{Gr}(5,\mathrm{V})\subset\mathbb{P}(\bigwedge^5\mathrm{V})$ corresponds to the square of the generator of the Picard group $\mathrm{Pic}(X)$.

This work is supported by the Russian Science Foundation under grant 14-50-00005.
AMS 2010 *Mathematics Subject Classification.* 14J40, 14J60, 14M17, 14C05, 14C15.

One of the most interesting features of the spinor tenfold X is its *projective self-duality*: the projective dual variety $X^\vee \subset \check{\mathbb{P}}^{15}$ is projectively isomorphic to X (here $\check{\mathbb{P}}^{15}$ is the dual projective space of $\mathbb{P}^{15}$, and these two spaces are projectivizations of the two half-spinor representations $\mathbb{S}$ and $\mathbb{S}^\vee$ of $\mathrm{Spin(V)}$). More canonically,

$$X^\vee \cong \mathrm{Spin(V)}/\mathbf{P}_4 \cong \mathrm{OGr}_-(5, \mathrm{V})$$

(hence $X^\vee$ is obtained from X by an outer automorphism of $\mathrm{Spin(V)}$ corresponding to the involution of the Dynkin diagram D_5, and one can also describe $X^\vee$ as the other connected component of the isotropic Grassmannian). The self-duality property is indeed very special. The only self-dual smooth projective varieties besides the spinor tenfold are the quadrics Q^n, the Segre varieties $\mathbb{P}^1 \times \mathbb{P}^n$ and the Grassmannian $\mathrm{Gr}(2, 5)$.

The projective self-duality of the spinor tenfold lifts to a higher homological level. In fact, it is also *homologically projectively self-dual* (see [1], § 6.2, [2], Theorem 5.5). This means that there is a nice relation (see Theorem 3.6) between the derived categories of coherent sheaves of the linear sections of X and $X^\vee$.

The goal of this paper is to initiate a systematic study of the geometry of linear sections

$$X_K = X \cap \mathbb{P}(K^\perp) \subset \mathbb{P}(\mathbb{S}) = \mathbb{P}^{15}.$$

Here $K \subset \mathbb{S}^\vee$ is a linear subspace and $K^\perp \subset \mathbb{S}$ is the orthogonal complement of K. We are mostly interested in smooth and dimensionally transverse linear sections of codimension at most 5. We describe the integral Chow motives of all these varieties, provide a classification in the cases of codimension 1 and codimension 2, discuss the most special sections of codimension 3 and introduce an important 'quadratic invariant' R_K of X_K, which will play an important role in subsequent papers.

Actually, a significant part of the results of this paper is known to experts, but the references are scattered (and some of the results are folklore) and use different approaches. For example, see [3]–[7]. Therefore we provide proofs of these results in an effort to keep the paper self-contained.

1.2. Smooth, complete and non-isotrivial families. Before explaining the content of the paper in detail, we mention an interesting property of the varieties X_K. By classical projective duality, in the case when $k = \dim K \leqslant 5$, the linear section X_K is smooth and dimensionally transverse if and only if the corresponding linear subspace $\mathbb{P}(K) \subset \mathbb{P}(\mathbb{S}^\vee) = \check{\mathbb{P}}^{15}$ possesses the property

$$X^\vee \cap \mathbb{P}(K) = \varnothing.$$

It follows that all intermediate linear sections $X_K \subset X_{K'} \subset X$ (sometimes referred to as *oversections* of X_K) are also smooth and dimensionally transverse. Moreover, this simple smoothness criterion has the following striking consequence.

Assume that B is a smooth projective variety and $\phi\colon B \to \mathrm{Gr}(k, \mathbb{S}^\vee)$ is a map such that

$$X^\vee \cap \mathbb{P}(K_b) = \varnothing$$

for every point $b \in B$, where $K_b = \phi(b)$ is the k-dimensional subspace of $\mathbb{S}^\vee$ associated with the point $b \in B$ under the map ϕ. Since $\operatorname{codim}_{\mathbb{P}(\mathbb{S}^\vee)} X^\vee = 5$, the assumptions above can easily be satisfied in the case when $\dim B+k-1<5$ (they hold generically in this case). Then every linear section $X_{K_b}\subset X$ is smooth and its codimension in X is equal to k. Consider the total family of these sections

$$\mathscr{X}_B := X\times_{\mathbb{P}(\mathbb{S})}\mathbb{P}_B(\phi^*(\mathcal{K}^\perp)),$$

where $\mathcal{K}\subset\mathbb{S}^\vee\otimes\mathscr{O}$ is the tautological subbundle on $\mathrm{Gr}(k,\mathbb{S}^\vee)$, and $\mathcal{K}^\perp\subset\mathbb{S}\otimes\mathscr{O}$ is the subbundle of its orthogonal complements. Then it follows that the morphism $\mathscr{X}_B\to B$ is smooth and, therefore, $\mathscr{X}_B\to B$ is a *complete family* of smooth projective varieties. One can easily choose k, B and ϕ in such a way that this family is not *isotrivial* (to do this, one must take $k\geqslant 2$). This gives us one of the rare known examples of a complete non-isotrivial family of smooth varieties.

1.3. Results. By homological projective duality, every smooth linear section X_K of X of codimension $k\leqslant 5$ comes with a full exceptional collection of vector bundles of length $2\dim(X)-4$. The existence of a full exceptional collection implies that the rational Chow motive of X_K is of Lefschetz type. However, it is unknown whether the existence of a full exceptional collection implies that the Chow motive with *integral* coefficients is of Lefschetz type (see [8] for some results in the 3-dimensional case).

Our first main result (Theorem 4.16) is a proof, by a geometric construction, that the integral Chow motive of X_K is of Lefschetz type. We actually show (Proposition 4.10) that the blow-up of the projective space $\mathbb{P}(K^\perp)$ along X_K is a Zariski piecewise-trivial fibration into projective spaces over the 8-dimensional quadric $\mathrm{Q}=\mathrm{Spin}(\mathrm{V})/\mathbf{P}_1$. Modifying this fibration (Proposition 4.14) to a projective bundle over a blow-up of Q, we conclude that the motive of X_K is a direct summand in a sum of Lefschetz motives, whence it follows immediately that it is itself a direct sum of Lefschetz motives. Another geometric argument (Corollary 7.6) proves that every section X_K is rational (it is expected, although not proved, that every smooth projective variety with a full exceptional collection is rational).

It may seem from the above that all smooth linear sections of X are uniform and boring. In the rest of the paper we show that this is far from being true, by exhibiting the rich and interesting geometry associated with them. Even more of this will appear in subsequent papers.

We first discuss hyperplane sections of X. It is well known (see § 2.3 in [6], or Corollary 4.2 and preceding references) that there are only two projective isomorphism classes of hyperplane sections: smooth and singular. We reprove this and give a convenient geometric description of hyperplane sections in both cases.

The description for the singular hyperplane section X_1' is as follows. We check that the singular locus of X_1' is a 4-space $\mathbb{P}^4\subset X$ and prove that the blow-up of X_1' along this 4-space is an explicit $\mathbb{P}^3$-bundle over $\mathrm{Gr}(2,5)$ (see Corollary 5.3 for details). In fact, we deduce this isomorphism from a more general result (Proposition 5.1) which identifies the blow-up of X along a 4-space and the blow-up of $\mathbb{P}^{10}$ along the Grassmannian $\mathrm{Gr}(2,5)$ (contained in a hyperplane $\mathbb{P}^9\subset\mathbb{P}^{10}$).

For the smooth hyperplane section X_1'' of X we similarly show that there is a unique 6-dimensional quadric Q^6 contained in X_1'' (Lemma 5.10) and that the blow-up of X_1'' along this quadric is isomorphic to a $\mathbb{P}^4$-bundle over a 5-dimensional quadric (Corollary 5.11, compare with Lemma 1.17 in [5]). Again, we deduce this from a more general result (Proposition 5.8) which identifies the blow-up of X along Q^6 and a $\mathbb{P}^4$-bundle over another 6-dimensional quadric.

Next, we consider smooth linear sections $X_K \subset X$ of codimension 2. We show that there are exactly two isomorphism classes of these. They are distinguished by the structure of the Hilbert schemes $F_p(X_K)$ of their linear p-dimensional subspaces. First, one has $F_4(X_K) = \varnothing$ for X_K of the first type and $F_4(X_K) \cong \mathbb{P}^1$ for the second (Proposition 6.1). Second, the scheme $F_1(X_K)$ is smooth for X_K of the first type and has a unique singular point for the second (Corollary 6.7).

We say that a section X_K of the second type is *special* and the line on X_K corresponding to the singular point of $F_1(X_K)$ is the *special line* of X_K. Geometrically, a special section X_K can be obtained by blowing up a quintic del Pezzo fourfold inside $\mathbb{P}^8$ (this fourfold is contained in a hyperplane $\mathbb{P}^7 \subset \mathbb{P}^8$) and then contracting the strict transform of this hyperplane (Proposition 6.1).

The existence of two isomorphism classes of smooth linear sections of codimension 2 on X has interesting geometric consequences. To describe them, we study the subvariety R_0 in the Grassmannian $\mathrm{Gr}(2, \mathbb{S}^\vee)$ of lines in $\mathbb{P}(\mathbb{S}^\vee)$ parametrizing the special linear sections of X. Consider its closure

$$R = \overline{R_0} \subset \mathrm{Gr}(2, \mathbb{S}^\vee).$$

We show (Lemma 6.13) that R is a *quadratic line complex*, that is, a hypersurface cut out on $\mathrm{Gr}(2, \mathbb{S}^\vee)$ by a quadric in the corresponding Plücker space $\mathbb{P}(\bigwedge^2 \mathbb{S}^\vee)$. We call R the *spinor quadratic line complex*.

The singular locus of R is shown (in Corollary 6.16) to be the variety of secant lines to $X^\vee$ (in particular, it follows that its codimension in R is equal to 7). In Lemma 6.17 we construct a nice resolution of singularities $\widetilde{R} \to R$, where $\widetilde{R}$ is isomorphic to a $\mathrm{Gr}(2, 8)$-bundle over $\mathrm{OGr}(3, \mathrm{V})$.

We use the spinor quadratic line complex R to define an interesting invariant for all linear sections of X of codimension at least 2. Given such an $X_K \subset X$, we define the *quadratic invariant* of X_K as

$$R_K := \mathrm{Gr}(2, K) \cap R \subset \mathrm{Gr}(2, \mathbb{S}^\vee).$$

It is easily shown (Lemma 7.1) that if $X_{K_1} \cong X_{K_2}$, then the associated quadratic invariants are also isomorphic: $R_{K_1} \cong R_{K_2}$. Special linear sections of codimension 2 can be characterized in terms of the quadratic invariant R_K: a section X_K is special if and only if R_K is non-empty (this is, of course, a tautological characterization). Associating with every linear section X_K its quadratic line complex R_K, we obtain a (rational) map from the moduli stack of linear sections $X_K \subset X$ of codimension k to the moduli stack of quadratic line complexes in $\mathrm{Gr}(2, k)$. It would be interesting to understand the relation between these moduli stacks.

We conclude the paper by discussing some properties of the varieties R_K. We show that $R_K \subset \mathrm{Gr}(2, K)$ is almost always a divisor (Lemma 7.11). The only exception (besides the special linear sections of codimension 2) is the case of a linear section $X_K \subset X$ of codimension 3 containing a 4-space, that is, the case when $F_4(X_K) \neq \varnothing$. We show (Proposition 7.7) that there is a unique isomorphism class of such varieties X_K and call them *very special*. Geometrically, a very special X_K can be obtained by blowing up a quintic del Pezzo threefold inside $\mathbb{P}^7$ (this threefold is contained in a hyperplane $\mathbb{P}^6 \subset \mathbb{P}^7$) and then contracting the strict transform of this hyperplane. This transformation and the birational isomorphisms (discussed above) between X and $\mathbb{P}^{10}$ or between a linear section of codimension 2 of X and $\mathbb{P}^8$ are particular cases of *special birational transformations of type* $(2,1)$, which were studied by Fu and Hwang in [7], Proposition 2.12.

1.4. Minifolds. Finally, we say some words about further results. Probably, one of the most interesting cases to be considered in subsequent papers is that of linear sections $X_K \subset X$ of codimension 5. These varieties are especially interesting since they are *minifolds* (see [9]), that is, their Hodge diamond is equal to that of $\mathbb{P}^5$ and their derived category of coherent sheaves is generated by the minimal possible number ($\dim X_K + 1 = 6$) of exceptional bundles. Besides X_K, the only known minifolds of dimension 5 are $\mathbb{P}^5$, Q^5, the adjoint G_2-Grassmannian and a hyperplane section of the Lagrangian Grassmannian $\mathrm{LGr}(3,6)$. Among them, X_K is the only minifold with non-trivial moduli. Moreover, besides two other examples in dimension 3 (the quintic del Pezzo threefold and prime Fano threefolds of genus 12), the only minifolds known to date are projective spaces and odd-dimensional quadrics.

One of our motivations in starting this project was the following strange observation. Consider the three 5-dimensional minifolds of index 3: the adjoint G_2-Grassmannian, a hyperplane section of the Lagrangian Grassmannian $\mathrm{LGr}(3,6)$, and a fivefold X_K. For each of them, the Hilbert scheme of lines is again a Fano variety of dimension 5. It is immediately seen that for the first of them, the Hilbert scheme of lines is isomorphic to the quadric Q^5. It is much less evident (see, however, [10], Corollary 6.7) that for the second of them, the Hilbert scheme of lines is isomorphic to the adjoint G_2-Grassmannian. In particular, the Hilbert scheme is a 5-dimensional minifold in both cases! Therefore one might hope that the Hilbert schemes $F_1(X_K)$ of lines on the fivefolds X_K give new examples of minifolds.

This appeared not to be the case, but still the geometry of $F_1(X_K)$ is quite interesting. We shall show in a forthcoming paper that there is a natural *Sarkisov link* between the Hilbert scheme $F_1(X_K)$ of lines and the quadratic line complex $R_K \subset \mathrm{Gr}(2,K)$, which in this case is a *Gushel–Mukai fivefold* (see [11]). Explicitly, there are natural $\mathbb{P}^1$-bundles over R_K and $F_1(X_K)$, related by a flop

$$\mathbb{P}^1_{R_K} \dashleftarrow\dashrightarrow \mathbb{P}^1_{F_1(X_K)}.$$

The flopping locus on both sides is a $\mathbb{P}^2$-bundle over the curve $F_2(X_K)$ (the Hilbert scheme of planes on X_K). This locus may also be described as the subvariety in $\mathrm{Gr}(3,K)$ of very special oversections of X_K of codimension 3. In particular, it follows that $F_1(X_K)$ is smooth if and only if R_K is smooth, and that the Hodge numbers of $F_1(X_K)$ and R_K are the same (see Proposition 3.1 in [15] for the Hodge numbers of R_K), whence $F_1(X_K)$ is not a minifold.

Actually, the Hilbert scheme $F_1(X_K)$ has already been considered in [4]. In particular, it was proved in Theorem 8.6 of [4] that $F_1(X_K)$ can be realized as the variety of sums of powers for a general cubic threefold (and some of the invariants of $F_1(X_K)$ have been calculated). It would be very interesting to understand the relation between the cubic threefold and the Gushel–Mukai fivefold associated with the minifold X_K.

We also note that the derived category of a smooth Gushel–Mukai fivefold has an interesting semiorthogonal decomposition ([16], Proposition 2.3) consisting of six exceptional vector bundles and an Enriques-type category ([16], Proposition 2.6). The relation between $F_1(X_K)$ and R_K suggests that the derived category $F_1(X_K)$ may have a semiorthogonal decomposition of the same type (thus the Enriques category is the obstruction for the minifold property). It would be interesting to find it.

1.5. Structure of the paper. The paper is organized as follows. In § 2 we give a short reminder of isotropic Grassmannians, spinor representations and bundles and prove a useful blow-up lemma (Lemma 2.5). In § 3 we introduce the spinor tenfold X and describe its Hilbert schemes of lines, planes, and other linear spaces. We also discuss a criterion for linear sections of X to be smooth, their semiorthogonal decompositions and consequences of these for the Chow motives with rational coefficients. In § 4 we prove that the blow-up of $\mathbb{P}^{15}$ along X is isomorphic to a $\mathbb{P}^7$-bundle over the 8-dimensional quadric Q and deduce many consequences of this result. Among these there is a description of the Hilbert schemes of quadrics on X and of the integral Chow motives of linear sections of X. In § 5 we prove that the blow-up of X along a 4-space is isomorphic to the blow-up of $\mathbb{P}^{10}$ along $\mathrm{Gr}(2,5)$ and extract from this a description of the singular hyperplane sections of X. We also prove that the blow-up of X along a 6-dimensional quadric is isomorphic to a $\mathbb{P}^4$-bundle over Q^6 and deduce from this a description of the smooth hyperplane sections of X as $\mathbb{P}^4$-bundles over Q^5. In § 6 we classify all smooth linear sections of X of codimension 2, define the spinor quadratic line complex R and briefly describe its geometry (in particular, we find its singular locus and construct a resolution of singularities). In § 7 we define the quadratic invariant R_K of a linear section $X_K \subset X$ and use it to answer some questions about the geometry of linear sections of X of codimension greater than 2.

1.6. Conventions. We work over a field $\Bbbk$, which is assumed to be algebraically closed of characteristic zero. By $\mathrm{Gr}(s,V)$ we denote the Grassmannian of s-dimensional vector subspaces of V. In particular, $\mathbb{P}(V) = \mathrm{Gr}(1,V)$ is the projectivization of a vector space V. Similarly, given a vector bundle $\mathscr{V}$ on a scheme S, we write

$$\mathbb{P}_S(\mathscr{V}) = \mathrm{Proj}\left(\bigoplus_{p=0}^{\infty} \mathrm{Sym}^p \mathscr{V}^\vee\right)$$

for the projectivization of $\mathscr{V}$. Let $\mathscr{O}_{\mathbb{P}_S(\mathscr{V})}(1)$ be the Grothendieck line bundle on $\mathbb{P}_S(\mathscr{V})$, normalized by the condition $\pi_*\mathscr{O}_{\mathbb{P}_S(\mathscr{V})}(1) \cong \mathscr{V}^\vee$. Its first Chern class is called the *relative hyperplane class* of $\mathbb{P}_S(\mathscr{V})$.

Given a vector bundle $\mathscr{E}$ over a scheme S and a point $s \in S$, we denote the fibre of $\mathscr{E}$ at s by $\mathscr{E}_s$. It is a vector space over the base field $\Bbbk$.

1.7. Acknowledgement. I am grateful to Dima Orlov and Yura Prokhorov for useful discussions and to Laurent Manivel for the reference [12]. When the first version of this paper was published, I was informed by Baohua Fu about similar results obtained in [5] and [7]; I am very grateful to him for these and other references. I would also like to thank the referee for his comments.

§ 2. Preliminaries

2.1. Isotropic orthogonal Grassmannians. Let V be a vector space over an algebraically closed field $\Bbbk$ of characteristic zero with a non-degenerate quadratic form $\mathbf{q}_V$. We denote by $\mathrm{OGr}(s,V) \subset \mathrm{Gr}(s,V)$ the subvariety of the Grassmannian that parametrizes $\mathbf{q}_V$-isotropic s-dimensional subspaces of V. In particular,

$$\mathrm{OGr}(1,V) = Q_V \subset \mathbb{P}(V)$$

is the smooth quadric defined by $\mathbf{q}_V$.

Of course, the isotropic Grassmannian is empty when $2s > \dim V$. Therefore we always assume that $2s \leqslant \dim V$. Every isotropic Grassmannian $\mathrm{OGr}(s,V)$ is a homogeneous space (or a disjoint union of homogeneous spaces) for the group $\mathrm{Spin}(V)$. Restricting ourselves by the case when $\dim V = 2m$ is even, so that $\mathrm{Spin}(V)$ is a group of Dynkin type D_m, we can write

$$\mathrm{OGr}(s,V) = \begin{cases} \mathrm{Spin}(V)/\mathbf{P}_s & \text{if } s \leqslant m-2, \\ \mathrm{Spin}(V)/\mathbf{P}_{m-1,m} & \text{if } s = m-1, \\ (\mathrm{Spin}(V)/\mathbf{P}_m) \sqcup (\mathrm{Spin}(V)/\mathbf{P}_{m-1}) & \text{if } s = m, \end{cases} \tag{2.1}$$

where $\mathbf{P}_I$ is the parabolic subgroup in $\mathrm{Spin}(V)$ corresponding to a set I of vertices of the Dynkin diagram. In particular, the maximal isotropic Grassmannian $\mathrm{OGr}(m,V)$ has two connected components, which are denoted by $\mathrm{OGr}_+(m,V)$ and $\mathrm{OGr}_-(m,V)$. We will use the convention

$$\mathrm{OGr}_+(m,V) = \mathrm{Spin}(V)/\mathbf{P}_m \quad \text{and} \quad \mathrm{OGr}_-(m,V) = \mathrm{Spin}(V)/\mathbf{P}_{m-1}.$$

Note that these varieties are abstractly isomorphic (an isomorphism is induced by an outer automorphism of $\mathrm{Spin}(V)$). The following property is useful when specifying the component containing a given isotropic space. Two maximal isotropic subspaces U', U'' belong to the same component if and only if $\dim(U' \cap U'') \equiv \dim U' \pmod 2$. We also mention the following identifications of isotropic Grassmannians for small m:

$$\begin{aligned} &\mathrm{OGr}_+(1,2) \cong \mathrm{Spec}(\Bbbk), \qquad && \mathrm{OGr}_+(2,4) \cong \mathbb{P}^1, \\ &\mathrm{OGr}_+(3,6) \cong \mathbb{P}^3, && \mathrm{OGr}_+(4,8) \cong Q^6, \end{aligned} \tag{2.2}$$

and similarly for $\mathrm{OGr}_-(m,2m)$. The last isomorphism is a manifestation of *triality*:

$$\mathrm{Spin}(8)/\mathbf{P}_1 \cong \mathrm{Spin}(8)/\mathbf{P}_3 \cong \mathrm{Spin}(8)/\mathbf{P}_4.$$

Besides the Grassmannians, we also need isotropic flag varieties. Given a vector space V of even dimension $2m$ with a non-degenerate quadratic form $\mathbf{q}_V$, and given a sequence $0 < s_1 < \cdots < s_r \leqslant m$ of integers, we write

$$\mathrm{OFl}(s_1, \ldots, s_r; V) = \mathrm{Fl}(s_1, \ldots, s_r; V) \times_{\mathrm{Gr}(s_r, V)} \mathrm{OGr}(s_r, V)$$

for the subvariety (in the flag variety) parametrizing $\mathbf{q}_V$-isotropic flags. When $s_r = m$, it has two connected components, which we denote by

$$\mathrm{OFl}_\pm(s_1, \ldots, s_{r-1}, m; V) = \mathrm{Fl}(s_1, \ldots, s_{r-1}, m; V) \times_{\mathrm{Gr}(m,V)} \mathrm{OGr}_\pm(m, V).$$

2.2. Spinor spaces and bundles. Concerning the content of this subsection, we refer to [13] in the case of quadrics and to [14], § 6, in general. Note that the conventions in the definition of a spinor bundle are opposite in these two references. Here we stick to the conventions used in [13].

We again assume that $\dim V = 2m$ and $\mathbf{q}_V$ is a non-degenerate quadratic form. Let ω_i be the fundamental weight of $\mathrm{Spin}(V)$ corresponding to the vertex i of the Dynkin diagram D_m, and let $\mathrm{V}^{\lambda}_{\mathrm{Spin}(V)}$ be the irreducible representation of $\mathrm{Spin}(V)$ with highest weight λ. Then

$$V \cong \mathrm{V}^{\omega_1}_{\mathrm{Spin}(V)}.$$

The irreducible representations

$$\mathbb{S} = \mathbb{S}_+ := (\mathrm{V}^{\omega_m}_{\mathrm{Spin}(V)})^\vee \quad \text{and} \quad \mathbb{S}_- := (\mathrm{V}^{\omega_{m-1}}_{\mathrm{Spin}(V)})^\vee,$$

which correspond to the last two weights, are called *half-spinor representations*. Their dimensions are equal to

$$\dim(\mathbb{S}) = \dim(\mathbb{S}_-) = 2^{m-1},$$

and they are swapped by outer automorphisms of $\mathrm{Spin}(V)$. Half-spinor representations are either self-dual or mutually dual depending on the parity of m. Explicitly,

$$\mathbb{S}^\vee \cong \mathbb{S}_{(-1)^m} \quad \text{and} \quad \mathbb{S}_-^\vee \cong \mathbb{S}_{(-1)^{m-1}}. \tag{2.3}$$

A similar construction can be used to define *spinor vector bundles* on the isotropic Grassmannians of V. Namely, for the maximal isotropic Grassmannians $\mathrm{OGr}_\pm(m, V)$, the spinor bundles are just the anti-ample generators of the Picard groups:

$$\begin{aligned} \mathcal{S}_1 = \mathcal{S}_{1,+} &:= \mathscr{O}(-\omega_m) \in \mathrm{Pic}(\mathrm{OGr}_+(m, V)), \\ \mathcal{S}_{1,-} &:= \mathscr{O}(-\omega_{m-1}) \in \mathrm{Pic}(\mathrm{OGr}_-(m, V)) \end{aligned}$$

(the integer subscripts stand for the ranks of the bundles). In what follows we denote $\mathcal{S}_{1,\pm}^\vee$ simply by $\mathscr{O}(1)$ and regard these line bundle as polarizations of $\mathrm{OGr}_\pm(m, V)$. Note that if $\mathscr{U}_\pm$ are the tautological bundles of rank m on $\mathrm{OGr}_\pm(m, V)$, then

$$\det \mathscr{U}_\pm^\vee \cong \mathscr{O}(2). \tag{2.4}$$

Similarly, when $s \leqslant m-2$ we consider the isotropic flag varieties $\mathrm{OFl}_\pm(s,m;V)$ with projections pr_s and $\mathrm{pr}_{m,\pm}$ to $\mathrm{OGr}(s,V)$ and $\mathrm{OGr}_\pm(m,V)$ respectively and define

$$\mathcal{S}_{2^{m-s-1}} = \mathcal{S}_{2^{m-s-1},+} := \bigl(\mathrm{pr}_{s*}(\mathrm{pr}_m^*(\mathcal{S}_1^\vee))\bigr)^\vee,$$
$$\mathcal{S}_{2^{m-s-1},-} := \bigl(\mathrm{pr}_{s*}(\mathrm{pr}_{m,-}^*(\mathcal{S}_{1,-}^\vee))\bigr)^\vee,$$

where the subscripts again specify the ranks. There is an analogue of the duality isomorphisms (2.3) for spinor bundles, but it involves a twist.

Lemma 2.1 (compare with [13], Theorem 2.8, [14], Corollary 6.5, Proposition 6.6). *Suppose that* $\dim V = 2m$. *If* $s \leqslant m-2$, *then the spinor bundles* $\mathcal{S}$ *and* $\mathcal{S}_-$ *on* $\mathrm{OGr}(s,V)$ *have the following properties*:

$$\mathcal{S}^\vee \cong \mathcal{S}_{(-1)^{m-s}}(1) \quad \textit{and} \quad \mathcal{S}_-^\vee \cong \mathcal{S}_{(-1)^{m-s-1}}(1).$$

In particular, $\det(\mathcal{S}_\pm) \cong \mathscr{O}(-2^{m-s-2})$.

Furthermore, there are identifications $\mathbb{S}_\pm^\vee = H^0(\mathrm{OGr}(s,V),\mathcal{S}_\pm^\vee)$. They induce canonical evaluation morphisms $\mathbb{S}_\pm^\vee \otimes \mathscr{O}_{\mathrm{OGr}(s,V)} \to \mathcal{S}_\pm^\vee$, which are surjective by the homogeneity of $\mathrm{OGr}(s,V)$. By duality, we obtain fibrewise monomorphisms $\mathcal{S}_\pm \hookrightarrow \mathbb{S}_\pm \otimes \mathscr{O}_{\mathrm{OGr}(s,V)}$. When $s=m$ this yields embeddings

$$\mathrm{OGr}_\pm(m,V) \to \mathbb{P}(\mathbb{S}_\pm). \tag{2.5}$$

When $m \in \{1,2,3\}$, we obtain the first three isomorphisms in (2.2). When $m=4$, we obtain the embedding $\mathrm{OGr}_\pm(4,8) \hookrightarrow \mathbb{P}^7$ as a quadric, thus giving the last isomorphism in (2.2). In general, (2.5) is called the *spinor embedding*. By (2.4), the Plücker embedding $\mathrm{OGr}_\pm(m,V) \hookrightarrow \mathrm{Gr}(m,V) \hookrightarrow \mathbb{P}(\bigwedge^m V)$ is the composition of the spinor embedding and the double Veronese embedding.

When $s=1$ (so that $\mathrm{OGr}(1,V) = Q = Q_V$), the embeddings $\mathcal{S}_\pm \to \mathbb{S}_\pm \otimes \mathscr{O}_Q$ extend to exact sequences.

Lemma 2.2 (see [13], Theorem 2.8). *If* $Q \subset \mathbb{P}(V)$ *is an even-dimensional quadric, then there are canonical exact sequences*

$$0 \to \mathcal{S} \to \mathbb{S} \otimes \mathscr{O}_Q \to \mathcal{S}_-(1) \to 0 \quad \textit{and} \quad 0 \to \mathcal{S}_- \to \mathbb{S}_- \otimes \mathscr{O}_Q \to \mathcal{S}(1) \to 0.$$

The situation is more complicated for $\mathrm{OGr}(s,V)$ when $s>1$. Instead of short exact sequences, one extends the spinor bundle to a filtration whose factors involve the spinor bundles $\mathcal{S}_\pm$ and the tautological vector bundle $\mathscr{U}_s$ on the isotropic Grassmannian.

Lemma 2.3 (compare with [14], Proposition 6.3). *Suppose that* $\dim V = 2m$. *If* $s \leqslant m-2$, *then the trivial vector bundle* $\mathbb{S} \otimes \mathscr{O}_{\mathrm{OGr}(s,V)}$ *has a natural filtration whose factors are isomorphic to*

$$\mathcal{S}_{(-1)^i} \otimes \bigwedge\nolimits^i \mathscr{U}_s^\vee, \qquad 0 \leqslant i \leqslant s.$$

Similarly, the trivial vector bundles $\mathbb{S} \otimes \mathscr{O}_{\mathrm{OGr}_+(m,V)}$ *and* $\mathbb{S}_- \otimes \mathscr{O}_{\mathrm{OGr}_+(m,V)}$ *have natural filtrations whose factors are isomorphic to*

$$\begin{aligned} &\mathcal{S} \otimes \textstyle\bigwedge^{2i} \mathscr{U}_m^\vee, && 0 \leqslant 2i \leqslant m, && \textit{for } \mathbb{S} \otimes \mathscr{O}_{\mathrm{OGr}_+(m,V)}, \\ &\mathcal{S} \otimes \textstyle\bigwedge^{2i+1} \mathscr{U}_m^\vee, && 0 \leqslant 2i+1 \leqslant m, && \textit{for } \mathbb{S}_- \otimes \mathscr{O}_{\mathrm{OGr}_+(m,V)}. \end{aligned}$$

Remark 2.4. By choosing a point $[U_m] \in \mathrm{OGr}_+(m,V)$ and trivializing the spinor line bundle $\mathcal{S}$ at this point, we obtain from Lemma 2.3 filtrations on the vector spaces $\mathbb{S}$ and $\mathbb{S}_-$ whose factors are isomorphic to $\bigwedge^{2i} U_m^\vee$ for the first and $\bigwedge^{2i+1} U_m^\vee$ for the second. Given a point $[U_{m,-}] \in \mathrm{OGr}_-(m,V)$, we similarly obtain filtrations on $\mathbb{S}$ and $\mathbb{S}_-$ whose factors are isomorphic to $\bigwedge^{2i+1} U_{m,-}^\vee$ for the first and $\bigwedge^{2i} U_{m,-}^\vee$ for the second.

We note that these filtrations are compatible with the duality between $\mathbb{S}$ and $\mathbb{S}^\vee$. In particular, when m is odd, the first term $\Bbbk = \bigwedge^0 U_{m,-}^\vee$ of the first filtration gives a vector in $\mathbb{S}^\vee = \mathbb{S}_-$ (which corresponds to the point $[U_{m,-}] \in \mathrm{OGr}_-(m,V)$ under the half-spinor embedding $\mathrm{OGr}_-(m,V) \hookrightarrow \mathbb{P}(\mathbb{S}_-) = \mathbb{P}(\mathbb{S}^\vee)$), and the corresponding hyperplane in $\mathbb{P}(\mathbb{S})$ corresponds to the projection $\mathbb{S} \to \bigwedge^m U_{m,-}^\vee$ onto the last factor of the second.

2.3. A blow-up lemma. The following result of Ein and Shepherd-Barron [17] will be used repeatedly to prove that certain birational isomorphisms are smooth blow-ups. Given a projective morphism $f\colon X \to Y$, we denote its relative Picard number by $\rho(f)$.

Lemma 2.5. *Assume that there is a commutative diagram*

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ Z \arrow[r, hook, "j"] & Y, \end{tikzcd}$$

where X, Y, Z *are smooth varieties,* $\operatorname{codim}_Y(Z) \geqslant 2$, E *is an irreducible divisor in* X, f *is a projective birational morphism,* p *is surjective, and* i *and* j *are closed embeddings. If* $\rho(f) = 1$, *then* f *is the blow-up of* Y *along* Z, $X \cong \mathrm{Bl}_Z(Y)$, *and* E *is the exceptional divisor of* f.

Proof. Since X and Y are smooth, the exceptional locus of f is a divisor. This divisor is irreducible because $\rho(f) = 1$. Since it contains E, we conclude that E is the exceptional locus of f. By Zariski's connectedness theorem, $f^{-1}(Z)$ lies in the exceptional locus of f, whence $f^{-1}(Z) = E$ set-theoretically, and $f\colon X \setminus E \to Y \setminus Z$ is an isomorphism. Thus Z is the base locus of f^{-1}. Since E is the set-theoretic pre-image of Z, Theorem 1.1 in [17] yields that f is a blow-up of Z and E is its exceptional divisor. □

The following assertion is also well known and very useful.

Lemma 2.6. *When* $f\colon X \to Y$ *is the projectivization of a vector bundle,* X *is smooth if and only if* Y *is smooth. When* $f\colon X \to Y$ *is the blow-up along* $Z \subset Y$

and Z is a locally complete intersection in Y, X is smooth if and only if both Y and Z are smooth.

Proof. The first part and one direction of the second part are evident. Hence we assume that $X = \mathrm{Bl}_Z(Y)$ is smooth and prove that Y and Z are smooth. Clearly, Y is smooth outside Z. Put $c = \mathrm{codim}_Y(Z)$ and let $f_1, \dots, f_c$ be the local equations of Z in Y. Then $X \subset Y \times \mathbb{P}^{c-1}$ is given by the equations $u_i f_j - u_j f_i = 0$, where $(u_1 : \dots : u_c)$ are the homogeneous coordinates on $\mathbb{P}^{c-1}$. These equations may be rewritten as $f_i - u_i f_c = 0$, $i = 1, \dots, c-1$, in the chart $u_c \neq 0$ (where we can put $u_c = 1$ and regard $u_1, \dots, u_{c-1}$ as coordinates). It follows that X is a locally complete intersection in $Y \times \mathbb{P}^{c-1}$. Since X is smooth, we conclude that $Y \times \mathbb{P}^{c-1}$ is smooth along X. In particular, it is smooth along the exceptional divisor of the blow-up, that is, along $Z \times \mathbb{P}^{c-1}$. Thus Y is smooth along Z and hence everywhere. Finally, the smoothness of Z can easily be deduced from that of X by comparing the Jacobian matrices corresponding to the equations of X in $Y \times \mathbb{P}^{c-1}$ and those of Z in Y. $\square$

Remark 2.7. One can also prove this lemma using derived categories. For example, if $Z \subset Y$ is a locally complete intersection, then the derived category $\mathbf{D}(X)$ of $X = \mathrm{Bl}_Z(Y)$ has a semiorthogonal decomposition with one component equivalent to $\mathbf{D}(Y)$, and the others equivalent to $\mathbf{D}(Z)$. If X is smooth, then the category $\mathbf{D}(X)$ is Ext-finite, whence its subcategories $\mathbf{D}(Y)$ and $\mathbf{D}(Z)$ are also Ext-finite and, therefore, Y and Z are smooth.

§ 3. The spinor tenfold and its linear sections

The spinor tenfold X and its projective dual variety $X^\vee$ (which is abstractly isomorphic to X) were described in the introduction. We first recall some notation introduced earlier.

3.1. Notation. We fix a vector space V of dimension 10 (in the notation of § 2 this means that $m = 5$) and a non-degenerate quadratic form $\mathbf{q}_\mathrm{V}$ on it. We will always identify V and $\mathrm{V}^\vee$ by means of $\mathbf{q}_\mathrm{V}$. Let $\mathbb{S}$ and $\mathbb{S}^\vee \cong \mathbb{S}_-$ be the corresponding 16-dimensional half-spinor representations (see (2.3)). Recall that

$$X := \mathrm{OGr}_+(5; \mathrm{V}) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_5 \subset \mathbb{P}(\mathbb{S}),$$
$$X^\vee := \mathrm{OGr}_-(5; \mathrm{V}) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_4 \subset \mathbb{P}(\mathbb{S}^\vee).$$

We usually write points of X as $[U_5]$, and points of $X^\vee$ as $[U_{5,-}]$, where $U_5, U_{5,-} \subset \mathrm{V}$ are the corresponding 5-dimensional isotropic subspaces. Accordingly, we denote the tautological vector bundles on X and $X^\vee$ by $\mathscr{U}_5$ and $\mathscr{U}_{5,-}$ respectively, often abbreviating this notation to $\mathscr{U}$ and $\mathscr{U}_-$. Furthermore, we put

$$\mathrm{Q} := \mathrm{OGr}(1, \mathrm{V}) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_1 \subset \mathbb{P}(\mathrm{V}),$$

which is a smooth 8-dimensional quadric, and

$$\mathscr{Q} := \mathrm{OFl}_+(1, 5; \mathrm{V}) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_{1,5} \subset \mathrm{Q} \times X,$$
$$\mathscr{Q}_- := \mathrm{OFl}_-(1, 5; \mathrm{V}) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_{1,4} \subset \mathrm{Q} \times X^\vee.$$

Then we have a diagram

$$\begin{tikzcd}[column sep=small] & \mathscr{Q} \arrow[dl] \arrow[dr] & & \mathscr{Q}_- \arrow[dl] \arrow[dr] & \\ X & & \mathrm{Q} & & X^\vee, \end{tikzcd} \tag{3.1}$$

whose outer (resp. inner) arrows are $\mathbb{P}^4$-fibrations (resp. fibrations into smooth 6-dimensional quadrics). To be more precise, on the one hand, we have isomorphisms

$$\mathscr{Q} \cong \mathbb{P}_X(\mathscr{U}), \qquad \mathscr{Q}_- \cong \mathbb{P}_{X^\vee}(\mathscr{U}_-) \tag{3.2}$$

and, on the other, there are canonical embeddings

$$\mathscr{Q} \hookrightarrow \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8), \qquad \mathscr{Q}_- \hookrightarrow \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_{8,-}) \tag{3.3}$$

in projectivizations of the spinor bundles. These embeddings are relative versions of the last embedding in (2.2). In most cases we regard $\mathscr{Q}$ and $\mathscr{Q}_-$ as families of 6-dimensional quadrics $\mathscr{Q}_v \subset X$ and $\mathscr{Q}_{v,-} \subset X^\vee$ parametrized by $v \in \mathrm{Q}$ (see also (3.4)).

Remark 3.1. We note for later use that the families of quadrics $\mathscr{Q}$ and $\mathscr{Q}_-$ in (3.1) have the following interpretation. Let $\mathscr{Q}_v$ and $\mathscr{Q}_{v,-}$ be the fibres of $\mathscr{Q}$ and $\mathscr{Q}_-$ over a point $v \in \mathrm{Q}$. The outer arrows in the diagram induce the following identifications of these 6-dimensional quadrics:

$$\mathscr{Q}_v = \mathrm{OGr}_+(4, v^\perp/v) \subset X \quad \text{and} \quad \mathscr{Q}_{v,-} = \mathrm{OGr}_-(4, v^\perp/v) \subset X^\vee \tag{3.4}$$

(the embeddings in X are defined as in (3.9) below). Thus $\mathscr{Q}_v$ parametrizes the subspaces $U_5 \subset \mathrm{V}$ such that $v \in U_5$ or, equivalently, $U_5 \subset v^\perp$, and similarly for $\mathscr{Q}_{v,-}$.

We recall that the two components X and $X^\vee$ of the Grassmannian $\mathrm{OGr}(5, \mathrm{V})$ can be distinguished by the parity of the dimension of the intersection of subspaces:

$$\dim(U_5' \cap U_5'') \equiv \begin{cases} 0 \pmod 2 & \text{if } U_5' \text{ and } U_5'' \text{ are in distinct components,} \\ 1 \pmod 2 & \text{if } U_5' \text{ and } U_5'' \text{ are in the same component.} \end{cases} \tag{3.5}$$

We write $\mathscr{O}_X(-1) = \mathcal{S}_1$, $\mathscr{O}_{X^\vee}(-1) = \mathcal{S}_{1,-}$ for the spinor line bundles on X and $X^\vee$. Then, by (2.4),

$$\det \mathscr{U} \cong \mathscr{O}_X(-2), \qquad \det \mathscr{U}_- \cong \mathscr{O}_{X^\vee}(-2). \tag{3.6}$$

The canonical line bundle of X can be written as

$$\omega_X \cong (\det \mathscr{U})^{\otimes 4} \cong \mathscr{O}_X(-8). \tag{3.7}$$

3.2. Linear spaces on the spinor tenfold. In this subsection we describe all linear spaces on the spinor tenfold. We write

$$F_d(X) = \mathrm{Hilb}^{(t+1)\cdots(t+d)/d!}(X) \tag{3.8}$$

for the Hilbert scheme of linearly embedded $\mathbb{P}^d \subset X \subset \mathbb{P}(\mathbb{S})$.

Let $U_s \subset \mathrm{V}$ be a $\mathbf{q}_{\mathrm{V}}$-isotropic subspace of dimension s. Denote by $U_s^{\perp} \subset \mathrm{V}$ its orthogonal complement with respect to $\mathbf{q}_{\mathrm{V}}$. Then $U_s \subset U_s^{\perp}$, and the quotient space $U_s^{\perp}/U_s$ is $(10-2s)$-dimensional and has a canonical quadratic form induced by $\mathbf{q}_{\mathrm{V}}$. Moreover, for every isotropic subspace of dimension d in $U_s^{\perp}/U_s$, its pre-image in $U_s^{\perp} \subset \mathrm{V}$ is a $\mathbf{q}_{\mathrm{V}}$-isotropic subspace of dimension $d+s$. In particular, we have a natural embedding

$$\mathrm{OGr}_+(5-s, U_s^{\perp}/U_s) \subset \mathrm{OGr}_+(5, \mathrm{V}) = X \tag{3.9}$$

and, under this embedding, the line bundle $\mathscr{O}_X(1)$ restricts to the ample generator of the Picard group. For example, (2.2) shows that for every isotropic 2-dimensional subspace $U_2 \subset \mathrm{V}$ the subvariety

$$\Pi^3_{U_2} := \mathrm{OGr}_+(3, U_2^{\perp}/U_2) \cong \mathbb{P}^3 \hookrightarrow X \tag{3.10}$$

is a linearly embedded 3-space and, for every isotropic 3-dimensional subspace $U_3 \subset \mathrm{V}$, the subvariety

$$L_{U_3} := \mathrm{OGr}_+(2, U_3^{\perp}/U_3) \hookrightarrow X \tag{3.11}$$

is a line on X. In the same vein we define a line on $X^{\vee}$ by

$$L^-_{U_3} := \mathrm{OGr}_-(2, U_3^{\perp}/U_3) \hookrightarrow X^{\vee}. \tag{3.12}$$

Note that the total space for the family of 3-spaces (3.10) on X is given by the diagram

$$\begin{tikzcd} & \mathrm{OFl}_+(2,5;\mathrm{V}) \arrow[r, equal, "\sim"] \arrow[dl] & \mathbb{P}_{\mathrm{OGr}(2,\mathrm{V})}(\mathcal{S}_4) \arrow[dr] & \\ X & & & \mathrm{OGr}(2,\mathrm{V}), \end{tikzcd} \tag{3.13}$$

and the total spaces for the families of lines (3.11) and (3.12) on X and $X^{\vee}$ are given by the diagram

$$\begin{tikzcd} \mathrm{OFl}_+(3,5;\mathrm{V}) \arrow[r, equal, "\sim"] \arrow[d] & \mathbb{P}_{\mathrm{OGr}(3,\mathrm{V})}(\mathcal{S}_2) \arrow[dr] & & \mathbb{P}_{\mathrm{OGr}(3,\mathrm{V})}(\mathcal{S}_{2,-}) \arrow[r, equal, "\sim"] \arrow[dl] & \mathrm{OFl}_-(3,5;\mathrm{V}) \arrow[d] \\ X & & \mathrm{OGr}(3,\mathrm{V}) & & X^{\vee}, \end{tikzcd} \tag{3.14}$$

where $\mathcal{S}_4$, $\mathcal{S}_2$ and $\mathcal{S}_{2,-}$ are the corresponding spinor bundles of ranks 4, 2 and 2 respectively, and the isomorphisms are relative versions of (2.2).

On the other hand, consider a point of $X^\vee$ and let $U_{5,-} \subset \mathrm{V}$ be the corresponding isotropic subspace. Then we have a natural embedding $\mathrm{Gr}(4, U_{5,-}) \subset \mathrm{OGr}(4, \mathrm{V})$. Furthermore, by (3.9) and (2.2), every isotropic subspace in V of dimension 4 extends uniquely to a 5-dimensional subspace corresponding to a point of X. This defines a regular map $\mathrm{OGr}(4, \mathrm{V}) \to X$. Combining these two observations, we obtain an embedding

$$\Pi^4_{U_{5,-}} := \mathrm{Gr}(4, U_{5,-}) \cong \mathbb{P}^4 \hookrightarrow X. \tag{3.15}$$

This is a linearly embedded 4-space that parametrizes the subspaces $U_5 \subset \mathrm{V}$ such that $\dim(U_5 \cap U_{5,-}) = 4$. The total space of this family is given by the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathbb{P}_X(\mathscr{U}^\vee(-1)) \arrow[dl] \arrow[r, equal, "\sim"] & \mathrm{OGr}(4, V) \arrow[dl] \arrow[dr] \arrow[r, equal, "\sim"] & \mathbb{P}_{X^\vee}(\mathscr{U}_-^\vee(-1)) \arrow[dr] & \\
X \arrow[r, equal] & \mathrm{OGr}_+(5, \mathrm{V}) & & \mathrm{OGr}_-(5, \mathrm{V}) \arrow[r, equal] & X^\vee.
\end{tikzcd} \tag{3.16}$$

Finally, we note that for every subspace $U_s \subset U_{5,-}$ there is a natural embedding

$$\Pi^{4-s}_{U_s, U_{5,-}} := \mathrm{Gr}(4-s, U_{5,-}/U_s) \hookrightarrow \mathrm{Gr}(4, U_{5,-}) \hookrightarrow X, \tag{3.17}$$

and this is a linear subspace of dimension $4-s$ on X. When $s = 3$, this yields an alternative description of the line (3.11):

$$L_{U_3} = \Pi^1_{U_3, U_{5,-}} \tag{3.18}$$

for every isotropic flag $U_3 \subset U_{5,-}$ (in particular, the line $\Pi^1_{U_3, U_{5,-}}$ is independent of $U_{5,-}$).

Theorem 3.2. *Every linear space on X is one of the following.*

1) *If $L \subset X$ is a line, then there is a unique isotropic 3-dimensional subspace $U_3 \subset \mathrm{V}$ such that $L = L_{U_3}$.*

2) *If $\Pi \subset X$ is a plane, then there is a unique isotropic flag $U_2 \subset U_{5,-} \subset \mathrm{V}$ such that $\Pi = \Pi^2_{U_2, U_{5,-}}$.*

3) *If $\Pi \subset X$ is a 3-space, then exactly one of the following two possibilities holds:*

a) *either there is a unique isotropic 2-dimensional subspace $U_2 \subset \mathrm{V}$ such that $\Pi = \Pi^3_{U_2}$,*

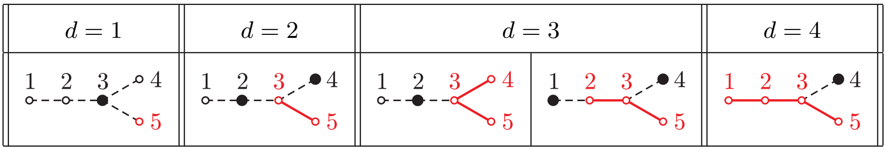
b) *or there is a unique isotropic flag $U_1 \subset U_{5,-} \subset \mathrm{V}$ such that $\Pi = \Pi^3_{U_1, U_{5,-}}$.*

4) *If $\Pi \subset X$ is a 4-space, then there is a unique isotropic subspace $U_{5,-} \subset \mathrm{V}$ such that $\Pi = \Pi^4_{U_{5,-}}$.*

In particular, there are no linear subspaces on X of dimension $d \geqslant 5$. Furthermore, the Hilbert schemes of linear spaces on X are the following Spin(V)*-varieties*:

$$\begin{aligned}
&F_1(X) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_3 \cong \mathrm{OGr}(3, V),\\
&F_2(X) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_{2,4} \cong \mathrm{OFl}_-(2, 5; \mathrm{V}),\\
&F_3(X) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_2 \sqcup \mathrm{Spin}(\mathrm{V})/\mathbf{P}_{1,4} \cong \mathrm{OGr}(2, \mathrm{V}) \sqcup \mathrm{OFl}_-(1, 5; \mathrm{V}),\\
&F_4(X) \cong \mathrm{Spin}(\mathrm{V})/\mathbf{P}_4 \cong \mathrm{OGr}_-(5, \mathrm{V}) = X^\vee.
\end{aligned} \tag{3.19}$$

Proof. This follows from a general result of Lansberg and Manivel ([12], Theorem 4.9). By this result, to describe $F_d(X)$, we should consider all minimal sets of vertices of the Dynkin diagram D_5 such that one of the connected components of their complements is a Dynkin diagram of type A_d with vertex 5 as an endpoint. The following picture shows all the possibilities:



Here solid segments form the subdiagram of type A_d with vertex 5 as an endpoint, and solid vertices form the minimal set whose complement contains A_d as a connected component. This gives (3.19). Furthermore, using the Tits construction explained in [12], § 4, we obtain the description of linear spaces on X stated in the theorem. □

3.3. Linear sections and their derived categories. The main characters of this paper are linear sections of X. Let

$$K \subset \mathbb{S}^\vee$$

be a vector subspace of dimension k and let

$$K^\perp := \operatorname{Ker}(\mathbb{S} \to K^\vee) \subset \mathbb{S}$$

be its orthogonal complement (of codimension k and dimension $16 - k$). We define

$$X_K := X \cap \mathbb{P}(K^\perp) \quad \text{and} \quad X_K^\vee := X^\vee \cap \mathbb{P}(K) \tag{3.20}$$

to be the corresponding linear sections of the spinor tenfold and its projective dual. If the intersections are dimensionally transverse, then

$$\dim X_K = 10 - k, \qquad \dim X_K^\vee = 10 - (16 - k) = k - 6$$

with the convention that the dimension of the empty set is an arbitrary negative number. The following simple observation is extremely useful.

Lemma 3.3. *A linear section X_K is smooth and dimensionally transverse if and only if $X_K^\vee$ is smooth and dimensionally transverse. In particular, when $k = \dim K \leqslant 5$, a linear section X_K is smooth and dimensionally transverse if and only if $X_K^\vee = \varnothing$.*

Proof. The proof is the same as in [11], Proposition 2.24. □

Remark 3.4. The same argument proves that if X_K is dimensionally transverse with hypersurface singularities, then the same holds for $X_K^\vee$ and there is a bijection between the singular points of X_K and $X_K^\vee$. This may lead to a simplification of the proof of the main result in [18] in the case when $g = 7$.

In what follows we often abbreviate 'smooth and dimensionally transverse' to just 'smooth'.

The tautological bundle $\mathscr{U}$ and the structure sheaf $\mathscr{O}_X$ give a very nice exceptional collection on X. We write $\mathbf{D}(X)$ for the bounded derived category of coherent sheaves on X.

Proposition 3.5 (see [1], § 6.2). *There is a full exceptional collection in* $\mathbf{D}(X)$ *of the form*

$$\mathbf{D}(X) = \langle \mathscr{O}_X, \mathscr{U}^\vee, \mathscr{O}_X(1), \mathscr{U}^\vee(1), \dots, \mathscr{O}_X(7), \mathscr{U}^\vee(7)\rangle.$$

This exceptional collection is Lefschetz and rectangular in the terminology of [19], [2], which simply means that it consists of several twists of the starting block $\langle \mathscr{O}_X, \mathscr{U}^\vee\rangle \subset \mathbf{D}(X)$. Moreover, the main result of § 6.2 in [1] (see also [2], Theorem 5.5) ensures that the classical projective duality between the spinor tenfolds $X \subset \mathbb{P}(\mathbb{S})$ and $X^\vee \subset \mathbb{P}(\mathbb{S}^\vee)$ extends to a homological projective duality. The main theorem of homological projective duality ([19], Theorem 6.3) then yields the following semiorthogonal decomposition relating the derived categories of X_K and $X_K^\vee$.

Theorem 3.6. *Assume that linear sections* X_K *and* $X_K^\vee$ *are dimensionally transverse and* $k = \dim K \leqslant 8$. *Denote the restriction* $\mathscr{U}|_{X_K}$ *of the tautological bundle by* $\mathscr{U}_{X_K}$. *Then there is a semiorthogonal decomposition*

$$\mathbf{D}(X_K) = \langle \mathbf{D}(X_K^\vee), \mathscr{O}_{X_K}, \mathscr{U}_{X_K}^\vee, \dots, \mathscr{O}_{X_K}(7-k), \mathscr{U}_{X_K}^\vee(7-k)\rangle.$$

The smoothness of X_K and $X_K^\vee$ is unnecessary in this theorem, but we shall usually assume it in what follows. When $k \geqslant 8$, there is a similar semiorthogonal decomposition with the roles of X_K and $X_K^\vee$ interchanged.

Remark 3.7. Let us spell out what these semiorthogonal decompositions tell us.

a) When $0 \leqslant k \leqslant 5$, the assumption of dimensional transversality ensures that X_K is a smooth Fano variety of dimension $10-k$ and $X_K^\vee = \varnothing$. Therefore the semiorthogonal decomposition reduces just to an exceptional collection of length $16 - 2k = 2\dim(X) - 4$,

$$\mathbf{D}(X_K) = \langle \mathscr{O}_{X_K}, \mathscr{U}_{X_K}^\vee, \dots, \mathscr{O}_{X_K}(7-k), \mathscr{U}_{X_K}^\vee(7-k)\rangle, \tag{3.21}$$

which may be regarded as a reduced replica of the original collection.

b) When $k = 6$, the assumption of dimensional transversality ensures that X_K is a Fano fourfold and $X_K^\vee$ is a finite scheme of length 12. Assuming also that $X_K^\vee$ is reduced (by Lemma 3.3, this is equivalent to the smoothness of X_K), we obtain a semiorthogonal decomposition

$$\mathbf{D}(X_K) = \langle \mathscr{E}_1, \mathscr{E}_2, \dots, \mathscr{E}_{12}, \mathscr{O}_{X_K}, \mathscr{U}_{X_K}^\vee, \mathscr{O}_{X_K}(1), \mathscr{U}_{X_K}^\vee(1)\rangle, \tag{3.22}$$

where $\mathscr{E}_1, \dots, \mathscr{E}_{12}$ is a completely orthogonal exceptional collection. One can check that each $\mathscr{E}_i$ is in fact a vector bundle of rank 2.

c) When $k = 7$, the assumption of dimensional transversality ensures that X_K is a Fano threefold and $X_K^\vee$ is a curve of arithmetic genus 7 (note that by the main theorem in [20] every smooth curve of genus 7 having no linear systems of type g^1_4 can be obtained in this way). The semiorthogonal decomposition takes the form

$$\mathbf{D}(X_K) = \langle \mathbf{D}(X_K^\vee), \mathscr{O}_{X_K}, \mathscr{U}_{X_K}^\vee\rangle. \tag{3.23}$$

In the smooth case one can check that $X_K^\vee$ is isomorphic to the moduli space of vector bundles of rank 2 on X_K, and the embedding of the derived category is given by the Fourier–Mukai functor with the universal bundle as kernel; see [21], Corollary 2.5, Theorem 4.4.

d) When $k=8$, the assumption of dimensional transversality ensures that X_K and $X_K^\vee$ are polarized K3-surfaces of degree 12. The semiorthogonal decomposition reduces to an equivalence of categories

$$\mathbf{D}(X_K) \cong \mathbf{D}(X_K^\vee). \tag{3.24}$$

In the smooth case, this is the classical equivalence discovered by Mukai ([22], Example 1.3); the surface $X_K^\vee$ can again be identified with the moduli space of vector bundles of rank 2 on X_K, and the equivalence of derived categories is given by the Fourier–Mukai functor with the universal bundle as kernel.

The semiorthogonal decompositions in Theorem 3.6 have many consequences for the geometry of the varieties involved. The simplest of these is a computation of the Grothendieck group:

$$\operatorname{rk} K_0(X_K) = \begin{cases} 16-2k & \text{when } \ 0 \leqslant k \leqslant 5, \\ 16 & \text{when } \ k=6 \end{cases} \tag{3.25}$$

(the first line follows from (3.21) and the second from (3.22)), as soon as X_K is smooth.

3.4. Rational Chow motives. Given a smooth projective variety Y, we write $\mathbf{M}(Y)$ for its Chow motive and $\mathbf{M}_{\mathbb{Q}}(Y)$ for its Chow motive with rational coefficients.

Corollary 3.8. *Let X_K be a smooth linear section of codimension $k \leqslant 6$ of the spinor tenfold. Then the rational Chow motive of X_K is of Lefschetz type. More precisely,* $\mathbf{M}_{\mathbb{Q}}(X_K)$ *is equal to*

$$\begin{cases} 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^2 \oplus 2\mathbf{L}_{\mathbb{Q}}^3 \oplus 2\mathbf{L}_{\mathbb{Q}}^4 \oplus 2\mathbf{L}_{\mathbb{Q}}^5 \oplus 2\mathbf{L}_{\mathbb{Q}}^6 \oplus 2\mathbf{L}_{\mathbb{Q}}^7 \oplus \mathbf{L}_{\mathbb{Q}}^8 \oplus \mathbf{L}_{\mathbb{Q}}^9 \oplus \mathbf{L}_{\mathbb{Q}}^{10} & \textit{when } \ k=0, \\ 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^2 \oplus 2\mathbf{L}_{\mathbb{Q}}^3 \oplus 2\mathbf{L}_{\mathbb{Q}}^4 \oplus 2\mathbf{L}_{\mathbb{Q}}^5 \oplus 2\mathbf{L}_{\mathbb{Q}}^6 \oplus \mathbf{L}_{\mathbb{Q}}^7 \oplus \mathbf{L}_{\mathbb{Q}}^8 \oplus \mathbf{L}_{\mathbb{Q}}^9 & \textit{when } \ k=1, \\ 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^2 \oplus 2\mathbf{L}_{\mathbb{Q}}^3 \oplus 2\mathbf{L}_{\mathbb{Q}}^4 \oplus 2\mathbf{L}_{\mathbb{Q}}^5 \oplus \mathbf{L}_{\mathbb{Q}}^6 \oplus \mathbf{L}_{\mathbb{Q}}^7 \oplus \mathbf{L}_{\mathbb{Q}}^8 & \textit{when } \ k=2, \\ 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^2 \oplus 2\mathbf{L}_{\mathbb{Q}}^3 \oplus 2\mathbf{L}_{\mathbb{Q}}^4 \oplus \mathbf{L}_{\mathbb{Q}}^5 \oplus \mathbf{L}_{\mathbb{Q}}^6 \oplus \mathbf{L}_{\mathbb{Q}}^7 & \textit{when } \ k=3, \\ 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^2 \oplus 2\mathbf{L}_{\mathbb{Q}}^3 \oplus \mathbf{L}_{\mathbb{Q}}^4 \oplus \mathbf{L}_{\mathbb{Q}}^5 \oplus \mathbf{L}_{\mathbb{Q}}^6 & \textit{when } \ k=4, \\ 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^2 \oplus \mathbf{L}_{\mathbb{Q}}^3 \oplus \mathbf{L}_{\mathbb{Q}}^4 \oplus \mathbf{L}_{\mathbb{Q}}^5 & \textit{when } \ k=5, \\ 1 \oplus \mathbf{L}_{\mathbb{Q}} \oplus 12\mathbf{L}_{\mathbb{Q}}^2 \oplus \mathbf{L}_{\mathbb{Q}}^3 \oplus \mathbf{L}_{\mathbb{Q}}^4 & \textit{when } \ k=6. \end{cases}$$

Moreover, $\mathrm{CH}^i(X_K) \otimes \mathbb{Q} \cong \mathbb{Q}^{n_i}$, *where the dimension n_i is equal to the multiplicity of the corresponding Lefschetz motive* $\mathbf{L}_{\mathbb{Q}}^i$ *in* $\mathbf{M}_{\mathbb{Q}}(X_K)$.

Proof. The motive of X_K is of Lefschetz type by Theorem 1.1 in [23]; see also the simplified proof in [9], Proposition 2.1. The multiplicities in the case $k=0$ can be read off the Hodge diamond of X, that is well known. Alternatively, one can argue as follows. Clearly,

$$\mathbf{M}_{\mathbb{Q}}(X_K) = \bigoplus_{i=0}^{10} n_i \mathbf{L}_{\mathbb{Q}}^i,$$

where $1 = n_0 \leqslant n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4 \leqslant n_5 \geqslant n_6 \geqslant n_7 \geqslant n_8 \geqslant n_9 \geqslant n_{10} = 1$ by Lefschetz' theorem. Moreover, $n_{10-i} = n_i$ by Poincaré duality, and $\sum n_i = 16$ since this sum is equal to the rank of the Grothendieck group $K_0(X)$; see (3.25). Thus, to determine the n_i, it suffices to verify that $n_5 \leqslant 2$. Assume, on the contrary, that $n_5 \geqslant 3$. Then for every smooth hyperplane section $X_1 \subset X$, since $\mathbf{M}_{\mathbb{Q}}(X_1)$ is of Lefschetz type, it follows from Lefschetz' theorem on hyperplane sections that

$$\mathbf{M}_{\mathbb{Q}}(X_1) = \left(\bigoplus_{i=0}^{4} n_i \mathbf{L}_{\mathbb{Q}}^i\right) \oplus \left(\bigoplus_{i=6}^{10} n_i \mathbf{L}_{\mathbb{Q}}^{i-1}\right).$$

By the assumption,

$$\sum_{i=0}^{4} n_i + \sum_{i=6}^{10} n_i = \sum_{i=0}^{10} n_i - n_5 \leqslant 16 - 3 = 13 < 14 = \operatorname{rk} K_0(X_1);$$

see (3.25). This contradiction proves that $n_5 \leqslant 2$ and thus gives the required expression for $\mathbf{M}_{\mathbb{Q}}(X)$.

The description of $\mathbf{M}_{\mathbb{Q}}(X_K)$ for $1 \leqslant k \leqslant 6$ now follows by combining Lefschetz' theorem on hyperplane sections (which enables us to determine the multiplicities of all the Lefschetz motives except possibly the middle one) and (3.25) (which enables us to determine the multiplicity of the middle Lefschetz motive when k is even). The result for the Chow groups follows immediately from the expression for the motive. $\square$

Remark 3.9. Most probably, an analogous result holds (in the Voevedsky category) for threefold linear sections of X with mild singularities:

$$\mathbf{M}_{\mathbb{Q}}(X_K) = 1 \oplus \mathbf{M}_{\mathbb{Q}}(X_K^\vee) \otimes \mathbf{L}_{\mathbb{Q}} \oplus \mathbf{L}_{\mathbb{Q}}^3.$$

Using motivic cohomology, one can deduce from this an isomorphism

$$\mathrm{CH}^1(X_K) \otimes \mathbb{Q} \cong \mathrm{CH}^0(X_K^\vee) \otimes \mathbb{Q}.$$

Hence the rank of the rational Weil class group $\mathrm{Cl}(X_K) \otimes \mathbb{Q}$ is equal to the number of irreducible components of the curve $X_K^\vee$. This relation should be useful for classifying G-Fano threefolds of genus 7; see [24].

§ 4. The blow-up of the spinor tenfold

In this section we discuss a description of the blow-up of the projective space $\mathbb{P}(\mathbb{S})$ along the spinor tenfold X and its consequences for linear sections of X.

4.1. The blow-up of the space of spinors along *X*. We use the notation of § 3.1. The following result can be extracted from Theorem III.3.8(4) in [3]. For completeness we provide a proof using the blow-up lemma (Lemma 2.5).

Proposition 4.1. *Let $X \subset \mathbb{P}(\mathbb{S})$ be the spinor tenfold, $\mathrm{Q} \subset \mathbb{P}(\mathrm{V})$ the corresponding 8-dimensional quadric, and $\mathcal{S} = \mathcal{S}_8$ the spinor bundle on* Q. *The left part of the diagram* (3.1) *extends to a commutative diagram*

$$\begin{tikzcd}
\mathrm{Bl}_X(\mathbb{P}(\mathbb{S})) \arrow[r, equal, "\sim"] \arrow[d, "f_{\mathbb{S}}"'] & \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \arrow[dl, "p"] & \mathscr{Q} \arrow[l, hook'] \arrow[dl] \arrow[dr] & \\
\mathbb{P}(\mathbb{S}) & X \arrow[l, hook'] & & \mathrm{Q}.
\end{tikzcd} \tag{4.1}$$

Under the isomorphism $\mathrm{Bl}_X(\mathbb{P}(\mathbb{S})) \cong \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$, the exceptional divisor of the blow-up morphism $f_{\mathbb{S}}$ coincides with the family of quadrics $\mathscr{Q} \subset \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$. Moreover, if $H_{\mathbb{S}}$ is the hyperplane class of $\mathbb{P}(\mathbb{S})$ and H_{Q} is the hyperplane class of Q, *then the class of the divisor $\mathscr{Q}$ in $\mathrm{Pic}(\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8))$ can be expressed as*

$$\mathscr{Q} = 2H_{\mathbb{S}} - H_{\mathrm{Q}}. \tag{4.2}$$

Proof. The canonical embedding of bundles $\mathcal{S}_8 \hookrightarrow \mathbb{S} \otimes \mathscr{O}_{\mathrm{Q}}$ induces a $\mathrm{Spin}(\mathrm{V})$-equivariant morphism

$$p\colon \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \to \mathbb{P}(\mathbb{S}). \tag{4.3}$$

We claim that this morphism is the blow-up along the spinor tenfold X.

First, let us check that the morphism p is birational. Indeed, by Lemmas 2.2 and 2.1, the quotient of $\mathbb{S} \otimes \mathscr{O}_{\mathrm{Q}}$ by $\mathcal{S}_8$ is isomorphic to $\mathcal{S}_{8,-}^\vee$ and, therefore, the image of $\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ in $\mathrm{Q} \times \mathbb{P}(\mathbb{S})$ is the zero locus of a global section of the vector bundle $\mathcal{S}_{8,-}^\vee \boxtimes \mathscr{O}(1)$. Hence the fibres of (4.3) are the zero loci of global sections of $\mathcal{S}_{8,-}^\vee$. Since $\mathcal{S}_{8,-}^\vee$ is globally generated of rank 8 with top Chern class equal to 1 (see [13], Remark 2.9), it follows that the generic fibre is a single point, whence p is birational.

Next, we apply the blow-up lemma to the morphism $p\colon \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \to \mathbb{P}(\mathbb{S})$. We have $\mathrm{Pic}(\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)) \cong \mathbb{Z}^2$ while $\mathrm{Pic}(\mathbb{P}(\mathbb{S})) \cong \mathbb{Z}$, whence the relative Picard number $\rho(p)$ is equal to 1. On the other hand, we have a natural embedding $\mathscr{Q} \hookrightarrow \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ (see (3.3)). Its composition with the map p is defined by the restriction of the relative hyperplane class from $\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$. The discussion in § 3.1 shows that this class is equal to the pullback of the hyperplane class of X. Hence the middle parallelogram in (4.1) is commutative. Since $\mathscr{Q} \subset \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ is a divisor and its image $p(\mathscr{Q}) = X \subset \mathbb{P}(\mathbb{S})$ is smooth of codimension 5, we conclude by Lemma 2.5 that p is the blow-up of X and $\mathscr{Q}$ is its exceptional divisor.

Finally, the equation of the relative quadric $\mathscr{Q} \subset \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ is induced by the self-duality isomorphism $\mathcal{S}_8^\vee \cong \mathcal{S}_8(H_{\mathrm{Q}})$ (see Lemma 2.1). This means that we have a linear equivalence $\mathscr{Q} = 2H_{\mathbb{S}} - H_{\mathrm{Q}}$, thus proving (4.2). □

Proposition 4.1 has several useful consequences for the geometry of X. First, it gives a simple proof of the transitivity of the $\mathrm{Spin}(\mathrm{V})$-action on $\mathbb{P}(\mathbb{S}) \setminus X$ (the fact of transitivity is well known; see, for example, Proposition 1.13 in [20], Proposition 2.1 in [6], Remark 2.13(1) in [7], and also Proposition 31 in [25]).

Corollary 4.2. *The action of* $\mathrm{Spin}(\mathrm{V})$ *on* $\mathbb{P}(\mathbb{S}) \setminus X$ *is transitive.*

Proof. The blow-up morphism $f_{\mathbb{S}}$ in (4.1) induces a Spin(V)-equivariant isomorphism
$$\mathbb{P}(\mathbb{S}) \setminus X \cong \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \setminus \mathscr{Q}.$$
Since the action of Spin(V) on the quadric Q is transitive, it is enough to check that the stabilizer of a point $v \in \mathrm{Q}$ in Spin(V) acts transitively on $\mathbb{P}(\mathcal{S}_{8,v}) \setminus \mathscr{Q}_v$. But this stabilizer contains the group $\mathrm{Spin}(v^{\perp}/v) \cong \mathrm{Spin}(\mathcal{S}_{8,v})$ as a subgroup, whence the claim. □

As another consequence, we find a resolution for the structure sheaf $\mathscr{O}_X$ on $\mathbb{P}(\mathbb{S})$, which was earlier computed using other tools (see [26], § 5.1).

Corollary 4.3. *There is an exact sequence*

$$0 \to \mathscr{O}(-8) \to \mathrm{V}(-6) \to \mathbb{S}^{\vee}(-5) \to \mathbb{S}(-3) \to \mathrm{V}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0.$$

In particular, if $X_K \subset X$ is a dimensionally transverse linear section of codimension $k \leqslant 10$, then X_K is an intersection of quadrics parametrized by the space $H^0(\mathbb{P}(K^{\perp}), I_{X_K}(2)) \cong \mathrm{V}$.

Proof. It was explained in the proof of Proposition 4.1 that the projective bundle $\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ can be written inside $\mathrm{Q} \times \mathbb{P}(\mathbb{S})$ as the zero locus of a global section of the vector bundle $\mathcal{S}_{8,-}^{\vee} \boxtimes \mathscr{O}(1)$. Therefore its structure sheaf has a Koszul resolution

$$0 \to \bigwedge^8 \mathcal{S}_{8,-} \boxtimes \mathscr{O}(-8) \to \bigwedge^7 \mathcal{S}_{8,-} \boxtimes \mathscr{O}(-7) \to \cdots \to \mathcal{S}_{8,-} \boxtimes \mathscr{O}(-1) \to \mathscr{O} \to \mathscr{O}_{\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)} \to 0.$$

By Proposition 4.1, the projective bundle $\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ is isomorphic to the blow-up of $\mathbb{P}(\mathbb{S})$ along X and, by (4.2), we have a linear equivalence $H_{\mathrm{Q}} = 2H_{\mathbb{S}} - \mathscr{Q}$. Hence the pushforward of $\mathscr{O}_{\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)}(H_{\mathrm{Q}})$ is isomorphic to $I_X(2)$. Below we compute this pushforward using the Koszul resolution.

The wedge products of $\mathcal{S}_{8,-}$ are direct sums of irreducible homogeneous vector bundles on Q. The corresponding weights of the group Spin(V) are listed in the second lines of the following two tables:

$\mathscr{O}$	$\mathcal{S}_{8,-}$	$\bigwedge^2 \mathcal{S}_{8,-}$	$\bigwedge^3 \mathcal{S}_{8,-}$	$\bigwedge^4 \mathcal{S}_{8,-}$	
0	$\omega_4 - \omega_1$	$\omega_3 - 2\omega_1$	$\omega_2 + \omega_5 - 3\omega_1$	$2\omega_2 - 4\omega_1$	$2\omega_5 - 3\omega_1$
V	$\mathbb{S}$	0	$\mathbb{S}^{\vee}[-1]$	$\mathrm{V}[-1]$	0

$\bigwedge^5 \mathcal{S}_{8,-}$	$\bigwedge^6 \mathcal{S}_{8,-}$	$\bigwedge^7 \mathcal{S}_{8,-}$	$\bigwedge^8 \mathcal{S}_{8,-}$
$\omega_2 + \omega_5 - 4\omega_1$	$\omega_3 - 4\omega_1$	$\omega_4 - 4\omega_1$	$-4\omega_1$
0	$\mathbb{k}[-2]$	0	0

The third lines of the tables contain the cohomology (computed by the Borel–Bott–Weil theorem) of the corresponding bundles on X twisted by $\mathscr{O}_{\mathrm{Q}}(H_{\mathrm{Q}}) \cong \mathscr{O}_{\mathrm{Q}}(\omega_1)$ with the cohomological degree in square brackets. As a result, we obtain an exact sequence

$$0 \to \mathscr{O}(-6) \to \mathrm{V}(-4) \to \mathbb{S}^{\vee}(-3) \to \mathbb{S}(-1) \to \mathrm{V} \otimes \mathscr{O} \to I_X(2) \to 0.$$

All the assertions of the corollary follow easily. □

A similar argument gives a resolution of the square I_X^2 of the ideal sheaf of X.

Corollary 4.4. *There is an exact sequence*

$$0 \to \mathbb{S}^\vee(-9) \to \bigwedge^2 \mathrm{V}(-8) \to \bigwedge^3 \mathrm{V}(-6) \to \frac{\mathbb{S}\otimes \mathrm{V}}{\mathbb{S}^\vee}(-5) \to \frac{\operatorname{Sym}^2 \mathrm{V}}{\Bbbk}(-4) \to I_X^2 \to 0.$$

In particular, if $X_K \subset X$ is a dimensionally transverse linear section of codimension $k \leqslant 10$, then there is a unique quadratic relation between the quadrics passing through X_K. It is given by $\mathbf{q}_{\mathrm{V}} \in \operatorname{Sym}^2 \mathrm{V}$.

Proof. By (4.2), the pushforward of $\mathscr{O}_{\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)}(2H_{\mathrm{Q}})$ is isomorphic to $I_X^2(4)$. Using the Koszul resolution and Borel–Bott–Weil theorem to compute it, we deduce the required exact sequence.

It follows from the resolution that $H^0(\mathbb{P}(K^\perp), I_{X_K}^2(4)) \cong (\operatorname{Sym}^2 \mathrm{V})/\Bbbk$, which means that there is a unique quadratic relation between the quadrics through X_K. In the case $K=0$ this relation is $\operatorname{Spin}(\mathrm{V})$-invariant and, therefore, is given by the element $\mathbf{q}_{\mathrm{V}} \in \operatorname{Sym}^2 \mathrm{V}$. By restriction, the same holds for all K. $\square$

The following result is also very useful (compare with the main theorem in [20]).

Corollary 4.5. *Let X_{K_1}, X_{K_2} be dimensionally transverse linear sections of X of codimension $k \leqslant 7$. If $X_{K_1} \cong X_{K_2}$, then there is an element $g \in \operatorname{Spin}(\mathrm{V})$ such that $g(X_{K_1}) = X_{K_2}$ and $g(K_1) = K_2$.*

Proof. Let $\varphi\colon X_{K_1} \to X_{K_2}$ be an isomorphism. By Lefschetz' theorem, $\operatorname{Pic}(X_{K_1})$ and $\operatorname{Pic}(X_{K_2})$ are generated by the restrictions H_1 and H_2 of the hyperplane class of $X \subset \mathbb{P}(\mathbb{S})$. Therefore,

$$\varphi^*(\mathscr{O}_{X_{K_2}}(H_2)) \cong \mathscr{O}_{X_{K_1}}(H_1).$$

By choosing such an isomorphism of line bundles, we obtain an isomorphism $\overline{\varphi}$ of vector spaces

$$K_1^\perp \cong H^0(X_{K_1}, \mathscr{O}_{X_{K_1}}(H_1))^\vee \cong H^0(X_{K_2}, \mathscr{O}_{X_{K_2}}(H_2))^\vee \cong K_2^\perp$$

such that the following diagram is commutative:

$$\begin{tikzcd} X_{K_1} \arrow[r, "\varphi"] \arrow[d] & X_{K_2} \arrow[d] \\ \mathbb{P}(K_1^\perp) \arrow[r, "\overline{\varphi}"] & \mathbb{P}(K_2^\perp), \end{tikzcd}$$

where the vertical arrows are the natural embeddings. By Corollary 4.3, the map $\overline{\varphi}$ induces an isomorphism

$$\mathrm{V} = H^0(\mathbb{P}(K_1^\perp), I_{X_{K_1}}(2)) \cong H^0(\mathbb{P}(K_2^\perp), I_{X_{K_2}}(2)) = \mathrm{V},$$

that is, an element $g' \in \operatorname{GL}(\mathrm{V})$. Since $\mathbf{q}_{\mathrm{V}} \in \operatorname{Sym}^2 \mathrm{V}$ corresponds by Corollary 4.4 to the unique relation between the quadrics passing through X_{K_1} and X_{K_2}, it follows that g preserves $\mathbf{q}_{\mathrm{V}}$ up to a scalar, whence $g' \in \operatorname{GO}(\mathrm{V})$. The element $g \in \operatorname{Spin}(\mathrm{V})$ may be defined as any lift to $\operatorname{Spin}(\mathrm{V})$ of the image of g' in $\operatorname{PSO}(\mathrm{V})$. We easily see that the induced action of g_{V} on $\mathbb{P}(\mathbb{S})$ takes X_{K_1} and K_1 to X_{K_2} and K_2 respectively. $\square$

The following well-known isomorphism is another useful consequence of the proposition.

Corollary 4.6. *The normal bundle of the spinor tenfold can be described as*

$$\mathscr{N}_{X/\mathbb{P}(\mathbb{S})} \cong \textstyle\bigwedge^4 \mathscr{U}^\vee \cong \mathscr{U}(2),$$

where $\mathscr{U}$ is the restriction of the tautological bundle.

Proof. Since, on the one hand, the exceptional divisor of a blow-up is isomorphic to the projectivization of the normal bundle of the blow-up centre and, on the other, $\mathscr{Q} \cong \mathbb{P}_X(\mathscr{U})$ by (3.2), it follows from Proposition 4.1 that the normal bundle of X is isomorphic to a twist of $\mathscr{U}$. By the adjunction formula and (3.7) we have $\det \mathscr{N}_{X/\mathbb{P}(\mathbb{S})} \cong \mathscr{O}_X(8)$ while $\det \mathscr{U} \cong \mathscr{O}_X(-2)$ by (3.6). Hence the required twist is given by $\mathscr{O}_X(2)$. $\square$

Of course, the proof of Proposition 4.1 applies to the blow-up of $\mathbb{P}(\mathbb{S}^\vee)$ along $X^\vee$, with a completely analogous result (or one can formally apply an outer automorphism of $\mathrm{Spin}(\mathrm{V})$ to the diagram (4.1)). In the next diagram we merge the resulting diagram with (4.1):

$$\begin{tikzcd}
 & & \mathscr{Q} \arrow[ddll] \arrow[d] & & & & \mathscr{Q}_- \arrow[d] \arrow[ddrr] & & \\
 & & \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \arrow[dl, "p"] \arrow[dr, "q"] & & & & \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_{8,-}) \arrow[dlll, "q_-"'] \arrow[dr, "p_-"] & & \\
X \arrow[r] & \mathbb{P}(\mathbb{S}) & & \mathrm{Q} & & & & \mathbb{P}(\mathbb{S}^\vee) \arrow[llll, dashed, "\gamma"] & X^\vee. \arrow[l]
\end{tikzcd} \tag{4.4}$$

The rational map $\gamma := q_- \circ p_- \colon \mathbb{P}(\mathbb{S}^\vee) \dashrightarrow \mathrm{Q}$ will play an important role in what follows.

4.2. Quadrics on the spinor tenfold. One can also use Proposition 4.1 to describe quadrics on X. Denote by

$$G_d(X) = \mathrm{Hilb}^{(t+1)\cdots(t+d-1)(2t+d)/d!}(X) \tag{4.5}$$

the Hilbert scheme of quadrics of dimension d on X. We recall that the family (3.1) of 6-dimensional quadrics $\mathscr{Q}_v \subset \mathbb{P}(\mathcal{S}_{8,v}) = q^{-1}(v)$ that are parametrized by the points $v \in \mathrm{Q}$ was defined in § 3.1; see Remark 3.1.

Corollary 4.7. *Let $Z \subset X$ be a quadric of dimension d. Then $d \leqslant 6$ and*

a) *either there is a unique point $v \in \mathrm{Q}$ and a linear subspace $\mathbb{P}^{d+1} \subset \mathbb{P}(\mathcal{S}_{8,v})$ such that*

$$Z = p(\mathbb{P}^{d+1} \cap \mathscr{Q}_v),$$

b) *or $d \leqslant 3$ and there is a unique linear space $\Pi^{d+1} \subset X$ such that $Z \subset \Pi^{d+1}$.*

In particular, all maximal (6-dimensional) quadrics on X are of the form $\mathscr{Q}_v$, where $v \in \mathrm{Q}$.

Proof. Let $\Pi := \langle Z\rangle \subset \mathbb{P}(\mathbb{S})$ be the linear span of Z in $\mathbb{P}(\mathbb{S})$. If Π is contained in X, we are in part b) and there is nothing to prove (since linear spaces in X have dimension at most 4 by Theorem 3.2, the dimension of such quadrics Z does not exceed 3).

Therefore we assume that $\Pi \not\subset X$. Since X is an intersection of quadrics (Corollary 4.3), we have a scheme-theoretic equality $\Pi \cap X = Z$. Hence the map q contracts the strict transform $\widetilde{\Pi}$ of Π in $\mathrm{Bl}_X(\mathbb{P}(\mathbb{S}))$ to a point of Q. Denoting this point by v, we see that $\widetilde{\Pi} \subset q^{-1}(v) = \mathbb{P}(\mathcal{S}_{8,v})$ and $Z = p(\widetilde{\Pi}) \cap X = p(\widetilde{\Pi} \cap \mathscr{Q}_v)$. At the same time $\widetilde{\Pi} \cong \mathrm{Bl}_Z(\Pi) \cong \Pi \cong \mathbb{P}^{d+1}$. $\square$

Remark 4.8. In fact, one can strengthen the results of Corollary 4.7 to get the following description of the Hilbert schemes $G_d(X)$ of d-dimensional quadrics on X:

$$\begin{aligned}
\mathrm{Bl}_{\mathbb{P}_{\mathrm{OGr}(3,\mathrm{V})}(\mathrm{Sym}^2(\mathcal{S}_2))}(G_0(X)) &\cong \mathrm{Bl}_{\mathrm{OGr}_{\mathrm{Q}}(2,\mathcal{S}_8)}(\mathrm{Gr}_{\mathrm{Q}}(2,\mathcal{S}_8)),\\
\mathrm{Bl}_{\mathbb{P}_{\mathrm{OFl}_-(2,5;\mathrm{V})}(\mathrm{Sym}^2(\mathscr{U}_{5,-}/\mathscr{U}_2))}(G_1(X)) &\cong \mathrm{Bl}_{\mathrm{OGr}_{\mathrm{Q}}(3,\mathcal{S}_8)}(\mathrm{Gr}_{\mathrm{Q}}(3,\mathcal{S}_8)),\\
\mathrm{Bl}_{\mathbb{P}_{\mathrm{OGr}(2,\mathrm{V})}(\mathrm{Sym}^2(\mathcal{S}_4))}(G_2(X)) &\cong \mathrm{Bl}_{\mathrm{OGr}_{\mathrm{Q}}(4,\mathcal{S}_8)}(\mathrm{Gr}_{\mathrm{Q}}(4,\mathcal{S}_8)),\\
G_3(X) &\cong \mathrm{Gr}_{\mathrm{Q}}(5,\mathcal{S}_8) \sqcup \mathbb{P}_{\mathrm{OGr}_-(5,\mathrm{V})}(\mathrm{Sym}^2(\mathscr{U}_{5,-})),\\
G_4(X) &\cong \mathrm{Gr}_{\mathrm{Q}}(6,\mathcal{S}_8),\\
G_5(X) &\cong \mathrm{Gr}_{\mathrm{Q}}(7,\mathcal{S}_8),\\
G_6(X) &\cong \mathrm{Q}.
\end{aligned}$$

However, we do not need these results, so we omit their proofs.

In what follows we shall need a description of the intersections of maximal quadrics $\mathscr{Q}_v$ with maximal linear spaces $\Pi^4_{U_{5,-}}$ (see (3.15)) on the spinor tenfold X. It turns out that these are either points or 3-spaces.

Lemma 4.9. *Let $\Pi^4_{U_{5,-}} \subset X$ be a linear 4-space and let $\mathscr{Q}_v \subset X$ be a 6-dimensional quadric on the spinor tenfold X. Then*

$$\Pi^4_{U_{5,-}} \cap \mathscr{Q}_v = \begin{cases} \mathrm{Spec}(\Bbbk) & \text{if } v \notin U_{5,-},\\ \Pi^3_{v,U_{5,-}} & \text{otherwise.}\end{cases}$$

Proof. Recall that $\mathscr{Q}_v = \mathrm{OGr}_+(4, v^\perp/v)$ (see (3.4)). Thus the intersection $\Pi^4_{U_{5,-}} \cap \mathscr{Q}_v$ parametrizes isotropic subspaces $U_5 \subset \mathrm{V}$ with $v \in U_5$ and $\dim(U_5 \cap U_{5,-}) = 4$.

We first assume that $v \notin U_{5,-}$. Then the vector v is not orthogonal to $U_{5,-}$ (since $U_{5,-}^\perp = U_{5,-}$) and, therefore, the intersection $U_{5,-} \cap v^\perp$ is 4-dimensional. On the other hand, if U_5 is an isotropic subspace containing v and having a 4-dimensional intersection with $U_{5,-}$, then this intersection is contained in $v^\perp$, whence U_5 is equal to the linear span $\langle v, U_{5,-} \cap v^\perp\rangle$, and this is the only intersection point of $\Pi^4_{U_{5,-}}$ and $\mathscr{Q}_v$.

We now assume that $v \in U_{5,-}$. Then the isotropic subspaces $U_5 \subset \mathrm{V}$ having a 4-dimensional intersection with the space $U_{5,-}$ are parametrized by the 4-space $\Pi^4_{U_{5,-}} = \mathrm{Gr}(4, U_{5-})$, and those of them that contain v are parametrized by the 3-space $\Pi^3_{v,U_{5,-}} \cong \mathrm{Gr}(3, U_{5,-}/v)$. $\square$

4.3. Blow-ups of linear sections. The diagram (4.4) induces similar diagrams for linear sections of the spinor tenfold. To state the result, we introduce the following notation.

Let $K \subset \mathbb{S}^\vee$ be a subspace of dimension k. Consider the following composition of morphisms of sheaves on Q:

$$\sigma_K \colon K \otimes \mathscr{O}_{\mathrm{Q}} \hookrightarrow \mathbb{S}^\vee \otimes \mathscr{O}_{\mathrm{Q}} \twoheadrightarrow \mathcal{S}_8^\vee, \tag{4.6}$$

where the first morphism is induced by the embedding $K \hookrightarrow \mathbb{S}^\vee$ and the second is the evaluation morphism for the natural identification $H^0(\mathrm{Q}, \mathcal{S}_8^\vee) \cong \mathbb{S}^\vee$ (see § 2.2). We denote by

$$\mathrm{Q} = \mathfrak{D}_{\geqslant 0}(\sigma_K) \supset \mathfrak{D}_{\geqslant 1}(\sigma_K) \supset \mathfrak{D}_{\geqslant 2}(\sigma_K) \supset \cdots$$

the discriminant stratification of the quadric Q by the corank strata of the morphism σ_K. In other words, $\mathfrak{D}_{\geqslant c}(\sigma_K)$ is the subscheme of Q where the corank of σ_K is not smaller than c (the ideal of this subscheme is generated by the minors of order $k - c + 1$ of the map (4.6)). We also put

$$\mathfrak{D}_c(\sigma_K) := \mathfrak{D}_{\geqslant c}(\sigma_K) \setminus \mathfrak{D}_{\geqslant c+1}(\sigma_K).$$

By a piecewise-Zariski locally trivial fibration we mean a morphism whose base admits a stratification such that the morphism is locally trivial over each stratum.

Proposition 4.10. *Assume that X_K and $X_K^\vee$ are dimensionally transverse linear sections of X and $X^\vee$ respectively. Then there is a commutative diagram*

$$\begin{tikzcd}
& & \mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)) \arrow[dl, "p"'] \arrow[dr, "q"] & & \mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)) \arrow[dl, "q_-"'] \arrow[dr, "p_-"] & \\
X_K \arrow[r, hook] & \mathbb{P}(K^\perp) & & \mathrm{Q} & & \mathbb{P}(K) \arrow[ll, dashed, "\gamma"] & X_K^\vee, \arrow[l, hook']
\end{tikzcd} \tag{4.7}$$

where p, q, p_- and q_- are the restrictions of the maps of the same name in (4.4). The maps p and p_- are blow-ups while q and q_- are piecewise-Zariski locally trivial fibrations whose fibres over the stratum $\mathfrak{D}_c(\sigma_K) \subset \mathrm{Q}$ are isomorphic to $\mathbb{P}^{7+c-k}$ and $\mathbb{P}^{c-1}$ respectively. In particular,

$$\mathfrak{D}_{\geqslant 1}(\sigma_K) = q_-(\mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K))).$$

Proof. Consider the diagram (4.4). By the transversality assumption,

$$p^{-1}(\mathbb{P}(K^\perp)) \cong \mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)).$$

On the other hand, the p-pre-image in $\mathrm{Bl}_X(\mathbb{P}(\mathbb{S})) \cong \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ of a hyperplane in $\mathbb{P}(\mathbb{S})$ is a relative hyperplane section of $q\colon \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \to \mathrm{Q}$. Therefore, $p^{-1}(\mathbb{P}(K^\perp))$ is the zero locus in $\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)$ of the natural section of the vector bundle $K^\vee \otimes \mathscr{O}_{\mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8)}(H_{\mathbb{S}})$ that corresponds to the morphism σ_K or, more precisely, to its dual

$$\sigma_K^\vee \colon \mathcal{S}_8 \hookrightarrow \mathbb{S} \otimes \mathscr{O}_{\mathrm{Q}} \twoheadrightarrow K^\vee \otimes \mathscr{O}_{\mathrm{Q}}. \tag{4.8}$$

Hence the fibre of the map q in (4.7) over a point $v \in \mathrm{Q}$ is the projectivization of the kernel of $\sigma_K^\vee$ at v. Therefore q is a piecewise-Zariski locally trivial fibration over Q and its fibre over the stratum $\mathfrak{D}_c(\sigma_K^\vee) = \mathfrak{D}_c(\sigma_K) \subset \mathrm{Q}$ is isomorphic to $\mathbb{P}^{7+c-k}$.

Furthermore, the first map in (4.8) is a fibrewise monomorphism and, by Lemmas 2.1 and 2.2, its cokernel is the natural epimorphism $\mathbb{S} \otimes \mathscr{O}_{\mathrm{Q}} \twoheadrightarrow \mathcal{S}_{8,-}^\vee$, while the second map in (4.8) is an epimorphism whose kernel is the natural fibrewise monomorphism $K^\perp \otimes \mathscr{O}_{\mathrm{Q}} \hookrightarrow \mathbb{S} \otimes \mathscr{O}_{\mathrm{Q}}$. Hence the rank stratification of (4.8) coincides with that of the composition

$$\sigma_K^\perp \colon K^\perp \otimes \mathscr{O}_{\mathrm{Q}} \hookrightarrow \mathbb{S} \otimes \mathscr{O}_{\mathrm{Q}} \twoheadrightarrow \mathcal{S}_{8,-}^\vee \tag{4.9}$$

of the kernel and cokernel maps mentioned above. Taking the dual of $\sigma_K^\perp$ and repeating these arguments, we conclude that q_- is also a piecewise-Zariski locally trivial fibration over Q and its fibre over the stratum $\mathfrak{D}_c(\sigma_K^\perp) \subset \mathrm{Q}$ is isomorphic to $\mathbb{P}^{c-1}$. But we have already checked that $\mathfrak{D}_c(\sigma_K^\perp) = \mathfrak{D}_c(\sigma_K)$. Since $\mathfrak{D}_{\geqslant 1}(\sigma_K^\perp)$ is the locus of the non-empty fibres of q_-, we conclude that $\mathfrak{D}_{\geqslant 1}(\sigma_K) = q_-(\mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)))$. $\square$

The following particular case of the proposition will be used very extensively. We recall Lemma 3.3.

Corollary 4.11. *Assume that the linear section X_K of X is smooth of codimension $k = \dim K \leqslant 5$ in X, so that $X_K^\vee = \varnothing$ and $\mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)) = \mathbb{P}(K)$. Then the rational map $q_- \circ p_-^{-1}$ in (4.7) is a regular closed embedding*

$$\gamma := q_- \circ p_-^{-1} \colon \mathbb{P}(K) \hookrightarrow \mathrm{Q}$$

such that $\gamma^*(\mathscr{O}_{\mathrm{Q}}(1)) \cong \mathscr{O}_{\mathbb{P}(K)}(2)$. In particular, $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$ is the isomorphic image of the Veronese variety $\mathbb{P}(K) \subset \mathbb{P}(\mathrm{Sym}^2 K)$ under a linear projection $\mathrm{Sym}^2 K \to \mathrm{V}$.

The map q has fibres $\mathbb{P}^{8-k}$ over $\gamma(\mathbb{P}(K))$ and $\mathbb{P}^{7-k}$ over its complement.

Proof. We first check that $\mathfrak{D}(\sigma_K)_{\geqslant 2} = \varnothing$. By Proposition 4.10 it is enough to check that no fibre of q_- contains a $\mathbb{P}^1$. But if $\mathbb{P}^1 \subset q_-^{-1}(v) \subset \mathbb{P}(\mathcal{S}_{8,-,v})$, then $q_-^{-1}(v) \cap \mathscr{Q}_{v,-} \neq \varnothing$ and, therefore,

$$\varnothing \neq p_-(q_-^{-1}(v) \cap \mathscr{Q}_{v,-}) \subset X^\vee \cap \mathbb{P}(K) = X_K^\vee,$$

contrary to the hypotheses. Thus, $q_- \colon \mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)) \to \mathrm{Q}$ is a closed embedding and, since the map $p_- \colon \mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)) \to \mathbb{P}(K)$ is an isomorphism, the composition $\gamma = q_- \circ p_-^{-1} \colon \mathbb{P}(K) \to \mathrm{Q}$ is also a closed embedding.

We further note that $\mathbb{P}(K) = \mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)) \subset \mathrm{Bl}_{X^\vee}(\mathbb{P}(\mathbb{S}^\vee))$ is disjoint from the exceptional divisor $\mathscr{Q}_- \subset \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_{8,-})$ and, therefore, the class of $\mathscr{Q}_-$ restricts trivially to $\mathbb{P}(K)$. Using the analogue of (4.2) for the right half of the diagram (4.4), we obtain an isomorphism $\gamma^*(\mathscr{O}_{\mathrm{Q}}(1)) \cong \mathscr{O}_{\mathbb{P}(K)}(2)$. The last assertion of the corollary follows immediately from Proposition 4.10. $\square$

Remark 4.12. We spell out the conclusion of Proposition 4.10 in the case when X_K (and hence also $X_K^\vee$) is smooth and dimensionally transverse of codimension $k \geqslant 6$.

1) Assume that $k=6$. Then $X_K^\vee$ is the set of 12 reduced points and the map q_- contracts the strict transforms of the 66 lines joining these points. Hence we have three strata: $\mathfrak{D}_2(\sigma_K)$ consists of 66 points (the images of the strict transforms of the lines), $\mathfrak{D}_{\geqslant 1}(\sigma_K) = q_-(\mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)))$, and $\mathfrak{D}_0(\sigma_K)$ is its open complement.

2) Assume that $k=7$. Then $X_K^\vee$ is a smooth (canonical) curve of genus 7 and the map q_- contracts the strict transforms of its secant lines. Hence, as before, we have three strata: $\mathfrak{D}_2(\sigma_K) \cong \mathrm{Sym}^2(X_K^\vee)$ is the image of the secant variety, $\mathfrak{D}_{\geqslant 1}(\sigma_K) = q_-(\mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K)))$, and $\mathfrak{D}_0(\sigma_K)$ is its open complement.

3) Assume that $k=8$. Then $X_K^\vee$ is a smooth K3-surface of degree 12 and the map q_- contracts the strict transforms of its secant lines and the strict transforms of planes intersecting $X_K^\vee$ along a conic. Hence we have at most four strata: $\mathfrak{D}_3(\sigma_K)$ is a finite (possibly empty) set of points (the images of the planes spanned by the conics in $X_K^\vee$), $\mathfrak{D}_{\geqslant 2}(\sigma_K)$ is the image of the secant variety (a contraction of $\mathrm{Sym}^2(X_K^\vee)$), $\mathfrak{D}_{\geqslant 1}(\sigma_K) = q_-(\mathrm{Bl}_{X_K^\vee}(\mathbb{P}(K))$, and $\mathfrak{D}_0(\sigma_K)$ is its open complement.

Remark 4.13. Considering the fibres of the map $q\colon \mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)) \to \mathrm{Q}$ in the case when $k=7$, one can also show that $\mathfrak{D}_{\geqslant 2}(\sigma_K)$ is isomorphic to $G_1(X_K)$, the Hilbert scheme of conics on the Fano threefold X_K. Combining this with Remark 4.12 yields a geometric construction of an isomorphism

$$G_1(X_K) \cong \mathrm{Sym}^2(X_K^\vee).$$

This isomorphism was first established in [21], Theorem 5.3, by means of derived categories.

The diagram (4.7) is sometimes inconvenient since the map q is not flat. However, when $k \leqslant 5$, this map can be transformed to a flat $\mathbb{P}^{7-k}$-bundle by another blow-up.

Proposition 4.14. *Let X_K be a smooth and dimensionally transverse linear section of X of codimension $k \leqslant 5$, so that $X_K^\vee = \varnothing$. Then there is a commutative diagram*

$$\begin{tikzcd}
& \mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)) \arrow[dl, "p"'] \arrow[dr, "q"] & & \mathrm{Bl}_{q^{-1}(\gamma(\mathbb{P}(K)))}(\mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp))) \arrow[ll, "\widetilde{r}"'] \arrow[dr, "\widetilde{q}"] & \\
X_K \hookrightarrow \mathbb{P}(K^\perp) & & \mathrm{Q} & & \mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q}), \arrow[ll, "r"']
\end{tikzcd} \tag{4.10}$$

where r and $\widetilde{r}$ are blow-up morphisms and $\widetilde{q}$ is the natural projection of the projectivization of a vector bundle of rank $8-k$.

Proof. Since the scheme-theoretic pre-image of the subscheme $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$ in $\mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp))$ is $q^{-1}(\gamma(\mathbb{P}(K)))$ (by definition), its scheme-theoretic pre-image in $\mathrm{Bl}_{q^{-1}(\gamma(\mathbb{P}(K)))}(\mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)))$ is the exceptional divisor of the blow-up $\widetilde{r}$. Therefore, by the universal property of blow-ups, the composition $q \circ \widetilde{r}$ factors through the blow-up $\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})$, thus defining the map $\widetilde{q}$ that makes the diagram commutative. It remains to show that $\widetilde{q}$ is the projectivization of a vector bundle.

Consider the morphisms $\sigma_K^\vee$ and $\sigma_K^\perp$ defined in (4.8) and (4.9). Arguing as in the proofs of Proposition 4.10 and Corollary 4.11, we see that the corank of $\sigma_K^\vee$

is everywhere less than or equal to 1, and its degeneration scheme $\mathfrak{D}_{\geqslant 1}(\sigma_K^\vee)$ coincides with the subscheme $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$. Therefore its cokernel is a line bundle on $\gamma(\mathbb{P}(K))$. Moreover, the proof of the equality $\mathfrak{D}_{\geqslant 1}(\sigma_K^\vee) = \mathfrak{D}_{\geqslant 1}(\sigma_K^\perp)$ shows that the cokernel sheaf is isomorphic to the γ-pushforward of the line bundle $\mathscr{O}_{\mathbb{P}(K)}(1)$. In other words, we have an exact sequence

$$\mathcal{S}_8 \xrightarrow{\sigma_K^\vee} K^\vee \otimes \mathscr{O}_{\mathrm{Q}} \to \gamma_*(\mathscr{O}_{\mathbb{P}(K)}(1)) \to 0.$$

Pulling it back to the blow-up $\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})$, we obtain an exact sequence

$$r^*(\mathcal{S}_8) \xrightarrow{r^*(\sigma_K^\vee)} K^\vee \otimes \mathscr{O}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})} \to i_* \operatorname{pr}^*(\mathscr{O}_{\mathbb{P}(K)}(1)) \to 0, \tag{4.11}$$

where $i\colon E \hookrightarrow \mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})$ is the embedding of the exceptional divisor of the blow-up r while $\operatorname{pr}\colon E \to \mathbb{P}(K)$ is the natural projection. Since E is a Cartier divisor, the image and the kernel of the morphism $r^*(\sigma_K^\vee)$ are vector bundles. We denote this kernel by $\mathscr{F}$ and rewrite the resulting exact sequence in the form

$$0 \to \mathscr{F} \to r^*(\mathcal{S}_8) \xrightarrow{r^*(\sigma_K^\vee)} K^\vee \otimes \mathscr{O}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})} \to i_* \operatorname{pr}^*(\mathscr{O}_{\mathbb{P}(K)}(1)) \to 0, \tag{4.12}$$

where the first map is a fibrewise monomorphism. Below we shall prove that the map $\widetilde{q}$ is the natural projection of the projectivization of the vector bundle $\mathscr{F}$.

First, we consider the composition

$$f\colon \mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(\mathscr{F}) \hookrightarrow \mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(r^*(\mathcal{S}_8)) \to \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_8) \cong \mathrm{Bl}_X(\mathbb{P}(\mathbb{S})),$$

where the first map is induced by the first map in (4.12), the second is induced by the blow-up r, and the third is the isomorphism in Proposition 4.1. Clearly, the image of f is the strict transform of $\mathbb{P}(K^\perp)$, that is, $\mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)) \subset \mathrm{Bl}_X(\mathbb{P}(\mathbb{S}))$. Moreover, the composition

$$\mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(\mathscr{F}) \xrightarrow{f} \mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp)) \xrightarrow{q} \mathrm{Q}$$

coincides by construction with the composition

$$\mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(\mathscr{F}) \to \mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q}) \xrightarrow{r} \mathrm{Q}.$$

Therefore, the scheme-theoretic pre-image of the subscheme $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$ under this composition is a Cartier divisor. It follows that the map f factors through a map

$$\widetilde{f}\colon \mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(\mathscr{F}) \to \mathrm{Bl}_{q^{-1}(\gamma(\mathbb{P}(K)))}(\mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp))).$$

We now note that $\widetilde{f}$ is a proper map between smooth varieties of the same dimension $15-k$ and an isomorphism over the open subset $\mathrm{Q} \setminus \gamma(\mathbb{P}(K)) \subset \mathrm{Q}$. Hence it is birational. Therefore it is the blow-up of an ideal. But these two varieties have the same Picard number 3, whence $\widetilde{f}$ is an isomorphism. □

Remark 4.15. Probably, one can construct a similar birational flattening of q in the case when $k=6$. A natural guess is to start by blowing up the 66-point set $\mathfrak{D}_2(\sigma_K)$ and then blow up the strict transform of $\mathfrak{D}_{\geqslant 1}(\sigma_K) = q_-(\mathrm{Bl}_{12}(\mathbb{P}(K)))$ (see Remark 4.12). This blow-up should carry a rank-2 vector bundle whose projectivization is also an iterated blow-up of $\mathrm{Bl}_{X_K}(\mathbb{P}(K^\perp))$. Such a description would be useful for describing the Chow motive of X_K.

4.4. Integral Chow motives. We use the blow-up relation obtained above to give a description of the integral Chow motive of X_K. To be precise, we prove an analogue of Corollary 3.8 at the integral level.

Theorem 4.16. *Let X_K be a smooth linear section of the spinor tenfold X of codimension $k \leqslant 5$. Then the integral Chow motive of X_K is of Lefschetz type. More precisely, $\mathbf{M}(X_K)$ is equal to*

$$\begin{cases} 1 \oplus \mathbf{L} \oplus \mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus 2\mathbf{L}^4 \oplus 2\mathbf{L}^5 \oplus 2\mathbf{L}^6 \oplus 2\mathbf{L}^7 \oplus \mathbf{L}^8 \oplus \mathbf{L}^9 \oplus \mathbf{L}^{10} & \textit{when} \quad k=0, \\ 1 \oplus \mathbf{L} \oplus \mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus 2\mathbf{L}^4 \oplus 2\mathbf{L}^5 \oplus 2\mathbf{L}^6 \oplus \mathbf{L}^7 \oplus \mathbf{L}^8 \oplus \mathbf{L}^9 & \textit{when} \quad k=1, \\ 1 \oplus \mathbf{L} \oplus \mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus 2\mathbf{L}^4 \oplus 2\mathbf{L}^5 \oplus \mathbf{L}^6 \oplus \mathbf{L}^7 \oplus \mathbf{L}^8 & \textit{when} \quad k=2, \\ 1 \oplus \mathbf{L} \oplus \mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus 2\mathbf{L}^4 \oplus \mathbf{L}^5 \oplus \mathbf{L}^6 \oplus \mathbf{L}^7 & \textit{when} \quad k=3, \\ 1 \oplus \mathbf{L} \oplus \mathbf{L}^2 \oplus 2\mathbf{L}^3 \oplus \mathbf{L}^4 \oplus \mathbf{L}^5 \oplus \mathbf{L}^6 & \textit{when} \quad k=4, \\ 1 \oplus \mathbf{L} \oplus \mathbf{L}^2 \oplus \mathbf{L}^3 \oplus \mathbf{L}^4 \oplus \mathbf{L}^5 & \textit{when} \quad k=5. \end{cases}$$

Moreover, $\mathrm{CH}^i(X_K) \cong \mathbb{Z}^{n_i}$, where the ranks n_i are equal to the multiplicities of the corresponding Lefschetz motives $\mathbf{L}^i$ in $\mathbf{M}(X_K)$.

Proof. By Proposition 4.14 we have an isomorphism

$$\mathrm{Bl}_{q^{-1}(\gamma(\mathbb{P}(K)))}(\mathrm{Bl}_{X_K}(\mathbb{P}(K^{\perp}))) \cong \mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(\mathscr{F}),$$

where $\mathscr{F}$ is a vector bundle of rank $8-k$. Using the blow-up formula for motives, we deduce that

$$\begin{aligned} \mathbf{M}(\mathrm{Bl}_{q^{-1}(\gamma(\mathbb{P}(K)))}(\mathrm{Bl}_{X_K}(\mathbb{P}(K^{\perp})))) &= \mathbf{M}(\mathbb{P}(K^{\perp})) \oplus \mathbf{M}(X_K) \otimes (\mathbf{L} \oplus \mathbf{L}^2 \oplus \mathbf{L}^3 \oplus \mathbf{L}^4) \\ &\quad \oplus \mathbf{M}(\mathbb{P}(K)) \otimes \mathbf{M}(\mathbb{P}^{7+c-k}) \otimes (\mathbf{L} \oplus \cdots \oplus \mathbf{L}^{7-k}). \end{aligned}$$

In a similar vein, using the blow-up formula and the projective bundle formula, we deduce that

$$\begin{aligned} \mathbf{M}(\mathbb{P}_{\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})}(\mathscr{F})) &= \mathbf{M}(\mathrm{Bl}_{\gamma(\mathbb{P}(K))}(\mathrm{Q})) \otimes \mathbf{M}(\mathbb{P}^{7-k}) \\ &= \big(\mathbf{M}(\mathrm{Q}) \oplus \mathbf{M}(\mathbb{P}(K)) \otimes (\mathbf{L} \oplus \cdots \oplus \mathbf{L}^{8-k})\big) \otimes (1 \oplus \mathbf{L} \oplus \cdots \oplus \mathbf{L}^{7-k}). \end{aligned}$$

The left-hand sides of the equalities are isomorphic. On the other hand, the right-hand side of the second equality is a sum of Lefschetz motives. Therefore, $\mathbf{M}(X_K) \otimes \mathbf{L}$, being a summand of the first equality, is a sum of Lefschetz motives. Hence so is $\mathbf{M}(X_K)$.

Of course, the multiplicities of the motives $\mathbf{L}^i$ in $\mathbf{M}(X_K)$ are determined by the multiplicities of $\mathbf{L}^i_{\mathbb{Q}}$ in the decomposition (computed in Corollary 3.8) of the motive $\mathbf{M}_{\mathbb{Q}}(X_K)$. This proves the desired formulae for $\mathbf{M}(X_K)$. The isomorphisms of the Chow groups of X_K follow immediately from the resulting expressions for the motive $\mathbf{M}(X_K)$. $\square$

Using the approach sketched in Remark 4.15, one can also show that

$$\mathbf{M}(X_K) = 1 \oplus \mathbf{L} \oplus 12\mathbf{L}^2 \oplus \mathbf{L}^3 \oplus \mathbf{L}^4$$

in the case when $k=6$.

§ 5. Linear sections of codimension 1

As already mentioned (Corollary 4.2), the $\mathrm{Spin}(\mathrm{V})$-action on the projective space $\mathbb{P}(\mathbb{S}^\vee)$ of hyperplanes in $\mathbb{P}(\mathbb{S})$ has exactly two orbits, the dual spinor variety $X^\vee \subset \mathbb{P}(\mathbb{S}^\vee)$ and its complement $\mathbb{P}(\mathbb{S}^\vee) \setminus X^\vee$. Hence there are two isomorphism classes of hyperplane sections of X, singular and smooth. In this section we give a geometric description of each.

5.1. Blow-up of a 4-space on *X*. Let $U_{5,-} \subset \mathrm{V}$ be the isotropic subspace corresponding to a point of $X^\vee$. We recall the definition of the associated 4-space $\Pi^4_{U_{5,-}} = \mathrm{Gr}(4, U_{5,-}) \cong \mathbb{P}(\bigwedge^4 U_{5,-}) \subset X \subset \mathbb{P}(\mathbb{S})$; see (3.15). Consider the corresponding embedding $\bigwedge^4 U_{5,-} \hookrightarrow \mathbb{S}$ and put

$$W := \mathbb{S}/\textstyle\bigwedge^4 U_{5,-}. \tag{5.1}$$

This is a vector space of dimension 11, and $\mathbb{P}(W) \cong \mathbb{P}^{10}$.

The following result can be found in [3] (Theorem III.3.8(5)) and the rational map $f_X \circ f_W^{-1}$ constructed below is an example of a special birational transformation of type $(2,1)$ in [7]. We give an independent proof using the blow-up lemma.

Proposition 5.1. *We have an embedding* $\mathrm{Gr}(2, U_{5,-}) \hookrightarrow \mathbb{P}(\bigwedge^2 U_{5,-}) \hookrightarrow \mathbb{P}(W)$, *an isomorphism of blow-ups* $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X) \cong \mathrm{Bl}_{\mathrm{Gr}(2,U_{5,-})}(\mathbb{P}(W))$ *and a diagram*

$$\begin{tikzcd}
& E_\Pi \arrow[r] \arrow[dl] & \mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X) \arrow[r, phantom, "\cong"] \arrow[dl, "f_X"] & \mathrm{Bl}_{\mathrm{Gr}(2,U_{5,-})}(\mathbb{P}(W)) \arrow[dr, "f_W"] & E_{\mathrm{Gr}} \arrow[l] \arrow[dr] & \\
\Pi^4_{U_{5,-}} \arrow[r] & X \arrow[rrr, dashed, "f_W \circ f_X^{-1}"] & & & \mathbb{P}(W) & \mathrm{Gr}(2, U_{5,-}). \arrow[l]
\end{tikzcd} \tag{5.2}$$

If H_X and H_W denote the hyperplane classes of X and $\mathbb{P}(W)$ while E_Π and E_{Gr} denote the exceptional divisors of the blow-ups, then we have linear equivalences

$$\begin{aligned} H_X &= 2H_W - E_{\mathrm{Gr}}, &\qquad H_W &= H_X - E_\Pi, \\ E_\Pi &= H_W - E_{\mathrm{Gr}}, &\qquad E_{\mathrm{Gr}} &= H_X - 2E_\Pi. \end{aligned} \tag{5.3}$$

Finally, the birational map $f_W \circ f_X^{-1} \colon X \dashrightarrow \mathbb{P}(W)$ is induced by the linear projection $\mathbb{S} \to W$ in (5.1) or, in other words, it is the linear projection centred at $\Pi^4_{U_{5,-}}$. Moreover, $f_W(E_\Pi) \subset \mathbb{P}(W)$ is the hyperplane containing $\mathrm{Gr}(2, U_{5,-})$.

The divisor $f_X(E_{\mathrm{Gr}})$ will also be described in (5.13) below.

Proof. Consider an abstract 5-dimensional vector space V_5 and define

$$W = \textstyle\bigwedge^2 V_5 \oplus \Bbbk \tag{5.4}$$

(later we shall identify V_5 with $U_{5,-}$, and the direct sum just defined will be identified with the quotient space $\mathbb{S}/\bigwedge^4 U_{5,-}$ in (5.1)). Then we have a natural embedding

$\mathrm{Gr}(2, V_5) \hookrightarrow \mathbb{P}(\bigwedge^2 V_5) \hookrightarrow \mathbb{P}(W)$. Consider the blow-up

$$f_W\colon \mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W)) \to \mathbb{P}(W).$$

Below we construct a map f_X from the blow-up $\mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W))$ to the spinor tenfold $X = \mathrm{OGr}_+(5, \mathrm{V})$ by producing an isotropic rank-5 vector subbundle in the trivial vector bundle with fibre V. Then we will check that f_X is birational and apply the blow-up lemma to show that f_X is the blow-up of a 4-space on X.

Recall the following natural resolution on $\mathbb{P}(\bigwedge^2 V_5)$:

$$0 \to \mathscr{O}(-5) \xrightarrow{\xi\wedge\xi} V_5^\vee(-3) \xrightarrow{\xi} V_5(-2) \xrightarrow{\xi\wedge\xi} \mathscr{O} \to \mathscr{O}_{\mathrm{Gr}(2,V_5)} \to 0,$$

where $\xi \in H^0(\mathbb{P}(\bigwedge^2 V_5), \bigwedge^2 V_5(1))$ is the tautological section and $\xi \wedge \xi$ is its exterior square in $H^0(\mathbb{P}(\bigwedge^2 V_5), \bigwedge^4 V_5(2))$. Combining this resolution with the Koszul resolution

$$0 \to \mathscr{O}_{\mathbb{P}(W)}(-1) \xrightarrow{\eta} \mathscr{O}_{\mathbb{P}(W)} \to \mathscr{O}_{\mathbb{P}(\wedge^2 V_5)} \to 0,$$

where $\eta \in H^0(\mathbb{P}(W), \mathscr{O}_{\mathbb{P}(W)}(1))$ is the equation of the hyperplane $\mathbb{P}(\bigwedge^2 V_5) \subset \mathbb{P}(W)$, we obtain the following resolution on $\mathbb{P}(W)$:

$$\begin{aligned} 0 \to \mathscr{O}_{\mathbb{P}(W)}(-6) &\xrightarrow{\left(\begin{smallmatrix} -\eta \\ \xi\wedge\xi \end{smallmatrix}\right)} \mathscr{O}_{\mathbb{P}(W)}(-5) \oplus V_5^\vee \otimes \mathscr{O}_{\mathbb{P}(W)}(-4) \\ &\xrightarrow{\left(\begin{smallmatrix} \xi\wedge\xi & \eta \\ 0 & \xi \end{smallmatrix}\right)} (V_5^\vee \oplus V_5) \otimes \mathscr{O}_{\mathbb{P}(W)}(-3) \\ &\xrightarrow{\left(\begin{smallmatrix} \xi & -\eta \\ 0 & \xi\wedge\xi \end{smallmatrix}\right)} V_5 \otimes \mathscr{O}_{\mathbb{P}(W)}(-2) \oplus \mathscr{O}_{\mathbb{P}(W)}(-1) \\ &\xrightarrow{(\xi\wedge\xi, \eta)} \mathscr{O}_{\mathbb{P}(W)} \to \mathscr{O}_{\mathrm{Gr}(2,V_5)} \to 0. \end{aligned} \tag{5.5}$$

We pull back the complex (5.5) to the blow-up $\mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W))$. Of course, it is no longer exact and, in fact, its cohomology sheaves are isomorphic to the exterior powers of the excess conormal bundle

$$\overline{\mathscr{N}}^\vee := \mathrm{Ker}\big(\,\mathrm{pr}^*\big(\mathscr{N}^\vee_{\mathrm{Gr}(2,V_5)/\mathbb{P}(W)}\big) \twoheadrightarrow \mathscr{O}_{E_{\mathrm{Gr}}}(-E_{\mathrm{Gr}})\big)$$

(we write $\mathrm{pr}\colon E_{\mathrm{Gr}} \to \mathrm{Gr}(2, V_5)$ for the projection of the exceptional divisor). In other words, we have the following exact sequences on $\mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W))$:

$$\begin{aligned} &0 \to \mathscr{F}' \to V_5 \otimes \mathscr{O}(-2H_W) \oplus \mathscr{O}(-H_W) \to \mathscr{O} \to \mathscr{O}_{E_{\mathrm{Gr}}} \to 0, \\ &0 \to \mathscr{F}'' \to (V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3H_W) \to \mathscr{F}' \to \overline{\mathscr{N}}^\vee \to 0, \\ &0 \to \mathscr{F}''' \to \mathscr{O}(-5H_W) \oplus V_5^\vee \otimes \mathscr{O}(-4H_W) \to \mathscr{F}'' \to {\textstyle\bigwedge}^2 \overline{\mathscr{N}}^\vee \to 0, \\ &0 \to \mathscr{O}(-6H_W) \to \mathscr{F}''' \to {\textstyle\bigwedge}^3 \overline{\mathscr{N}}^\vee \to 0. \end{aligned} \tag{5.6}$$

Consider the vector space $V_5^\vee \oplus V_5$ with its natural non-degenerate quadratic form (induced by the pairing between the summands). We claim that the sheaf

$$\mathscr{F}'' := \mathrm{Ker}\left((V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3H_W) \xrightarrow{\left(\begin{smallmatrix} \xi & -\eta \\ 0 & \xi\wedge\xi \end{smallmatrix}\right)} V_5 \otimes \mathscr{O}(-2H_W) \oplus \mathscr{O}(-H_W)\right), \tag{5.7}$$

which is defined by the second exact sequence in (5.6), is an isotropic subbundle of rank 5 in $(V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3H_W)$ and therefore defines a regular map $\mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W)) \to \mathrm{OGr}_+(5, V_5^\vee \oplus V_5) = X$.

Indeed, the sheaf $\mathscr{O}_{E_{\mathrm{Gr}}}$ in the first sequence in (5.6) is locally free on a divisor, whence the sheaf $\mathscr{F}'$ is locally free of rank 5. Similarly, the sheaf $\overline{\mathscr{N}}^\vee$ in the second sequence in (5.6) is locally free on a divisor, whence the kernel of the map $\mathscr{F}' \to \overline{\mathscr{N}}^\vee$ is locally free of rank 5. Therefore the sheaf $\mathscr{F}''$ is locally free of rank 5 and its embedding in $(V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3H_W)$ is a fibrewise monomorphism.

Let us show that the subbundle $\mathscr{F}''$ of $(V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3H_W)$ is isotropic. Clearly, it suffices to verify this on the open subset

$$\mathbb{P}(W) \setminus \mathbb{P}(\textstyle\bigwedge^2 V_5) \subset \mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W)), \tag{5.8}$$

the complement of the linear span of the Grassmannian. Another way to describe this open subset is by the inequality $\eta \neq 0$. Hence, by rescaling, we may assume that $\eta = 1$ on (5.8) and use ξ as a coordinate.

On the open set (5.8), the complex (5.5) is acyclic and, therefore, the bundle $\mathscr{F}''$ is just the image of the second map of the complex. Since the image of the first map $\binom{-1}{\xi\wedge\xi}$ surjects over the first summand $\mathscr{O}(-5)$ of the second term $\mathscr{O}(-5) \oplus V_5^\vee \otimes \mathscr{O}(-4)$, we see that $\mathscr{F}''$ is the image of the second summand. Thus,

$$\mathscr{F}''|_{\mathbb{P}(W)\setminus\mathbb{P}(\wedge^2 V_5)} = \mathrm{Im}\Big(V_5^\vee \otimes \mathscr{O}(-4) \xrightarrow{\binom{1}{\xi}} (V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3)\Big).$$

In other words, $\mathscr{F}''|_{\mathbb{P}(W)\setminus\mathbb{P}(\wedge^2 V_5)}$ is the graph of the map

$$V_5^\vee \otimes \mathscr{O}(-4) \xrightarrow{\xi} V_5 \otimes \mathscr{O}(-3).$$

The map ξ is skew-symmetric by definition and, therefore, its graph is isotropic with respect to the natural quadratic form.

Next, we choose an isomorphism

$$\mathrm{V} \cong V_5^\vee \oplus V_5$$

compatible with the quadratic forms on these vector spaces and such that the isotropic subspace $V_5 \subset V_5^\vee \oplus V_5 \cong \mathrm{V}$ corresponds to the point $[U_{5,-}]$ of $X^\vee$ (this identifies the subspaces V_5 and $U_{5,-}$). Then we obtain a map

$$f_X \colon \mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W)) \to X$$

such that $f_X^*(\mathscr{U}) \cong \mathscr{F}''(3H_W)$. Indeed, the map to $\mathrm{OGr}(5, \mathrm{V})$ is given by the universal property of the Grassmannian, and its image lies in the connected component X because the graph of any map $V_5^\vee \to V_5$ is disjoint from the subspace $V_5 \subset V_5^\vee \oplus V_5$ and hence, when isotropic, corresponds by (3.5) to a point of X. This proves that the image of the open subset (5.8) lies in X. Hence the same holds for the whole scheme $\mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W))$ by continuity.

To show that f_X is birational, merely note that its restriction to the open subset (5.8) is an isomorphism onto the open subset of $X \cong \mathrm{OGr}_+(5, V_5^\vee \oplus V_5)$ parametrizing the isotropic subspaces disjoint from the subspace V_5. Indeed, every such subspace is the graph of a map $V_5^\vee \to V_5$, and a graph is isotropic if and only if the corresponding map is skew-symmetric.

Using the exact sequences (5.6), we easily compute that

$$\begin{aligned} \mathrm{c}_1(\mathscr{F}'(3H_W)) &= 4H_W + E_{\mathrm{Gr}}, \\ \mathrm{c}_1(\mathscr{F}''(3H_W)) &= -(4H_W + E_{\mathrm{Gr}}) + 3E_{\mathrm{Gr}} = -2(2H_W - E_{\mathrm{Gr}}). \end{aligned}$$

Since $\mathrm{c}_1(\mathscr{U}) = -2H_X$ by (3.6), it follows that $H_X = 2H_W - E_{\mathrm{Gr}}$. This proves the first relation in (5.3). In other words, the rational map $f_X \circ f_W^{-1}$ is given by quadrics passing through $\mathrm{Gr}(2, V_5)$.

We now consider the restriction of the map $f_X \circ f_W^{-1}$ to $\mathbb{P}(\bigwedge^2 V_5) \subset \mathbb{P}(W)$, the linear span of the Grassmannian. It is well known that the map given by the Plücker quadrics determines an isomorphism

$$\mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(\textstyle\bigwedge^2 V_5)) \cong \mathbb{P}_{\mathrm{Gr}(4,V_5)}(\textstyle\bigwedge^2 \mathscr{U}_4), \tag{5.9}$$

where $\mathscr{U}_4$ is the tautological vector bundle of rank 4 on $\mathrm{Gr}(4, V_5)$ (actually, this is an analogue for $\mathrm{Gr}(2, 5)$ of the isomorphism in Proposition 4.1). Hence the map f_X contracts the strict transform of the hyperplane $\mathbb{P}(\bigwedge^2 V_5)$ onto $\mathrm{Gr}(4, V_5) \subset X$. In other words, we have a commutative diagram

$$\begin{tikzcd} \mathbb{P}_{\mathrm{Gr}(4,V_5)}(\bigwedge^2 \mathscr{U}_4) \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{\mathrm{Gr}(2,V_5)}(\mathbb{P}(W)) \arrow[d, "f_X"] \\ \mathrm{Gr}(4, V_5) \arrow[r, hook] & X. \end{tikzcd}$$

Clearly, the relative Picard number for f_X is equal to 1. Therefore, by Lemma 2.5 we conclude that f_X is the blow-up of $\mathrm{Gr}(4, V_5) \subset X$ and

$$E_\Pi := \mathbb{P}_{\mathrm{Gr}(4,V_5)}(\textstyle\bigwedge^2 \mathscr{U}_4)$$

is the exceptional divisor of f_X. Moreover, the above argument shows that E_Π is the strict transform of the hyperplane $\mathbb{P}(\bigwedge^2 V_5) \subset \mathbb{P}(W)$ containing $\mathrm{Gr}(2, V_5)$. Hence $f_W(E_\Pi) = \mathbb{P}(\bigwedge^2 V_5)$ and we have a linear equivalence $E_\Pi = H_W - E_{\mathrm{Gr}}$. Combining it with the linear equivalence $H_X = 2H_W - E_{\mathrm{Gr}}$ proved above, we deduce (5.3). This completes the proof of the first two parts of the proposition. It remains to identify the image $\mathrm{Gr}(4, V_5) = f_X(E_\Pi) \subset X$ with $\Pi^4_{U_{5,-}}$, the space (5.4) with (5.1), and the rational map $f_W \circ f_X^{-1}$ with the linear projection from $\Pi^4_{U_{5,-}}$.

First, by (5.3) we have an isomorphism

$$\mathbb{S}^\vee \cong H^0(X, \mathscr{O}_X(H_X)) \cong H^0(\mathbb{P}(W), I_{\mathrm{Gr}(2,V_5)}(2H_W)).$$

The right-hand side is the space of quadrics in $\mathbb{P}(W)$ through $\mathrm{Gr}(2, V_5)$. Hence (5.5) yields an exact sequence

$$0 \to W^\vee \xrightarrow{\eta} \mathbb{S}^\vee \to \textstyle\bigwedge^4 V_5^\vee \to 0,$$

whose first term is the space of quadrics containing the hyperplane $\mathbb{P}(\bigwedge^2 V_5) \subset \mathbb{P}(W)$ and the last term is the space of quadrics in $\mathbb{P}(\bigwedge^2 V_5)$. After dualization we obtain

$$0 \to \bigwedge\nolimits^4 V_5 \to \mathbb{S} \to W \to 0 \tag{5.10}$$

and, therefore, the image $f_X(E_\Pi)$ of the exceptional divisor E_Π is identified with $\mathrm{Gr}(4, V_5) = \mathbb{P}(\bigwedge^4 V_5) \subset \mathbb{P}(\mathbb{S})$. This gives an identification of the space W (defined in (5.4)) with the quotient (5.1), using the identification (established above) of the spaces V_5 and $U_{5,-}$. Moreover, the composition of f_X and the linear projection $\mathbb{P}(\mathbb{S}) \dashrightarrow \mathbb{P}(W)$ coincides with the map given by the linear system of quadrics in $\mathbb{P}(W)$ containing the hyperplane $\mathbb{P}(\bigwedge^2 V_5)$, which is equivalent to the linear system of all hyperplanes. This proves that the map $f_W \circ f_X^{-1}$ is the linear projection from $\mathbb{P}(\bigwedge^4 V_5)$.

Finally, it remains to check that by putting $U_{5,-} = V_5$, we obtain an identification of $\mathbb{P}(\bigwedge^4 V_5)$ with the 4-space $\Pi^4_{U_{5,-}}$. To do this, we restrict the map $f_X \circ f_W^{-1}$ to the subscheme $\mathbb{P}(\bigwedge^2 V_5) \setminus \mathrm{Gr}(2, V_5) \subset E_\Pi$. On this subscheme of $\mathbb{P}(W)$, we have $\eta = 0$ and ξ is a skew-symmetric matrix of rank 4. By (5.7), the intersection of each fibre of the subbundle $\mathscr{F}'' \subset (V_5^\vee \oplus V_5) \otimes \mathscr{O}(-3H_W)$ with the subspace $V_5 \subset V_5^\vee \oplus V_5$ is the kernel of $\xi \wedge \xi \colon V_5 \otimes \mathscr{O}(-3H_W) \to \mathscr{O}(-H_W)$, that is, a 4-dimensional subspace of V_5. Hence the image $f_X(E_\Pi)$ is contained in the locus of those subspaces $U_5 \subset \mathrm{V}$ that have a 4-dimensional intersection with V_5. Thus, $\mathbb{P}(\bigwedge^4 V_5) \subset \Pi^4_{V_5} = \Pi^4_{U_{5,-}}$. Since both sides are 4-spaces, this is an equality. □

Remark 5.2. Note that the direct-sum decomposition (5.4), which was used in the proof of the proposition, is not canonical (actually, it corresponds to the isotropic direct sum decomposition $\mathrm{V} = V_5^\vee \oplus V_5$ obtained from (5.4) in the course of the proof). On the other hand, there is a canonical exact sequence

$$0 \to \bigwedge\nolimits^2 U_{5,-} \to W \to \Bbbk \to 0, \tag{5.11}$$

where we identify $V_5 = U_{5,-}$ as in the proof. Indeed, the subspace $\bigwedge^2 U_{5,-}$ corresponds to the linear span of the Grassmannian $\mathrm{Gr}(2, U_{5,-}) \subset \mathbb{P}(W)$ (the centre of the blow-up f_W).

5.2. Blow-ups of 4-spaces on hyperplane sections of X. As before we let $U_{5,-} \subset \mathrm{V}$ be an isotropic subspace corresponding to a point of $X^\vee$, and let W be the space defined by (5.1). Consider the pre-image in $\mathbb{S}$ of the hyperplane $\bigwedge^2 U_{5,-} \subset W$ (see (5.11)) with respect to the linear projection $\mathbb{S} \to W$ of (5.1). This is a hyperplane in $\mathbb{S}$. We denote the corresponding hyperplane section of X by $X_{U_{5,-}} \subset X$. Denote by $\mathscr{U}_2$ the tautological rank-2 bundle on $\mathrm{Gr}(2, U_{5,-})$ and by $\mathscr{U}_2^\perp$ the tautological rank-3 bundle on the same Grassmannian.

Corollary 5.3. *The singular locus of the hyperplane section $X_{U_{5,-}} \subset X$ is the 4-space $\Pi^4_{U_{5,-}} \subset X$. Moreover, there is an isomorphism*

$$\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_{U_{5,-}}) \cong \mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{U}_2^\perp \oplus \mathscr{O}(-1))$$

such that the exceptional divisor E'_Π of the blow-up is identified with the projective bundle $\mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{U}_2^{\perp})$, and there is a commutative diagram

$$\begin{tikzcd}
 & & E'_\Pi \arrow[r, equal, "\sim"] \arrow[d] \arrow[ddll] & \mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{U}_2^{\perp}) \arrow[r, equal, "\sim"] \arrow[d] & \mathrm{Fl}(2,4;U_{5,-}) \arrow[ddl, bend left] \\
 & & \mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_{U_{5,-}}) \arrow[r, phantom, "\cong"] \arrow[dl] & \mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{U}_2^{\perp} \oplus \mathscr{O}(-1)) \arrow[d] & \\
\Pi^4_{U_{5,-}} \arrow[r] & X_{U_{5,-}} \arrow[rr, dashed, "f_W \circ f_X^{-1}"] & & \mathrm{Gr}(2,U_{5,-}). &
\end{tikzcd} \tag{5.12}$$

Proof. We use the notation introduced in the proof of Proposition 5.1 with the identification $V_5 = U_{5,-}$. Consider the pre-image $f_W^{-1}(\mathbb{P}(\bigwedge^2 U_{5,-})) \subset \mathrm{Bl}_{\mathrm{Gr}(2,U_{5,-})}(\mathbb{P}(W))$ of the hyperplane $\mathbb{P}(\bigwedge^2 U_{5,-}) \subset \mathbb{P}(W)$. Since $\mathrm{Gr}(2,U_{5,-}) \subset \mathbb{P}(\bigwedge^2 U_{5,-})$, this pre-image contains the exceptional divisor E_{Gr}. On the other hand, the strict transform of this hyperplane is the exceptional divisor E_Π. Since $E_\Pi + E_{\mathrm{Gr}} = H_W$ by (5.3), it follows that there are no other components in the pre-image, that is,

$$f_W^{-1}(\mathbb{P}(\textstyle\bigwedge^2 U_{5,-})) = E_\Pi \cup E_{\mathrm{Gr}}.$$

The morphism f_X contracts E_Π to the 4-space $\Pi^4_{U_{5,-}}$ and maps E_{Gr} to a hyperplane section of the spinor tenfold X, singular along $\Pi^4_{U_{5,-}}$ (this follows from the linear equivalence $E_{\mathrm{Gr}} = H_X - 2E_\Pi$). On the other hand, since the rational map $f_W \circ f_X^{-1} \colon X \dashrightarrow \mathbb{P}(W)$ in Proposition 5.1 is the linear projection induced by the map $\mathbb{S} \to W$, the strict transform of the hyperplane $\mathbb{P}(\bigwedge^2 U_{5,-})$ in X is the hyperplane section $X_{U_{5,-}} \subset X$. Hence the morphism f_X induces a birational map $E_{\mathrm{Gr}} \to X_{U_{5,-}}$, so that

$$X_{U_{5,-}} = f_X(E_{\mathrm{Gr}}). \tag{5.13}$$

Furthermore, E_{Gr} is the strict transform of $X_{U_{5,-}}$ in $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X)$ and, therefore,

$$\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_{U_{5,-}}) \cong E_{\mathrm{Gr}} \cong \mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{N}_{\mathrm{Gr}(2,U_{5,-})/\mathbb{P}(W)}).$$

It remains to show that $X_{U_{5,-}} \setminus \Pi^4_{U_{5,-}}$ is smooth and that the normal bundle of the Grassmannian $\mathrm{Gr}(2,U_{5,-})$ in $\mathbb{P}(W)$ is isomorphic to a twist of $\mathscr{U}_2^{\perp} \oplus \mathscr{O}(-1)$. Clearly, the first follows from the smoothness of E_{Gr}. For the second we use the exact sequence

$$0 \to \mathscr{N}_{\mathrm{Gr}(2,U_{5,-})/\mathbb{P}(\wedge^2 U_{5,-})} \to \mathscr{N}_{\mathrm{Gr}(2,U_{5,-})/\mathbb{P}(W)} \to \mathscr{O}_{\mathbb{P}(W)}(1)|_{\mathrm{Gr}(2,U_{5,-})} \to 0$$

whose last term is the restriction of the normal bundle of the hyperplane $\mathbb{P}(\bigwedge^2 U_{5,-}) \subset \mathbb{P}(W)$. It is well known that the first term is isomorphic to $\mathscr{U}_2^{\perp}(2)$ (see, for example, [11], Proposition A.7). Hence the middle term is an extension of $\mathscr{O}(1)$ by $\mathscr{U}_2^{\perp}(2)$. On the other hand, the Borel–Bott–Weil theorem yields that

$$\mathrm{Ext}^1(\mathscr{O}(1), \mathscr{U}_2^{\perp}(2)) \cong H^1(\mathrm{Gr}(2,U_{5,-}), \mathscr{U}_2^{\perp}(1)) = 0,$$

whence the extension splits and we deduce an isomorphism

$$\mathscr{N}_{\mathrm{Gr}(2,U_{5,-})/\mathbb{P}(W)} \cong \mathscr{U}_2^{\perp}(2) \oplus \mathscr{O}(1).$$

Since projectivization is not affected by a twist, we obtain the required isomorphism.

Finally, we have to describe the exceptional divisor E'_{Π}. The argument above shows that $E'_{\Pi} = E_{\Pi} \cap E_{\mathrm{Gr}}$, whence E'_{Π} is nothing but the exceptional divisor of the blow-up of $\mathbb{P}(\bigwedge^2 U_{5,-})$ along $\mathrm{Gr}(2, U_{5,-})$. Therefore,

$$E'_{\Pi} \cong \mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{N}_{\mathrm{Gr}(2,U_{5,-})/\mathbb{P}(\wedge^2 U_{5,-})}) \cong \mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{U}_2^{\perp}),$$

which embeds in $\mathbb{P}_{\mathrm{Gr}(2,U_{5,-})}(\mathscr{U}_2^{\perp} \oplus \mathscr{O}(-1))$ as the projectivization of the first summand. $\square$

Remark 5.4. It is easy to see that the hyperplane section $X_{U_{5,-}}$ considered above is nothing but the singular hyperplane section of X associated with the point $[U_{5,-}] \in X^{\vee}$ (recall that $X^{\vee}$ is the projective dual of X). This can be done as follows. By projective duality, every singular hyperplane section of X corresponds to a point of $X^{\vee}$. We denote the point corresponding to $X_{U_{5,-}}$ by $[U'_{5,-}]$. Then the map $[U_{5,-}] \mapsto [U'_{5,-}]$ is an automorphism of $X^{\vee}$, which is canonical and, therefore, Spin(V)-equivariant. Hence it belongs to the centre of the group $\mathrm{Aut}(X^{\vee}) \cong \mathrm{PSO(V)}$, which is trivial.

As an application of our results, we describe the Hilbert scheme $F_4(X_{U_{5,-}})$ of 4-spaces on $X_{U_{5,-}}$ and the Hilbert scheme $G_6(X_{U_{5,-}})$ of 6-quadrics on $X_{U_{5,-}}$.

Corollary 5.5. *If $X_{U_{5,-}} \subset X$ is the singular hyperplane section of X corresponding to an isotropic subspace $U_{5,-} \subset \mathrm{V}$, then $F_4(X_{U_{5,-}}) \cong \mathrm{Cone}(\mathrm{Gr}(3, U_{5,-}))$ (the cone in the Plücker embedding).*

Proof. By Theorem 3.2, every 4-space on X is equal to $\Pi^4_{U'_{5,-}}$ for some isotropic subspace $U'_{5,-} \subset \mathrm{V}$. Let us check that the following three conditions are equivalent:

1) $\Pi^4_{U'_{5,-}} \subset X_{U_{5,-}}$;

2) $\dim(U'_{5,-} \cap U_{5,-}) \geqslant 3$;

3) $\Pi^4_{U'_{5,-}} \cap \Pi^4_{U_{5,-}} \neq \varnothing$.

2) $\Rightarrow$ 3). If $\dim(U'_{5,-} \cap U_{5,-}) > 3$, then by (3.5) we have $U'_{5,-} = U_{5,-}$ and there is nothing to prove. Therefore we assume that $\dim(U'_{5,-} \cap U_{5,-}) = 3$ and let U_3 be the intersection. By (3.18) we have

$$\Pi^1_{U_3,U_{5,-}} = L_{U_3} = \Pi^1_{U_3,U'_{5,-}},$$

whence $L_{U_3} \subset \Pi^4_{U'_{5,-}} \cap \Pi^4_{U_{5,-}}$ and 3) holds.

3) $\Rightarrow$ 2). Let $[U_5] \in \Pi^4_{U'_{5,-}} \cap \Pi^4_{U_{5,-}}$ be a point in the intersection of the 4-spaces. Using their definition (3.15), we conclude that $\dim(U_5 \cap U_{5,-}) = \dim(U_5 \cap U'_{5,-}) = 4$, which yields 2).

1) $\Rightarrow$ 3). Assume that $\Pi^4_{U'_{5,-}} \subset X_{U_{5,-}}$ but $\Pi^4_{U'_{5,-}} \cap \Pi^4_{U_{5,-}} = \varnothing$. Then the linear projection $f_W \circ f_X^{-1}$ in (5.2) is regular on $\Pi^4_{U'_{5,-}}$ and its image is a 4-space in $\mathbb{P}(W)$,

which is contained in $\mathrm{Gr}(2, U_{5,-})$ by Corollary 5.3. But this Grassmannian contains no 4-spaces. The resulting contradiction proves the implication.

3) $\Rightarrow$ 1). If $\Pi^4_{U'_{5,-}} \cap \Pi^4_{U_{5,-}} \neq \varnothing$, then the linear projection $f_W \circ f_X^{-1}$ restricted to $\Pi^4_{U'_{5,-}}$ has fibres of positive dimension. Hence the strict transform of $\Pi^4_{U'_{5,-}}$ in $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X)$ is contained in the exceptional divisor E_{Gr} of f_W and, therefore, $\Pi^4_{U'_{5,-}} \subset f_X(E_{\mathrm{Gr}})$, which coincides with $X_{U_{5,-}}$ by (5.13).

We conclude from the equivalence of 1) and 2) that

$$F_4(X_{U_{5,-}}) = \{U'_{5,-} \mid \dim(U'_{5,-} \cap U_{5,-}) \geqslant 3\}.$$

It remains to show that this is a cone over the Grassmannian. To do this, we consider the scheme $\widetilde{F}_4(X_{U_{5,-}})$ parametrizing pairs of subspaces $(U_3, U'_{5,-})$ such that $U_3 \subset U'_{5,-} \cap U_{5,-}$. Forgetting $U'_{5,-}$ defines a regular map $\varphi\colon \widetilde{F}_4(X_{U_{5,-}}) \to \mathrm{Gr}(3, U_{5,-}) \subset \mathrm{OGr}(3, \mathrm{V})$ whose fibre over a point $[U_3] \in \mathrm{Gr}(3, U_{5,-})$ is the line $L^-_{U_3} \subset X^\vee$ defined in (3.12). The description of the universal line L^- in the right half of (3.14) shows that

$$\widetilde{F}_4(X_{U_{5,-}}) = \mathbb{P}_{\mathrm{Gr}(3,U_{5,-})}(\varphi^*(\mathcal{S}_{2,-})),$$

where $\mathcal{S}_{2,-}$ is the spinor bundle on $\mathrm{OGr}(3, \mathrm{V})$. By Remark 2.4, the restriction of this spinor bundle to $\mathrm{Gr}(3, U_{5,-})$ admits a filtration which takes the form of a short exact sequence

$$0 \to \mathscr{O} \to \mathcal{S}_{2,-}|_{\mathrm{Gr}(3,U_{5,-})} \to \textstyle\bigwedge^2 (U_{5,-}/\mathscr{U}_3)^\vee \to 0.$$

Clearly, the last term is isomorphic to $\mathscr{O}(-1)$ and since there are no non-trivial extensions between $\mathscr{O}(-1)$ and $\mathscr{O}$ on $\mathrm{Gr}(3, U_{5,-})$, the restriction of the spinor bundle is isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$. Therefore,

$$\widetilde{F}_4(X_{U_{5,-}}) \cong \mathbb{P}_{\mathrm{Gr}(3,U_{5,-})}(\mathscr{O} \oplus \mathscr{O}(-1)).$$

The projection $\widetilde{F}_4(X_{U_{5,-}}) \to F_4(X_{U_{5,-}})$, of course, contracts the exceptional section of this projective bundle (that parametrizes the pairs $(U_3, U'_{5,-})$ with $U'_{5,-} = U_{5,-}$) to the point of $F_4(X_{U_{5,-}})$ corresponding to the subspace $U_{5,-}$. The result of this contraction is the cone $\mathrm{Cone}(\mathrm{Gr}(3, U_{5,-}))$. $\square$

Lemma 5.6. *If $X_{U_{5,-}} \subset X$ is the singular hyperplane section of X corresponding to an isotropic subspace $U_{5,-} \subset \mathrm{V}$, then $G_6(X_{U_{5,-}}) \cong \mathbb{P}(U_{5,-})$.*

Proof. Recall that every 6-dimensional quadric on X is equal to $\mathscr{Q}_v$ (Corollary 4.7) and that the intersection $\Pi^4_{U_{5,-}} \cap \mathscr{Q}_v$ is either a point or a 3-space (Lemma 4.9).

In the first case, the image of $\mathscr{Q}_v$ under the linear projection $f_W \circ f_X^{-1}$ from $\Pi^4_{U_{5,-}}$ is a $\mathbb{P}^6$, and if $\mathscr{Q}_v \subset X_{U_{5,-}}$, then Corollary 5.3 yields that this $\mathbb{P}^6$ is contained in $\mathrm{Gr}(2, U_{5,-})$, which is absurd.

In the second case, the linear projection $f_W \circ f_X^{-1}$, restricted to $\mathscr{Q}_v$, has fibres of positive dimension. Therefore, the strict transform of $\mathscr{Q}_v$ in $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X)$ is contained in the exceptional divisor E_{Gr} of f_W, whence $\mathscr{Q}_v \subset f_X(E_{\mathrm{Gr}})$, which coincides with $X_{U_{5,-}}$ by (5.13).

It remains to note that, by Lemma 4.9, the intersection $\Pi^4_{U_{5,-}} \cap \mathscr{Q}_v$ is a 3-space if and only if $v \in \mathbb{P}(U_{5,-})$. Hence we have an isomorphism $G_6(X_{U_{5,-}}) \cong \mathbb{P}(U_{5,-})$. $\square$

One can also use Proposition 5.1 to give a description of smooth hyperplane sections of X. The following birational transformation is another example of a special birational transformation of type $(2,1)$ in [7].

Corollary 5.7. *Let $\kappa \in \mathbb{P}(\mathbb{S}^\vee) \setminus X^\vee$ be a point, and let $X_\kappa \subset X$ be the corresponding smooth hyperplane section of X. If $\Pi^4_{U_{5,-}} \subset X_\kappa$, then we have an isomorphism $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_\kappa) \cong \mathrm{Bl}_{Z_\kappa}(\mathbb{P}(W_\kappa))$ and a diagram*

$$\begin{tikzcd} E_{\Pi,\kappa} \arrow[r] \arrow[d] & \mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_\kappa) \arrow[r, equal, "\sim"] \arrow[dl] & \mathrm{Bl}_{Z_\kappa}(\mathbb{P}(W_\kappa)) \arrow[dr] & E_{Z_\kappa} \arrow[l] \arrow[dr] & \\ \Pi^4_{U_{5,-}} \arrow[r] & X_\kappa & & \mathbb{P}(W_\kappa) & Z_\kappa, \arrow[l] \end{tikzcd} \tag{5.14}$$

where $W_\kappa \subset W$ is the hyperplane corresponding to κ, and $Z_\kappa = \mathrm{Gr}(2, U_{5,-}) \cap \mathbb{P}(W_\kappa)$ is a smooth hyperplane section of the Grassmannian.

Proof. Since the map $f_W \circ f_X^{-1} \colon X \dashrightarrow \mathbb{P}(W)$ is a linear projection centred at $\Pi^4_{U_{5,-}}$, the hyperplanes in $\mathbb{P}(\mathbb{S})$ containing $\Pi^4_{U_{5,-}}$ correspond to the hyperplanes in $\mathbb{P}(W)$. Let $W_\kappa \subset W$ be the hyperplane corresponding to κ. Note that it is distinct from the hyperplane $\bigwedge^2 U_{5,-} \subset W$ since the latter corresponds to a singular hyperplane section of X. Therefore the intersection $Z_\kappa = \mathrm{Gr}(2, U_{5,-}) \cap \mathbb{P}(W_\kappa)$ is dimensionally transverse.

The pre-image of $\mathbb{P}(W_\kappa)$ in $\mathrm{Bl}_{\mathrm{Gr}(2,U_{5,-})}(\mathbb{P}(W))$ is isomorphic to the blow-up $\mathrm{Bl}_{Z_\kappa}(\mathbb{P}(W_\kappa))$ and, at the same time, it is the strict transform of X_κ and, therefore, is isomorphic to the blow-up $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_\kappa)$. This gives the required diagram.

It remains to check that Z_κ is smooth. To do this, we note that $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_\kappa)$ is smooth and Z_κ is a locally complete intersection. Therefore Lemma 2.6 applies. $\square$

Later we will show that $F_4(X_\kappa) \neq \varnothing$ (Corollary 5.12), so the above description is applicable.

5.3. Blow-up of a 6-quadric on *X*. We present another description of the spinor tenfold X by projecting from a maximal quadric and use it to give an alternative description of its smooth hyperplane sections. Recall that for each point $v \in \mathrm{Q}$ there is an exact sequence

$$0 \to \mathcal{S}_{8,v} \to \mathbb{S} \to \mathcal{S}_{8,-,v} \to 0 \tag{5.15}$$

(this is the fibre at v of the sequence in Lemma 2.2). In this sequence, the projective spaces $\mathbb{P}(\mathcal{S}_{8,v})$ and $\mathbb{P}(\mathcal{S}_{8,-,v})$ contain smooth 6-dimensional quadrics

$$\mathscr{Q}_v \subset \mathbb{P}(\mathcal{S}_{8,v}) \quad \text{and} \quad \mathscr{Q}_{v,-} \subset \mathbb{P}(\mathcal{S}_{8,-,v});$$

see (3.3). Recall also that

$$\mathscr{Q}_v = \mathrm{OGr}_+(4, v^\perp/v) \subset X \quad \text{and} \quad \mathscr{Q}_{v,-} = \mathrm{OGr}_-(4, v^\perp/v) \subset X^\vee;$$

see (3.4). We write $\mathscr{U}_4$ and $\mathscr{U}_{4,-}$ for the tautological bundles on $\mathscr{Q}_v$ and $\mathscr{Q}_{v,-}$ regarded as isotropic Grassmannians.

Proposition 5.8. *There is an isomorphism*

$$\mathrm{Bl}_{\mathscr{Q}_v}(X) \cong \mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{O}(-1) \oplus \mathscr{U}_{4,-}^\vee(-1))$$

such that the exceptional divisor $E_{\mathscr{Q}}$ of the blow-up is identified with $\mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{U}_{4,-}^\vee(-1))$, so that we have a commutative diagram

$$\begin{tikzcd}
 & E_{\mathscr{Q}} \arrow[r, equal, "\sim"] \arrow[d] \arrow[ddl] & \mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{U}_{4,-}^\vee(-1)) \arrow[d] \\
 & \mathrm{Bl}_{\mathscr{Q}_v}(X) \arrow[r, equal, "\sim"] \arrow[dl, "g_X"] & \mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{O}(-1) \oplus \mathscr{U}_{4,-}^\vee(-1)) \arrow[d, "g_{\mathscr{Q}}"'] \\
\mathscr{Q}_v \arrow[r] & X \arrow[r, dashed, "g_{\mathscr{Q}} \circ g_X^{-1}"] & \mathscr{Q}_{v,-}.
\end{tikzcd} \tag{5.16}$$

If H_X and $H_{\mathscr{Q}}$ denote the hyperplane classes of X and $\mathscr{Q}_{v,-}$ while $E_{\mathscr{Q}}$ denotes the exceptional divisor of the blow-up g_X, then there is a linear equivalence

$$H_{\mathscr{Q}} = H_X - E_{\mathscr{Q}}. \tag{5.17}$$

The rational map $g_{\mathscr{Q}} \circ g_X^{-1} \colon X \dashrightarrow \mathscr{Q}_{v,-}$ in the diagram is induced by the linear projection $\mathbb{P}(\mathbb{S}) \dashrightarrow \mathbb{P}(\mathcal{S}_{8,-,v})$ given by the second map in (5.15).

Proof. Consider the diagram

$$\begin{tikzcd}
\mathscr{Q}_{v,-} \times_{X^\vee} \mathrm{OGr}(4, \mathrm{V}) \arrow[r] \arrow[d] & \mathrm{OGr}(4, \mathrm{V}) \arrow[r] \arrow[d] & X \\
\mathscr{Q}_{v,-} \arrow[r] & X^\vee &
\end{tikzcd} \tag{5.18}$$

obtained from the right half of the diagram (3.16) by base change along the embedding $\mathscr{Q}_{v,-} \hookrightarrow X^\vee$. Then

$$\mathscr{Q}_{v,-} \times_{X^\vee} \mathrm{OGr}(4, \mathrm{V}) \cong \mathbb{P}_{\mathscr{Q}_{v,-}}\big(\mathscr{U}_{5,-}^\vee(-1)|_{\mathscr{Q}_{v,-}}\big).$$

On the other hand, since $\mathscr{Q}_{v,-} = \mathrm{OGr}_-(4, v^\perp/v)$ parametrizes those isotropic subspaces $U_{5,-} \subset \mathrm{V}$ that contain the vector v, we have an exact sequence

$$0 \to \mathscr{O} \xrightarrow{v} \mathscr{U}_{5,-}|_{\mathscr{Q}_{v,-}} \to \mathscr{U}_{4,-} \to 0. \tag{5.19}$$

Using the Borel–Bott–Weil theorem, we see that $\mathrm{Ext}^1(\mathscr{U}_{4,-}, \mathscr{O}) = 0$. Hence the sequence splits and after dualization and twist we obtain an isomorphism

$$\mathscr{U}_{5,-}^\vee(-1)|_{\mathscr{Q}_{v,-}} \cong \mathscr{O}(-1) \oplus \mathscr{U}_{4,-}^\vee(-1).$$

Composing the arrows in the top row of (5.18), we obtain a map

$$g_X\colon \mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{O}(-1)\oplus\mathscr{U}_{4,-}^{\vee}(-1))\cong\mathscr{Q}_{v,-}\times_{X^\vee}\mathrm{OGr}(4,\mathrm{V})\to X.$$

By definition, its fibre over a point $[U_5]$ of X is the intersection of the 4-space $\mathrm{Gr}(4,U_5)\subset X^\vee$ with the 6-quadric $\mathrm{OGr}_-(4,v^\perp/v)$. The argument in Lemma 4.9 (applied to $X^\vee$ instead of X) shows that this intersection consists of a single point (unless $v\in U_5$). Hence the map g_X is birational (and it is an isomorphism over the complement of $\mathscr{Q}_v=\mathrm{OGr}_+(4,v^\perp/v)\subset X$, which parametrizes the subspaces U_5 containing the vector v).

Finally, consider the scheme

$$E_{\mathscr{Q}}:=\mathrm{OGr}(3,v^\perp/v)\cong\mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{U}_{4,-}^{\vee}(-1))\cong\mathbb{P}_{\mathscr{Q}_v}(\mathscr{U}_4^{\vee}(-1)).$$

Clearly, it is a subscheme in $\mathrm{OGr}(4,\mathrm{V})$ and its projection to $X^\vee$ equals $\mathscr{Q}_{v,-}$. Hence it is contained in the fibre product in (5.18) and is a divisor in it. On the other hand, its projection to X equals $\mathscr{Q}_v$ and, therefore, we have the following commutative diagram:

$$\begin{tikzcd}
E_{\mathscr{Q}} \arrow[r, equal] & \mathbb{P}_{\mathscr{Q}_v}(\mathscr{U}_4^{\vee}(-1)) \arrow[r, hook] \arrow[d] & \mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{O}(-1)\oplus\mathscr{U}_{4,-}^{\vee}(-1)) \arrow[d, "g_X"] \\
& \mathscr{Q}_v \arrow[r, hook] & X.
\end{tikzcd}$$

It is easy to see that the relative Picard number of the map g_X is equal to 1. Since $E_{\mathscr{Q}}$ is a divisor in $\mathbb{P}_{\mathscr{Q}_{v,-}}(\mathscr{O}(-1)\oplus\mathscr{U}_{4,-}^{\vee}(-1))$, it follows from Lemma 2.5 that g_X is the blow-up of $\mathscr{Q}_v\subset X$ and $E_{\mathscr{Q}}$ is its exceptional divisor.

Under the identification $E_{\mathscr{Q}}\cong\mathbb{P}_{\mathscr{Q}_v}(\mathscr{U}_4^{\vee}(-1))$, the divisor $E_{\mathscr{Q}}$ is equal to the zero locus of the natural map

$$\mathscr{O}(-H_X)=g_X^*\mathscr{O}(-1)\to g_{\mathscr{Q}}^*(\mathscr{O}(-1)\oplus\mathscr{U}_{4,-}^{\vee}(-1))\to g_{\mathscr{Q}}^*\mathscr{O}(-1)=\mathscr{O}(-H_{\mathscr{Q}}).$$

We deduce that $E_{\mathscr{Q}}=H_X-H_{\mathscr{Q}}$. This proves (5.17).

Finally, (5.17) shows that the map $g_{\mathscr{Q}}\circ g_X^{-1}$ is given by the complete linear system $|H_X-E_{\mathscr{Q}}|$. Hence it is a linear projection from the quadric $\mathscr{Q}_v$ and is induced by the linear projection of $\mathbb{P}(\mathbb{S})$ from its linear span $\mathbb{P}(\mathcal{S}_{8,v})$. $\square$

Remark 5.9. Consider the quadratic cone

$$\widetilde{\mathscr{Q}}_{v,-}:=\mathrm{Cone}_{\mathbb{P}(\mathcal{S}_{8,v})}(\mathscr{Q}_{v,-})\subset\mathbb{P}(\mathbb{S}) \tag{5.20}$$

over $\mathscr{Q}_{v,-}\subset\mathbb{P}(\mathcal{S}_{8,-,v})$ with vertex $\mathbb{P}(\mathcal{S}_{8,v})\subset\mathbb{P}(\mathbb{S})$ (with respect to the linear projection in (5.15)). Since the projection of X from $\mathbb{P}(\mathcal{S}_{8,v})$ is contained in $\mathscr{Q}_{v,-}$ by Proposition 5.8, the quadric $\widetilde{\mathscr{Q}}_{v,-}$ contains X. This is a geometric way of describing some of the quadrics passing through X; see Corollary 4.3.

Similarly, the quadratic cone

$$\widetilde{\mathscr{Q}}_v:=\mathrm{Cone}_{\mathbb{P}(\mathcal{S}_{8,-,v})}(\mathscr{Q}_v)\subset\mathbb{P}(\mathbb{S}^\vee) \tag{5.21}$$

is a quadric containing $X^\vee$.

5.4. Blow-ups of 6-quadrics on smooth hyperplane sections of X. Proposition 5.8 can be applied when describing smooth hyperplane sections of X. Before doing this, we check that every such section contains a 6-quadric. Recall the map $\gamma = q_- \circ p_- \colon \mathbb{P}(\mathbb{S}^\vee) \setminus X^\vee \to \mathrm{Q}$ defined in the diagram (4.4).

Lemma 5.10. *Suppose that $\kappa \in \mathbb{P}(\mathbb{S}^\vee) \setminus X^\vee$ and let X_κ be the corresponding smooth hyperplane section of X. Then X_κ contains a unique 6-dimensional quadric, that is,*

$$G_6(X_\kappa) \cong \operatorname{Spec}(\Bbbk),$$

and this quadric is nothing but $\mathscr{Q}_v = \mathrm{OGr}_+(4, v^\perp/v)$, where $v = \gamma(\kappa) \in \mathrm{Q}$.

Proof. It follows from the description of 6-dimensional quadrics in Corollary 4.7 that the Hilbert scheme $G_6(X_\kappa)$ is equal, as a subscheme of $G_6(X) \cong \mathrm{Q}$, to the zero locus of the global section of the vector bundle

$$q_{-*}p_-^*(\mathscr{O}_{\mathbb{P}(\mathbb{S}^\vee)}(1)) \cong \mathcal{S}_8^\vee$$

on Q (see the diagram (4.4)) that corresponds to the vector $\kappa \in \mathbb{S}^\vee = H^0(\mathrm{Q}, \mathcal{S}_8^\vee)$. But this zero locus is just the point $\gamma(\kappa)$; this can be explained by an argument that is completely analogous to the argument in Proposition 4.1 with X replaced by $X^\vee$, and $\mathcal{S}_{8,-}^\vee$ replaced by $\mathcal{S}_8^\vee$. □

Suppose that $\kappa \in \mathbb{P}(\mathbb{S}^\vee) \setminus X^\vee$ and put $v = \gamma(\kappa)$. Then we have an inclusion $\kappa \in q_-^{-1}(v) = \mathbb{P}(\mathcal{S}_{8,-,v})$. Define a 5-quadric

$$\mathscr{Q}_{\kappa,-} = \mathscr{Q}_{v,-} \cap \mathbb{P}(\kappa^\perp), \tag{5.22}$$

where $\mathbb{P}(\kappa^\perp)$ is the orthogonal in the space $\mathbb{P}(\mathcal{S}_{8,-,v})$ to the point κ with respect to the natural quadratic form on this space. This is a hyperplane section of the smooth quadric $\mathscr{Q}_{v,-}$ and, as will be seen below, it is itself smooth. Recall that by triality the vector bundle $\mathscr{U}_{4,-}$ on the 6-dimensional quadric $\mathscr{Q}_{v,-}$ can be identified with one of its spinor bundles, and the bundle $\mathscr{U}_{4,-}^\vee(-1)$ with the other. The restrictions of both bundles to the 5-dimensional quadric $\mathscr{Q}_{\kappa,-}$ are then identified with the unique spinor bundle $\mathcal{S}_4$ on it.

Combining Proposition 5.8 and Lemma 5.10, we obtain the following result that was also mentioned in [5], Lemma 1.17.

Corollary 5.11. *If X_κ is a smooth hyperplane section of X, $v = \gamma(\kappa)$, and $\mathscr{Q}_v \subset X$ is the 6-dimensional quadric contained in X_κ, then we have an isomorphism $\mathrm{Bl}_{\mathscr{Q}_v}(X_\kappa) \cong \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$ and a commutative diagram*

$$\begin{tikzcd}
 & & E'_{\mathscr{Q}} \arrow[r, equal, "\sim"] \arrow[d] \arrow[ddll] & \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathcal{S}_4) \arrow[r, equal, "\sim"] \arrow[d] & \mathrm{OFl}(1,3;7) \arrow[ddl, bend left] \\
 & & \mathrm{Bl}_{\mathscr{Q}_v}(X_\kappa) \arrow[r, phantom, "\cong"] \arrow[dl, "g_X"'] & \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4) \arrow[d, "g_{\mathscr{Q}}"'] & \\
\mathscr{Q}_v \arrow[r] & X_\kappa & & \mathscr{Q}_{\kappa,-}\,. &
\end{tikzcd} \tag{5.23}$$

Moreover, the 5-dimensional quadric $\mathscr{Q}_{\kappa,-}$ is smooth.

Proof. Since the map $g_{\mathscr{Q}} \circ g_X^{-1} \colon X \dashrightarrow \mathscr{Q}_{v,-}$ constructed in Proposition 5.8 is a linear projection, the hyperplane $\mathbb{P}(\kappa^\perp) \subset \mathbb{P}(\mathbb{S})$ defined by $\kappa \in \mathbb{P}(\mathbb{S}^\vee)$ is the pre-image of a hyperplane in the ambient space $\mathbb{P}(\mathcal{S}_{8,-,v})$ of the quadric $\mathscr{Q}_{v,-}$. To understand which hyperplane it is, we recall that besides (5.15) we have the following exact sequence (the second sequence in Lemma 2.2 for $\mathbb{S}^\vee \cong \mathbb{S}_-$):

$$0 \to \mathcal{S}_{8,-,v} \to \mathbb{S}^\vee \to \mathcal{S}_{8,v} \to 0. \tag{5.24}$$

The duality between $\mathbb{S}$ and $\mathbb{S}^\vee$ is compatible with these sequences, that is, the sequence above is dual to (5.15), and the induced pairing on $\mathcal{S}_{8,-,v}$ coincides with the one given by the natural quadratic form on it (whose associated quadric is $\mathscr{Q}_{v,-}$). This proves that the hyperplane in $\mathbb{P}(\mathcal{S}_{8,-,v})$ corresponding to the hyperplane in $\mathbb{P}(\mathbb{S})$ defined by κ is orthogonal to κ with respect to the natural quadratic form.

Taking the strict transform of the hyperplane section X_κ in the left half of the diagram (5.16) and the pre-image of the hyperplane section $\mathscr{Q}_{\kappa,-}$ of the quadric $\mathscr{Q}_{v,-}$ in the right half, we deduce the isomorphism stated in the corollary and obtain the diagram (5.23).

Since X_κ and $\mathscr{Q}_v$ are smooth, the blow-up $\mathrm{Bl}_{\mathscr{Q}_v}(X_\kappa)$ is smooth and, therefore, the quadric $\mathscr{Q}_{\kappa,-}$ is also smooth by Lemma 2.6. $\square$

Corollary 5.12. *If $X_\kappa \subset X$ is a smooth hyperplane section of X, then*

$$F_4(X_\kappa) \cong \mathscr{Q}_{\kappa,-}$$

and the universal family of 4-spaces on X_κ is given by $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$.

Proof. Assume that $\Pi = \Pi^4_{U_{5,-}}$ is a 4-space lying on X_κ. By Lemma 4.9, its intersection with $\mathscr{Q}_v$ is either a point or a 3-space. If it is a point, then the image of Π in the smooth 5-dimensional quadric $\mathscr{Q}_{\kappa,-}$ must be a 3-space, which is of course impossible. Therefore the intersection is a 3-space, the image of Π in $\mathscr{Q}_{\kappa,-}$ is just a point, and Π is a fibre of the map $g_{\mathscr{Q}} \colon \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4) \to \mathscr{Q}_{\kappa,-}$. $\square$

The isomorphism of the corollary gives an alternative proof of Theorem 4.16 for smooth hyperplane sections of X.

§ 6. Linear sections of codimension 2 and the spinor quadratic line complex

The situation with linear sections of codimension 2 is more interesting than the one with hyperplane sections. We shall show that there are two isomorphism classes of smooth linear sections of codimension 2. We use this result to define an important subvariety of $\mathrm{Gr}(2, \mathbb{S}^\vee)$, which we call the spinor quadratic line complex.

6.1. Special linear sections of codimension 2. Let $K \subset \mathbb{S}^\vee$ be a subspace of dimension $\dim K = 2$ such that $X^\vee \cap \mathbb{P}(K) = \varnothing$, so that X_K is a smooth linear section of X of codimension 2. The easiest way to distinguish between different isomorphism classes of X_K is by looking at their Hilbert scheme $F_4(X_K)$ of 4-spaces.

We need some preparation.[1] Recall that the trivial vector bundle on $\mathrm{Q} = \mathrm{OGr}(1, \mathrm{V})$ with fibre $\mathbb{S}^\vee$ has a natural filtration with factors $\mathcal{S}_{8,-}$ and $\mathcal{S}_8^\vee$ respectively. Similarly, the trivial vector bundle on $\mathrm{OGr}(3, \mathrm{V})$ with the same fibre $\mathbb{S}^\vee$ has a natural filtration with factors $\mathcal{S}_{2,-}$, $\mathcal{S}_2 \otimes \mathscr{U}_3^\vee$, $\mathcal{S}_{2,-} \otimes \bigwedge^2 \mathscr{U}_3^\vee$ and $\mathcal{S}_2 \otimes \bigwedge^3 \mathscr{U}_3^\vee$ respectively (see the discussion before Lemma 6.17 below). Pulling back these two filtrations to $\mathrm{OFl}(1, 3; \mathrm{V})$ we obtain the common refinement with factors

$$\begin{array}{llll} \mathcal{S}_{2,-}, & \mathcal{S}_2 \otimes (\mathscr{U}_3/\mathscr{U}_1)^\vee, & \mathcal{S}_{2,-} \otimes \bigwedge^2 (\mathscr{U}_3/\mathscr{U}_1)^\vee, & \\ & \mathcal{S}_2 \otimes \mathscr{U}_1^\vee, & \mathcal{S}_{2,-} \otimes \mathscr{U}_1^\vee \otimes (\mathscr{U}_3/\mathscr{U}_1)^\vee, & \mathcal{S}_2 \otimes \mathscr{U}_1^\vee \otimes \bigwedge^2 (\mathscr{U}_3/\mathscr{U}_1)^\vee, \end{array}$$

where $\mathscr{U}_1 \subset \mathscr{U}_3 \subset \mathrm{V} \otimes \mathscr{O}$ is the tautological flag, the rows collect into the two factors $\mathcal{S}_{8,-}$ and $\mathcal{S}_8^\vee$ of the first filtration, and the columns into the four factors of the second filtration. In particular, the first two columns are the factors of the natural filtration (6.14) of the pullback of the subbundle $\mathscr{W}_- \subset \mathbb{S}^\vee \otimes \mathscr{O}_{\mathrm{OGr}(3,\mathrm{V})}$ defined in (6.13) below.

Recall also the closed embedding $\gamma\colon \mathbb{P}(K) \to \mathrm{Q}$ defined in Corollary 4.11. Since $\gamma^* \mathscr{O}_Q(1) \cong \mathscr{O}_{\mathbb{P}(K)}(2)$, its image $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$ is a conic. We also recall the 11-dimensional quotient space (5.1) associated with an isotropic subspace $U_{5,-} \subset \mathrm{V}$. Finally, we recall the line $L^-_{U_3} \subset X^\vee$ associated with an isotropic subspace $U_3 \subset \mathrm{V}$; see (3.12). The birational transformation in the following proposition is another example of a special birational transformation of type $(2, 1)$ in [7].

Proposition 6.1. *Let X_K be a smooth dimensionally transverse linear section of X of codimension 2. Then the following conditions are equivalent*:

1) *the Hilbert scheme $F_4(X_K)$ of linear 4-spaces on X_K is non-empty*;
2) *the linear span of the conic $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$ is contained in* Q.

If these conditions hold, then

$$F_4(X_K) = L^-_{U_3} \tag{6.1}$$

as subschemes of $F_4(X) \cong X^\vee$, where $\mathbb{P}(U_3) \subset \mathbb{P}(\mathrm{V})$ is the linear span of the conic $\gamma(\mathbb{P}(K))$. Moreover, if $\Pi^4_{U_{5,-}}$ is a 4-space on X_K, then we have an isomorphism $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_K) \cong \mathrm{Bl}_{Z_K}(\mathbb{P}(W_K))$ and a commutative diagram

$$\begin{tikzcd} E_{\Pi,K} \arrow[r] \arrow[d] & \mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_K) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Bl}_{Z_K}(\mathbb{P}(W_K)) \arrow[d] & E_{Z_K} \arrow[l] \arrow[d] \\ \Pi^4_{U_{5,-}} \arrow[r] & X_K & \mathbb{P}(W_K) & Z_K, \arrow[l] \end{tikzcd} \tag{6.2}$$

where $W_K \subset W$ is the 9-dimensional vector subspace corresponding to K, and $Z_K = \mathrm{Gr}(2, U_{5,-}) \cap \mathbb{P}(W_K)$ is a smooth linear section of the Grassmannian of codimension 2.

Proof. For each point $\kappa \in \mathbb{P}(K)$, the hyperplane section X_κ of X is smooth by Lemma 3.3. By Corollary 5.12, the Hilbert scheme $F_4(X_\kappa)$ of 4-spaces in X_κ is

[1] *Editor's Note.* Pages 735, 736 contain corrections made by the author at the proof stage.

equal to the 5-dimensional quadric $\mathscr{Q}_{\kappa,-}$ as a subscheme in $F_4(X) \cong X^\vee$; see (3.19). Since

$$F_4(X_K) = \bigcap_{\kappa \in \mathbb{P}(K)} F_4(X_\kappa) = \bigcap_{\kappa \in \mathbb{P}(K)} \mathscr{Q}_{\kappa,-} \tag{6.3}$$

and $\mathscr{Q}_{\kappa,-} \subset \mathscr{Q}_{v,-} = \mathrm{OGr}_-(4, v^\perp/v)$, where $v = \gamma(\kappa)$, we see that condition 1) implies the existence of a common point $[U_{5,-}]$ of all $\mathrm{OGr}_-(4, v^\perp/v) \subset X^\vee$ when $v \in \gamma(\mathbb{P}(K))$. In other words, this means that the conic $\gamma(\mathbb{P}(K))$ is contained in the 4-space $\mathbb{P}(U_{5,-}) \subset \mathrm{Q}$. Therefore, the linear span of the conic is contained in Q. This proves the implication 1) ⇒ 2).

For the converse, assume that the linear span of the conic $\gamma(\mathbb{P}(K))$ is contained in Q. Then it is equal to $\mathbb{P}(U_3)$ for an isotropic subspace $U_3 \subset \mathrm{V}$. It follows that the intersection of all $\mathrm{OGr}_-(4, v^\perp/v)$, where $v = \gamma(\kappa)$ and κ ranges over $\mathbb{P}(K)$, is equal to the line $L^-_{U_3}$. This proves that

$$F_4(X_K) \subset L^-_{U_3}.$$

We now show that this embedding is an equality. Regard the plane $\mathbb{P}(U_3)$ as the fibre of $\mathrm{OFl}(1,3;\mathrm{V})$ over the point $[U_3] \in \mathrm{OGr}(3,\mathrm{V})$. The restriction to this plane of the filtration of $\mathbb{S}^\vee \otimes \mathscr{O}$ discussed before the proposition takes the form

$$\begin{array}{llll} \mathcal{S}_{2,-,U_3} \otimes \mathscr{O}_{\mathbb{P}(U_3)}, & \mathcal{S}_{2,U_3} \otimes \Omega_{\mathbb{P}(U_3)}(1), & \mathcal{S}_{2,-,U_3} \otimes \mathscr{O}_{\mathbb{P}(U_3)}(-1), & \\ & \mathcal{S}_{2,U_3} \otimes \mathscr{O}_{\mathbb{P}(U_3)}(1), & \mathcal{S}_{2,-,U_3} \otimes T_{\mathbb{P}(U_3)}(-1), & \mathcal{S}_{2,U_3} \otimes \mathscr{O}_{\mathbb{P}(U_3)}, \end{array}$$

and again, the first line corresponds to a filtration of $\mathcal{S}_{8,-}|_{\mathbb{P}(U_3)} \subset \mathbb{S}^\vee \otimes \mathscr{O}_{\mathbb{P}(U_3)}$, while the first two columns correspond to a filtration of $\mathscr{W}_{-,U_3} \otimes \mathscr{O}_{\mathbb{P}(U_3)} \subset \mathbb{S}^\vee \otimes \mathscr{O}_{\mathbb{P}(U_3)}$. Recall that by definition the map γ factors through the embedding

$$\mathbb{P}(K) \subset \mathbb{P}_{\mathbb{P}(U_3)}(\mathcal{S}_{8,-}|_{\mathbb{P}(U_3)}) \subset \mathbb{P}(U_3) \times \mathbb{P}(\mathbb{S}^\vee),$$

and therefore the tautological embedding $\mathscr{O}_{\mathbb{P}(K)}(-1) \subset K \otimes \mathscr{O}_{\mathbb{P}(K)} \subset \mathbb{S}^\vee \otimes \mathscr{O}_{\mathbb{P}(K)}$ factors through the subbundle $\gamma^*\mathcal{S}_{8,-}$, that is, it sits in the first row of the above filtration. Since $\gamma(\mathbb{P}(K)) \subset \mathbb{P}(U_3)$ is a conic, the restriction of this filtration to $\gamma(\mathbb{P}(K))$ takes the form

$$\mathcal{S}_{2,-,U_3} \otimes \mathscr{O}_{\mathbb{P}(K)}, \qquad \mathcal{S}_{2,U_3} \otimes \mathscr{O}_{\mathbb{P}(K)}(-1)^{\oplus 2}, \qquad \mathcal{S}_{2,-,U_3} \otimes \mathscr{O}_{\mathbb{P}(K)}(-2).$$

Since $\mathrm{Hom}(\mathscr{O}_{\mathbb{P}(K)}(-1), \mathscr{O}_{\mathbb{P}(K)}(-2)) = 0$, it follows that $\mathscr{O}_{\mathbb{P}(K)}(-1)$ sits in the first two factors of this filtration, hence in the first two columns of the previous filtration, hence in the subspace $\mathscr{W}_{-,U_3} \otimes \mathscr{O}_{\mathbb{P}(K)} \subset \mathbb{S}^\vee \otimes \mathscr{O}_{\mathbb{P}(K)}$. So we conclude that

$$K \subset \mathscr{W}_{-,U_3} \subset \mathbb{S}^\vee.$$

We now recall from the proof of Lemma 6.17 below that the 4-spaces parameterized by the line $L^-_{U_3}$ span the subspace $\mathscr{W}_{U_3} \subset \mathbb{S}$, and that this subspace is annihilated by the subspace $\mathscr{W}_{-,U_3} \subset \mathbb{S}^\vee$. Therefore, it is also annihilated by K. This means that all these 4-spaces are contained in the linear section X_K, hence

$$L^-_{U_3} \subset F_4(X_K).$$

This proves that the conditions 1) and 2) are equivalent and that (6.1) holds.

We now assume that conditions 1) and 2) hold, and let $U_{5,-}$ be the subspace corresponding to a point of $F_4(X_K)$. Consider the isomorphism of Proposition 5.1. Let

$$\widetilde{X}_K \cong \mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_K)$$

be the strict transform of X_K in the blow-up of X along $\Pi^4_{U_{5,-}}$. Every hyperplane in $\mathbb{S}$ corresponding to a point of $\mathbb{P}(K) \subset \mathbb{P}(\mathbb{S}^\vee)$ contains the 4-space $\Pi^4_{U_{5,-}}$. Hence, by (5.3), it corresponds to a hyperplane in $\mathbb{P}(W)$ that intersects the linear span of the Grassmannian $\mathrm{Gr}(2, U_{5,-})$ transversally. Therefore $\widetilde{X}_K$ is isomorphic to the blow-up of a codimension-2 subspace $\mathbb{P}(W_K) \subset \mathbb{P}(W)$ along the corresponding linear section Z_K of the Grassmannian $\mathrm{Gr}(2, U_{5,-})$.

If the linear section Z_K is not dimensionally transverse, then its pre-image in $\widetilde{X}_K$ is an irreducible component of the latter, which is absurd. Since $\widetilde{X}_K$ is smooth, it follows that Z_K is also smooth; see Lemma 2.6. □

Definition 6.2. A smooth linear section $X_K \subset X$ of codimension 2 in the spinor tenfold is said to be *special* if the equivalent conditions of Proposition 6.1 hold for X_K.

Remark 6.3. The birational transformation in Proposition 6.1 shows that a special section X_K of X is unique up to isomorphism (and hence up to the Spin(V)-action; see Corollary 4.5). Indeed, this follows from the classical fact that a smooth linear section of $\mathrm{Gr}(2, 5)$ of codimension 2 is unique up to projective isomorphism.

Here is another characterization of special linear sections.

Lemma 6.4. *A smooth linear section $X_K \subset X$ of codimension 2 is special if and only if there is a line $L \subset X_K$ such that*

$$\mathscr{N}_{L/X_K} \cong \mathscr{O}_L(-2) \oplus \mathscr{O}_L(1)^{\oplus 6}. \tag{6.4}$$

Moreover, such a line is unique and coincides with the intersection of all 4-spaces lying on X_K.

Proof. We will use the notation introduced in the proof of Proposition 6.1. In particular, let $\mathbb{P}(U_3) \subset \mathrm{Q}$ be the linear span of the conic $\gamma(\mathbb{P}(K)) \subset \mathrm{Q}$.

We first prove that the line $L = L_{U_3}$ is contained in X_K and has the required normal bundle. To do this, we note that there is a pencil of 4-spaces passing through L (it corresponds to the pencil of the spaces $U_{5,-}$ containing U_3) and L is the scheme-theoretic intersection of any two distinct 4-spaces Π_1 and Π_2 in this pencil. Hence there is a natural embedding of vector bundles

$$\mathscr{N}_{L/\Pi_1} \oplus \mathscr{N}_{L/\Pi_2} \hookrightarrow \mathscr{N}_{L/X_K}.$$

We note that the left-hand side is isomorphic to $\mathscr{O}_L(1)^{\oplus 6}$ while the right-hand side is a bundle of rank $8 - 1 = 7$ and degree $6 - 2 = 4$. Therefore the cokernel of the embedding is isomorphic to $\mathscr{O}_L(-2)$. This gives the required formula for the normal bundle of L.

For the converse, assume that $L = L_{U_3} \subset X_K$ is a line with normal bundle as in (6.4). Let Π be any 4-space on X containing L (such 4-spaces correspond to the subspaces $U_{5,-}$ containing U_3 and hence form a pencil). If we can show that Π is contained in X_K, then this will mean that $F_4(X_K) \neq \varnothing$ and, therefore, X_K is special. It also means that L is contained in the intersection of the pencil of 4-spaces on X_K and, therefore, it is the only such line on X_K. So we consider the following diagram:

$$\begin{tikzcd} & & \mathscr{N}_{L/\Pi} \arrow[dl, dotted] \arrow[d] & & \\ 0 \arrow[r] & \mathscr{N}_{L/X_K} \arrow[r] & \mathscr{N}_{L/X} \arrow[r] & \mathscr{N}_{X_K/X}|_L \arrow[r] & 0. \end{tikzcd}$$

Its bottom row can be rewritten as

$$0 \longrightarrow \mathscr{O}_L(-2) \oplus \mathscr{O}_L(1)^{\oplus 6} \longrightarrow \mathscr{O}_L^{\oplus 3} \oplus \mathscr{O}_L(1)^{\oplus 6} \longrightarrow \mathscr{O}_L(1)^{\oplus 2} \longrightarrow 0 \tag{6.5}$$

(for the first term we use the hypothesis of the lemma and for the second we use Lemma 8.1 in [4]). Tensoring it by $\mathscr{O}_L(-1)$ and passing to cohomology, we obtain an exact sequence

$$0 \to \Bbbk^6 \to \Bbbk^6 \to \Bbbk^2 \to \Bbbk^2 \to 0,$$

where the first three spaces come from the multiplicities of $\mathscr{O}_L(1)$ in (6.5) and the last space is equal to $H^1(L, \mathscr{O}_L(-3))$. It follows that the map $\Bbbk^6 \to \Bbbk^6$ is an isomorphism and, therefore, the first map in (6.5) is an isomorphism on the summands $\mathscr{O}_L(1)$. Since moreover $\mathscr{N}_{L/\Pi} \cong \mathscr{O}_L(1)^{\oplus 3}$, it follows that the vertical arrow in the diagram factors through the dotted arrow. Geometrically, this means that the tangent space to Π at each point of L is contained in the tangent space to X_K. But since $\Pi \subset X$ and X_K is a linear section of X, it follows that $\Pi \subset X_K$. $\square$

Definition 6.5. The line L on a special linear section $X_K \subset X$ of codimension 2 such that (6.4) holds is called its *special line*.

One can also describe the normal bundle for non-special lines.

Lemma 6.6. *Let X_K be a smooth linear section of X of codimension 2, and let L be a non-special line on X_K. Then*

$$\mathscr{N}_{L/X_K} \cong \mathscr{O}_L^{\oplus 3} \oplus \mathscr{O}_L(1)^{\oplus 4} \quad \textit{or} \quad \mathscr{N}_{L/X_K} \cong \mathscr{O}_L(-1) \oplus \mathscr{O}_L \oplus \mathscr{O}_L(1)^{\oplus 5}.$$

Proof. As in the proof of Lemma 6.4, we have an exact sequence

$$0 \longrightarrow \mathscr{N}_{L/X_K} \longrightarrow \mathscr{O}_L^{\oplus 3} \oplus \mathscr{O}_L(1)^{\oplus 6} \longrightarrow \mathscr{O}_L(1)^{\oplus 2} \longrightarrow 0. \tag{6.6}$$

The restriction of the right map to the second summand $\mathscr{O}_L(1)^{\oplus 6} \to \mathscr{O}_L(1)^{\oplus 2}$ is a map of constant rank. If the rank equals 2, then $\mathscr{N}_{L/X_K} \cong \mathscr{O}_L^{\oplus 3} \oplus \mathscr{O}_L(1)^{\oplus 4}$. If the rank equals 1, then $\mathscr{N}_{L/X_K} \cong \mathscr{O}_L(-1) \oplus \mathscr{O}_L \oplus \mathscr{O}_L(1)^{\oplus 5}$. Finally, if the rank equals 0, then $\mathscr{N}_{L/X_K} \cong \mathscr{O}_L(-2) \oplus \mathscr{O}_L(1)^{\oplus 6}$ and the corresponding line is special. $\square$

The characterization in Lemma 6.4 may be restated as follows.

Corollary 6.7. *Let $X_K \subset X$ be a smooth linear section of codimension 2. The Hilbert scheme $F_1(X_K)$ of lines on X_K is singular if and only if X_K is special. Moreover, if X_K is special, then the singular locus of $F_1(X_K)$ consists of a single point and this point corresponds to the special line on X_K.*

Proof. By Lemma 6.6 we have $H^1(L, \mathscr{N}_{L/X_K}) \neq 0$, that is, the Hilbert scheme $F_1(X_K)$ is singular at a point $[L]$ if and only if (6.4) holds and, therefore, X_K is special and L is its special line. □

6.2. Non-special linear sections of codimension 2. In this subsection we show that there is a unique isomorphism class of non-special smooth linear sections $X_K \subset X$ of codimension 2.

Given a subspace $K \subset \mathbb{S}$ (of any codimension) and a point $\kappa \in \mathbb{P}(K) \setminus X^\vee$, we consider the quadric

$$\mathscr{Q}_{\kappa,K} := \mathscr{Q}_v \cap \mathbb{P}(K^\perp), \tag{6.7}$$

where $v = \gamma(\kappa)$ and $\mathscr{Q}_v = \mathrm{OGr}_+(4, v^\perp/v)$. Note that the quadric $\mathscr{Q}_v$ is contained in the hyperplane $\mathbb{P}(\kappa^\perp)$ (Lemma 5.10). Hence $\mathscr{Q}_{\kappa,K}$ is a linear section of $\mathscr{Q}_v$ of codimension at most $k-1$, where $k = \dim K$.

We also recall the smooth 5-dimensional quadric $\mathscr{Q}_{\kappa,-}$ defined in (5.22). Finally, recall the spinor bundle $\mathcal{S}_4$ on the 5-quadric $\mathscr{Q}_{\kappa,-}$ and note that $\mathrm{c}_4(\mathcal{S}_4) = 0$ (see [13], Remark 2.9). Since $\mathcal{S}_4^\vee$ is globally generated, it is standard (see, for example, [27], § 4, Lemma 4.3.2) that a general morphism $\mathcal{S}_4 \to \mathscr{O}_{\mathscr{Q}_{\kappa,-}}$ is surjective. Denoting the kernel of such a morphism by $\overline{\mathcal{S}}_4$, we have an exact sequence

$$0 \to \overline{\mathcal{S}}_4 \to \mathcal{S}_4 \to \mathscr{O}_{\mathscr{Q}_{\kappa,-}} \to 0. \tag{6.8}$$

Then $\overline{\mathcal{S}}_4$ is a vector bundle of rank 3 on $\mathscr{Q}_{\kappa,-}$. Since the group Spin(7) acts transitively on the open subset of $\mathbb{P}(\mathrm{Hom}(\mathcal{S}_4, \mathscr{O}_{\mathscr{Q}_{\kappa,-}}))$ corresponding to surjective morphisms (see the proof of Proposition 6.8 for more details), this bundle is defined uniquely up to the action of Spin(7) on $\mathscr{Q}_{\kappa,-}$.

Proposition 6.8. *Let X_K be a smooth dimensionally transverse linear section of X of codimension 2. Then the blow-up $\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K)$ is a relative hyperplane section in the $\mathbb{P}^4$-bundle $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$ over $\mathscr{Q}_{\kappa,-}$. It is a flat $\mathbb{P}^3$-bundle if and only if X_K is non-special, that is, $F_4(X_K) = \varnothing$. In this case,*

$$\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K) \cong \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \overline{\mathcal{S}}_4),$$

where $\overline{\mathcal{S}}_4$ is the bundle of rank 3 defined in (6.8). In particular, the non-special section X_K of codimension 2 is unique up to isomorphism.

Proof. Let $\kappa, \kappa' \in \mathbb{P}(K)$ be a basis. We put $v = \gamma(\kappa)$, so that $\mathscr{Q}_{\kappa,K} = \mathscr{Q}_v \cap \mathbb{P}(\kappa'^\perp)$. Consider the isomorphism $\mathrm{Bl}_{\mathscr{Q}_v}(X_\kappa) \cong \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$ in Corollary 5.11. Then the strict transform of $X_K = X_\kappa \cap \mathbb{P}(\kappa'^\perp)$ is isomorphic to the blow-up of X_K along $\mathscr{Q}_{\kappa,K}$. On the other hand, it is a relative hyperplane section of the projective bundle $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$ corresponding to the composition

$$\mathscr{O}(-1) \oplus \mathcal{S}_4 \hookrightarrow \mathbb{S} \otimes \mathscr{O} \xrightarrow{\kappa'} \mathscr{O}. \tag{6.9}$$

Hence we only have to check that the composition (6.9) is surjective if and only if $F_4(X_K)=\varnothing$. Indeed, if the morphism is not surjective at some point, then the fibre $\mathbb{P}^4$ of the bundle $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1)\oplus\mathcal{S}_4)$ over this point is contained in $\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K)$ and hence gives a 4-space on X_K. Conversely, if $\Pi\subset X_K$ is a 4-space, then $\Pi\subset X_\kappa$ and we know by Corollary 5.12 that Π is the image of a fibre of $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1)\oplus\mathcal{S}_4)$ over some point of $\mathscr{Q}_{\kappa,-}$. Furthermore, this fibre is equal to the strict transform of Π in $\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K)$ and, therefore, the morphism (6.9) is zero at this point.

We now assume that the composition (6.9) is surjective. Its component $\mathcal{S}_4\to\mathscr{O}$ is determined by a global section of the bundle $\mathcal{S}_4^\vee$ on $\mathscr{Q}_{\kappa,-}$. The space of such global sections is the 8-dimensional spinor representation of $\mathrm{Spin}(7)$ and can be identified with $\mathcal{S}_{8,v}$. The $\mathrm{Spin}(7)$-action on its projectivization has two orbits, the smooth 6-dimensional quadric $\mathscr{Q}_v\subset\mathbb{P}(\mathcal{S}_{8,v})$ and its open complement. It is easy to see that each global section of $\mathcal{S}_4^\vee$ corresponding to a point of the closed orbit $\mathscr{Q}_v$ vanishes on a certain plane $\mathbb{P}^2\subset\mathscr{Q}_{\kappa,-}$ and, therefore, its extension to a morphism (6.9) is not surjective (at least along a line $\mathbb{P}^1\subset\mathbb{P}^2$). Thus, if (6.9) is surjective, then its kernel is an extension of $\mathscr{O}(-1)$ by $\overline{\mathcal{S}}_4$. It is easy to check that $\mathrm{Ext}^1(\mathscr{O}(-1),\overline{\mathcal{S}}_4)=0$ on the 5-quadric $\mathscr{Q}_{\kappa,-}$ and, therefore, the kernel of (6.9) is isomorphic to $\mathscr{O}(-1)\oplus\overline{\mathcal{S}}_4$.

Hence X_K is the image of $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1)\oplus\overline{\mathcal{S}}_4)$ under the map given by the linear system of relative hyperplane sections. Its uniqueness up to isomorphism follows from the uniqueness of $\overline{\mathcal{S}}_4$ up to the action of $\mathrm{Spin}(7)$. $\square$

Combining this with Remark 6.3, we obtain the following assertion.

Corollary 6.9. *There are exactly two isomorphism classes of smooth linear sections $X_K\subset X$ of codimension 2, the special and the non-special. They are described in Propositions 6.1 and 6.8 respectively.*

Using Corollary 4.5, we can restate this in the following form.

Corollary 6.10. *There are exactly two orbits of the $\mathrm{Spin}(\mathrm{V})$-action on the open subset of $\mathrm{Gr}(2,\mathbb{S}^\vee)$ parametrizing lines that do not intersect the spinor tenfold $X^\vee$. One of them is open and the other is closed.*

This corollary, together with Lemma 6.13 below, provides a refinement of Proposition 32 in [25].

By using the description of singular hyperplane sections, one can also describe the orbits of $\mathrm{Spin}(\mathrm{V})$ on the subset of $\mathrm{Gr}(2,\mathbb{S}^\vee)$ parametrizing lines that intersect $X^\vee$. We leave this to the interested reader.

For future use, we also prove the following fact about the quadrics $\mathscr{Q}_{\kappa,K}$ defined in (6.7).

Corollary 6.11. *If $\dim K=2$ and X_K is a non-special smooth linear section of X, then the quadric $\mathscr{Q}_{\kappa,K}$ is smooth for every $\kappa\in K$. If, on the contrary, X_K is special, then the quadric $\mathscr{Q}_{\kappa,K}$ is singular for every $\kappa\in K$.*

Proof. The first assertion follows immediately from Lemma 2.6 since the blow-up of X_K along $\mathscr{Q}_{\kappa,K}$ is a $\mathbb{P}^3$-bundle over a smooth quadric $\mathscr{Q}_{\kappa,-}$ and is therefore smooth.

To prove the second, note that the zero locus of the morphism (6.9) is equal to $F_4(X_K) \cong \mathbb{P}^1$. Hence the corresponding relative hyperplane section $\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_\kappa)$ in $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$ is not smooth and, therefore, the blow-up centre $\mathscr{Q}_{\kappa,K}$ is not smooth. $\square$

Remark 6.12. We easily see that $G_6(X_K) = \varnothing$ for all X_K of codimension 2 while $G_5(X_K) \cong \mathbb{P}(K)$ and the corresponding family of 5-quadrics consists of the quadrics $\mathscr{Q}_{\kappa,K}$. Thus the smoothness of the universal family of 5-quadrics gives another characterization of non-special linear sections of X of codimension 2.

6.3. The spinor quadratic line complex. We write

$$R_0 \subset \mathrm{Gr}(2, \mathbb{S}^\vee)$$

for the Spin(V)-orbit in $\mathrm{Gr}(2, \mathbb{S}^\vee)$ parametrizing the 2-dimensional subspaces $K \subset \mathbb{S}^\vee$ such that X_K is smooth and special, and let

$$R := \overline{R_0}$$

be its closure in $\mathrm{Gr}(2, \mathbb{S}^\vee)$.

Lemma 6.13. *The subscheme* $R \subset \mathrm{Gr}(2, \mathbb{S}^\vee)$ *is a divisor cut out by a smooth* Spin(V)*-invariant quadric in* $\mathbb{P}(\bigwedge^2 \mathbb{S}^\vee)$.

Proof. Let $\mathcal{K}_2 \subset \mathbb{S}^\vee \otimes \mathscr{O}$ denote the tautological bundle of rank 2 on the Grassmannian $\mathrm{Gr}(2, \mathbb{S}^\vee)$. Consider the composition

$$\mathrm{V} \otimes \mathscr{O} \to \mathrm{Sym}^2 \mathbb{S} \otimes \mathscr{O} \to \mathrm{Sym}^2 \mathcal{K}_2^\vee, \tag{6.10}$$

where the first map is induced by the embedding of V regarded as the space of quadratic equations of $X^\vee$ (see Corollary 4.3) and the second is tautological. Then the dual map $\mathrm{Sym}^2 \mathcal{K}_2 \to \mathrm{V} \otimes \mathscr{O}$ of (6.10) induces a universal version of the quadratic map $\gamma\colon \mathbb{P}(K_2) \to \mathrm{Q} \subset \mathbb{P}(\mathrm{V})$ discussed in Corollary 4.11.

We now consider the composition

$$\mathscr{O} \xrightarrow{\mathbf{q}_\mathrm{V}} \mathrm{Sym}^2 \mathrm{V} \otimes \mathscr{O} \to \mathrm{Sym}^2(\mathrm{Sym}^2 \mathcal{K}_2^\vee) \to \mathrm{Sym}^4 \mathcal{K}_2^\vee, \tag{6.11}$$

where the first map is given by the equation of the quadric Q (using the identification $\mathrm{V} \cong \mathrm{V}^\vee$ given by $\mathbf{q}_\mathrm{V}$), the second is the symmetric square of (6.10), and the last is the multiplication map. The composition map is identically equal to zero because for general $[K_2] \in \mathrm{Gr}(2, \mathbb{S}^\vee)$ we have $\gamma(\mathbb{P}(K_2)) \subset \mathrm{Q}$ by Corollary 4.11. Hence the composition of the first two arrows factors through the kernel of the third, which is nothing but

$$\mathrm{Ker}\left(\mathrm{Sym}^2(\mathrm{Sym}^2 \mathcal{K}_2^\vee) \to \mathrm{Sym}^4 \mathcal{K}_2^\vee\right) \cong \mathrm{Sym}^2(\textstyle\bigwedge^2 \mathcal{K}_2^\vee) \cong \mathscr{O}_{\mathrm{Gr}(2,\mathbb{S}^\vee)}(2),$$

and thus gives a global section of $\mathscr{O}_{\mathrm{Gr}(2,\mathbb{S}^\vee)}(2)$ which determines a quadratic divisor on $\mathrm{Gr}(2, \mathbb{S}^\vee)$. Furthermore, for general $[K_2]$, this global section vanishes at the point $[K_2]$ if and only if the composition of the first two arrows in (6.11) vanishes at $[K_2]$, that is, if and only if the linear span of the conic $\gamma(\mathbb{P}(K_2)) \subset \mathrm{Q}$ is contained in Q.

By Proposition 6.1 this is equivalent to the speciality of X_{K_2}. Thus the resulting global section of $\mathscr{O}_{\mathrm{Gr}(2,\mathbb{S}^\vee)}(2)$ defines the subscheme R.

Since the divisor $R \subset \mathrm{Gr}(2, \mathbb{S}^\vee)$ is $\mathrm{Spin}(\mathrm{V})$-invariant, we conclude that the quadric in $\mathbb{P}(\bigwedge^2 \mathbb{S}^\vee)$ corresponding to the constructed global section of $\mathscr{O}_{\mathrm{Gr}(2,\mathbb{S}^\vee)}(2)$ is also $\mathrm{Spin}(\mathrm{V})$-invariant. On the other hand, $\bigwedge^2 \mathbb{S}^\vee$ is an irreducible representation of $\mathrm{Spin}(\mathrm{V})$ (actually, its highest weight is ω_3 and thus it is isomorphic to $\bigwedge^3 \mathrm{V}$). Hence every $\mathrm{Spin}(\mathrm{V})$-invariant quadratic form on $\mathbb{P}(\bigwedge^2 \mathbb{S}^\vee)$ is non-degenerate and the corresponding quadric is smooth. □

Quadratic divisors on Grassmannians of lines are traditionally referred to as quadratic line complexes. The divisor R constructed in this lemma is very important for the geometry of linear sections of the spinor tenfold. Therefore we suggest the following terminology.

Definition 6.14. The quadratic divisor $R \subset \mathrm{Gr}(2, \mathbb{S}^\vee)$ described in Lemma 6.13 is called the *spinor quadratic line complex*.

Before proceeding, we discuss some properties of the spinor quadratic line complex R.

For every point $\kappa \in \mathbb{P}(\mathbb{S}^\vee)$ there is a natural isomorphism between the projective space $\mathbb{P}(\mathbb{S}^\vee/\kappa) \cong \mathbb{P}^{14}$ and the closed subvariety of $\mathrm{Gr}(2, \mathbb{S}^\vee)$ parametrizing the lines through κ in $\mathbb{P}(\mathbb{S}^\vee)$. We shall use this isomorphism implicitly and regard $\mathbb{P}(\mathbb{S}^\vee/\kappa)$ as a subvariety of $\mathrm{Gr}(2, \mathbb{S}^\vee)$. We write

$$R_\kappa := R \cap \mathbb{P}(\mathbb{S}^\vee/\kappa) \subset \mathbb{P}(\mathbb{S}^\vee/\kappa). \tag{6.12}$$

The following lemma describes these subschemes of $\mathbb{P}(\mathbb{S}^\vee/\kappa)$. We recall the quadratic cone $\widetilde{\mathscr{Q}}_v \subset \mathbb{P}(\mathbb{S}^\vee)$ (see (5.21)) where $v = \gamma(\kappa)$, and note that $\kappa \in \mathbb{P}(\mathcal{S}_{8,-,v})$ lies at its vertex.

Lemma 6.15. *Suppose that $\kappa \in \mathbb{P}(\mathbb{S}^\vee) \setminus X^\vee$ and put $v = \gamma(\kappa)$. Then the subscheme $R_\kappa \subset \mathbb{P}(\mathbb{S}^\vee/\kappa)$ is the image of the quadratic cone $\widetilde{\mathscr{Q}}_v$ under the linear projection from κ, so that $\widetilde{\mathscr{Q}}_v = \mathrm{Cone}_\kappa(R_\kappa)$. In particular, R_κ is equal to $\mathrm{Cone}_{\mathbb{P}(\mathcal{S}_{8,-,v}/\kappa)}(\mathscr{Q}_v)$ and contains the image of the projection of $X^\vee$ from κ.*

Proof. Let $X_\kappa \subset X$ be the smooth hyperplane section of X associated with κ. By the definition of R, the subscheme R_κ is the closure of the locus of all hyperplanes in $\mathbb{P}(\kappa^\perp) \subset \mathbb{P}(\mathbb{S})$ such that the corresponding hyperplane section of X_κ is special. By Corollary 6.11, such a linear section is special if and only if the hyperplane section $\mathscr{Q}_v \cap \mathbb{P}(\kappa'^\perp)$ of the smooth quadric $\mathscr{Q}_v$ is singular (see (6.7)). Thus R_κ is the cone over the projective dual quadric $\mathscr{Q}_v^\vee$ whose vertex is the orthogonal complementof the linear span of $\mathscr{Q}_v$.

Since the linear span of $\mathscr{Q}_v$ is $\mathbb{P}(\mathcal{S}_{8,v}) \subset \mathbb{P}(\mathbb{S})$, its orthogonal in $\mathbb{P}(\mathbb{S}^\vee)$ is $\mathbb{P}(\mathcal{S}_{8,-,v})$ and its orthogonal in $\mathbb{P}(\mathbb{S}^\vee/\kappa)$ is $\mathbb{P}(\mathcal{S}_{8,-,v}/\kappa)$. On the other hand, the quadric $\mathscr{Q}_v$ is self-dual under the identification of $\mathbb{P}(\mathcal{S}_{8,v}^\vee)$ with $\mathbb{P}(\mathcal{S}_{8,v})$ in terms of the natural quadratic form. Thus the scheme $R_\kappa = \mathrm{Cone}_{\mathbb{P}(\mathcal{S}_{8,-,v}/\kappa)}(\mathscr{Q}_v)$ is the image of $\widetilde{\mathscr{Q}}_v$. Finally, R_κ contains the projection of $X^\vee$ since the cone over R_κ with vertex at κ is the quadric $\widetilde{\mathscr{Q}}_v$, which contains $X^\vee$ by Remark 5.9. □

Corollary 6.16. *The spinor quadratic line complex R contains the locus of lines in $\mathbb{P}(\mathbb{S}^\vee)$ intersecting $X^\vee$. The singular locus of R is the variety of secant lines of $X^\vee$, that is, the image of the relative Grassmannian $\mathrm{Gr}_{\mathrm{Q}}(2,\mathcal{S}_{8,-})$ under the natural map $\mathrm{Gr}_{\mathrm{Q}}(2,\mathcal{S}_{8,-}) \to \mathrm{Gr}(2,\mathbb{S}^\vee)$. In particular,* $\operatorname{codim}_R(\operatorname{Sing}(R)) = 7$.

Proof. To verify the first assertion, it suffices to note that, by Lemma 6.15, the quadric R_κ contains the projection of $X^\vee$ for all $\kappa \notin X^\vee$. To prove the second, we consider $\mathbb{P}_R(\mathcal{K}_2)$ and the natural map $\mathbb{P}_R(\mathcal{K}_2) \to \mathbb{P}(\mathbb{S}^\vee)$, where $\mathcal{K}_2$ is the restriction to R of the tautological vector bundle of rank 2 on $\mathrm{Gr}(2,\mathbb{S}^\vee)$. This is a (non-flat) quadratic fibration whose fibres over the points outside $X^\vee$ are the quadrics $R_\kappa \subset \mathbb{P}^{14}$ described above and whose fibres over the points of $X^\vee$ are the spaces $\mathbb{P}^{14}$ (thus $X^\vee$ is the non-flat locus). Note that the singular locus of $\mathbb{P}_R(\mathcal{K}_2)$ is equal to $\mathbb{P}_{\operatorname{Sing}(R)}(\mathcal{K}_2)$. On the other hand, it certainly contains the $\mathbb{P}^6$-fibration $\mathbb{P}_{\mathbb{P}(\mathbb{S}^\vee)\setminus X^\vee}(\gamma^*\mathcal{S}_{8,-}/\mathcal{O}(-1))$ over $\mathbb{P}(\mathbb{S}^\vee)\setminus X^\vee$ swept out by the singular loci of the quadrics R_κ. Its projection to $\mathrm{Gr}(2,\mathbb{S}^\vee)$ is the variety of secant lines of $X^\vee$ and, therefore, lies in $\operatorname{Sing}(R)$. It is also easy to see that R is smooth outside the locus of secant lines.

To show that the locus of secant lines is equal to the image of the Grassmannian $\mathrm{Gr}_{\mathrm{Q}}(2,\mathcal{S}_{8,-})$, it is enough to note that each secant line of $X^\vee$ is contracted by the rational map $q_-\circ p_-^{-1}$ to a point and, therefore, its strict transform in $\mathrm{Bl}_{X^\vee}(\mathbb{P}(\mathbb{S}^\vee)) \cong \mathbb{P}_{\mathrm{Q}}(\mathcal{S}_{8,-})$ is contained in a fibre over Q, that is, it corresponds to a point of the relative Grassmannian $\mathrm{Gr}_{\mathrm{Q}}(2,\mathcal{S}_{8,-})$.

This enables us to compute the codimension of $\operatorname{Sing}(R)$. Indeed,

$$\dim(\mathrm{Gr}_{\mathrm{Q}}(2,\mathcal{S}_{8,-})) = 8 + 2\cdot 6 = 20 \quad\text{and}\quad \dim R = \dim(\mathrm{Gr}(2,\mathbb{S}^\vee)) - 1 = 2\cdot 14 - 1 = 27,$$

whence $\operatorname{codim}_R(\operatorname{Sing}(R)) = 7$. □

We conclude this section by constructing a useful resolution of singularities of R. For every point U_3 of the isotropic Grassmannian $\mathrm{OGr}(3,\mathrm{V})$ we consider the induced filtration on $\mathbb{S}\otimes\mathcal{O}$ (see Lemma 2.3) with factors $\mathcal{S}_2$, $\mathcal{S}_{2,-}\otimes\mathcal{U}_3^\vee$, $\mathcal{S}_2\otimes\bigwedge^2\mathcal{U}_3^\vee$, and $\mathcal{S}_{2,-}\otimes\bigwedge^3\mathcal{U}_3^\vee$, where $\mathcal{S}_2$ and $\mathcal{S}_{2,-}$ are the spinor bundles of rank 2 and $\mathcal{U}_3$ is the tautological bundle of rank 3. Let $\mathcal{W}\subset\mathbb{S}\otimes\mathcal{O}$ be the subbundle of rank 8 generated by the first two factors. We put $\mathcal{W}_- := (\mathbb{S}/\mathcal{W})^\vee \subset \mathbb{S}^\vee\otimes\mathcal{O}$, which is an analogous subbundle of rank 8 in the dual space. Thus we have a collection of exact sequences:

$$0\to\mathcal{W}\to\mathbb{S}\otimes\mathcal{O}\to\mathcal{W}_-^\vee\to 0,\qquad 0\to\mathcal{W}_-\to\mathbb{S}^\vee\otimes\mathcal{O}\to\mathcal{W}^\vee\to 0 \tag{6.13}$$

(the second sequence is dual to the first) and, in view of Lemma 2.1,

$$0\to\mathcal{S}_2\to\mathcal{W}\to\mathcal{S}_{2,-}\otimes\mathcal{U}_3^\vee\to 0,\qquad 0\to\mathcal{S}_{2,-}\to\mathcal{W}_-\to\mathcal{S}_2\otimes\mathcal{U}_3^\vee\to 0. \tag{6.14}$$

Recall that $R_0\subset R$ denotes the open subset parametrizing the points $[K_2]\in R$ with $X^\vee\cap\mathbb{P}(K_2)=\varnothing$, that is, those corresponding to smooth special linear sections of X of codimension 2.

Lemma 6.17. *The natural map*

$$\widetilde{R} := \mathrm{Gr}_{\mathrm{OGr}(3,\mathrm{V})}(2,\mathcal{W}_-) \to \mathrm{Gr}(2,\mathbb{S}^\vee)$$

is a birational isomorphism onto the hypersurface $R \subset \mathrm{Gr}(2, \mathbb{S}^\vee)$ *and an isomorphism over the open subset* $R_0 \subset R$.

Proof. Choose a point $[U_3] \in \mathrm{OGr}(3, \mathrm{V})$ and let $L = L_{U_3} \subset X$ be the corresponding line (3.11). Recall from Lemma 6.4 that L is the special line of a special linear section $X_K \subset X$ of codimension 2 if and only if X_K contains all 4-spaces in X containing L.

The 4-spaces in X containing L are parametrized by the line $L^-_{U_3} \subset X^\vee$; see (3.12). Let us check that their linear span in $\mathbb{S}$ is equal to the fibre $\mathscr{W}_{U_3}$ of the subbundle $\mathscr{W} \subset \mathbb{S} \otimes \mathscr{O}$ at the point $[U_3]$. Indeed, for every point $s \in \mathbb{P}(\mathcal{S}_{2,-,U_3}) = L^-_{U_3}$ let $U_{5,-}$ be the corresponding point of the line $L^-_{U_3}$. Then the tautological exact sequence $0 \to U_3 \to U_{5,-} \to U_{5,-}/U_3 \to 0$ gives an exact sequence

$$0 \to (U_{5,-}/U_3) \otimes \det(U_3) \to \textstyle\bigwedge^4 U_{5,-} \to \det(U_{5,-}/U_3) \otimes \det(U_3) \otimes U_3^\vee \to 0,$$

which can be rewritten as the sequence

$$0 \to \mathcal{S}_{2,U_3} \to \textstyle\bigwedge^4 U_{5,-} \to s \otimes U_3^\vee \to 0$$

induced by the restriction of the first sequence in (6.14) to the point $[U_3]$ by means of the embedding $s \otimes U_3^\vee \subset \mathcal{S}_{2,-,U_3} \otimes U_3^\vee$. Since the linear span of the pencil of planes $\mathbb{P}(s \otimes U_3^\vee) \subset \mathbb{P}(\mathcal{S}_{2,-,U_3} \otimes U_3^\vee)$ parametrized by $s \in \mathbb{P}(\mathcal{S}_{2,-,U_3})$ is the 5-space $\mathbb{P}(\mathcal{S}_{2,-,U_3} \otimes U_3^\vee)$, it follows that the linear span of the pencil of 4-spaces $\Pi^4_{U_{5,-}}$ is the fibre $\mathscr{W}_{U_3}$ of the subbundle $\mathscr{W} \subset \mathbb{S} \otimes \mathscr{O}$ at $[U_3]$.

It follows that X_K contains all 4-spaces containing L if and only if K is contained in the orthogonal of $\mathscr{W}_{U_3} \subset \mathbb{S}$, and this orthogonal is nothing but the fibre $\mathscr{W}_{-,U_3}$ of the bundle $\mathscr{W}_-$ at the point $[U_3]$. Thus the relative Grassmannian $\widetilde{R}$ parametrizes all pairs (L, K) where L is a line on X and $K \subset \mathbb{S}^\vee$ is a two-dimensional subspace such that X_K contains the pencil of 4-spaces on X containing L. In particular, the image of $\widetilde{R}$ in $\mathrm{Gr}(2, \mathbb{S}^\vee)$ contains the open subset R_0 of the hypersurface R and (by the uniqueness of the special line on a special variety X_K; see Lemma 6.4) the map from $\widetilde{R}$ is an isomorphism over it. Since $\widetilde{R}$ is irreducible and

$$\dim \widetilde{R} = \dim \mathrm{OGr}(3, 10) + \dim \mathrm{Gr}(2, 8) = 15 + 12 = 27 = \dim R,$$

it follows that the image is equal to R and the map is birational. Finally, since $\widetilde{R}$ is smooth, the map $\widetilde{R} \to R$ is a resolution of singularities. □

Remark 6.18. It is easy to check that the intersection of $X^\vee$ with the projectivized fibre $\mathbb{P}(\mathscr{W}_{-,U_3}) \subset \mathbb{P}(\mathbb{S}^\vee)$ of the bundle $\mathscr{W}_-$ is the 5-dimensional cubic Segre cone

$$\mathrm{Cone}_{\mathbb{P}(\mathcal{S}_{2,-,U_3})}(\mathbb{P}(\mathcal{S}_{2,U_3}) \times \mathbb{P}(U_3^\vee)) \subset \mathbb{P}(\mathscr{W}_{-,U_3})$$

(it is swept out by the pencil of 4-spaces in $X^\vee$ passing through the line $L^-_{U_3}$). In particular, the pre-image of the complement $R \backslash R_0$ is the subvariety of $\widetilde{R}$ parametrizing the secant lines to this cone. Hence it is a divisor in $\widetilde{R}$ of relative degree 3 over $\mathrm{OGr}(3, \mathrm{V})$. Thus the resolution $\widetilde{R} \to R$ is not small and is not an identity over the smooth locus of R.

The assertion of the lemma may be expressed by a commutative diagram

$$\begin{tikzcd} R_0 \arrow[r, hook] \arrow[dr, hook] & \widetilde{R} \arrow[r, equal] \arrow[d] & \mathrm{Gr}_{\mathrm{OGr}(3,\mathrm{V})}(2, \mathscr{W}_-) \arrow[dr] & \\ & R \arrow[rr, dashed] & & \mathrm{OGr}(3,\mathrm{V}). \end{tikzcd}$$

It determines a rational map (the dashed arrow at the diagram)

$$\lambda\colon R \dashrightarrow \mathrm{OGr}(3,\mathrm{V}), \tag{6.15}$$

which is regular on the open subset $R_0 \subset R$.

Corollary 6.19. *We have an isomorphism* $\lambda^*\mathscr{O}_{\mathrm{OGr}(3,\mathrm{V})}(1) \cong \mathscr{O}_{\mathrm{Gr}(2,\mathbb{S}^\vee)}(3)|_{R_0}$ *of line bundles on* R_0.

Proof. Let us calculate the canonical class of $\widetilde{R}$. If H' is the hyperplane class of $\mathrm{Gr}(2,\mathbb{S}^\vee)$ and H'' is the hyperplane class of $\mathrm{OGr}(3,\mathrm{V})$, then $K_{\mathrm{OGr}(3,\mathrm{V})} = -6H''$ and $\mathrm{c}_1(\mathscr{W}_-) = -2H''$. Hence,

$$K_{\widetilde{R}} = -6H'' + 4H'' - 8H' = -8H' - 2H''.$$

On the other hand, $K_R = -14H'$ by adjunction. Hence the discrepancy is $6H' - 2H'' = 2(3H' - H'')$. This shows that the exceptional divisor of the resolution $\widetilde{R} \to R$ is linearly equivalent to $3H' - H''$ (multiplicity 2 in the discrepancy corresponds to $\mathrm{codim}_R(R \setminus R_0) = 3$). Since the exceptional divisor is disjoint from R_0, the restriction of $3H' - H''$ to R_0 is equivalent to zero, whence the required relation. □

Remark 6.20. The map λ endows R_0 with a bunch of vector bundles. Besides the tautological vector bundle $\mathcal{K}_2$ of rank 2 (restricted from $\mathrm{Gr}(2,\mathbb{S})$), these are the pullback of the tautological bundle $\mathscr{U}_3$ of rank 3 and the pullbacks of the spinor bundles $\mathcal{S}_{2,\pm}$ of rank 2 from $\mathrm{OGr}(3,\mathrm{V})$.

§ 7. Linear sections of bigger codimension

In this section we discuss some results about linear sections of X of codimension higher than 2.

7.1. The quadratic invariant. Let $K \subset \mathbb{S}^\vee$ be a vector subspace of dimension $k \geqslant 2$ and let X_K be the corresponding linear section of X (in most cases we assume that $X^\vee \cap \mathbb{P}(K) = \varnothing$, so that X_K is smooth, but this is not always necessary). We write

$$R_K := R \cap \mathrm{Gr}(2,K) \subset \mathrm{Gr}(2,\mathbb{S}^\vee) \tag{7.1}$$

for the intersection of the spinor quadratic line complex $R \subset \mathrm{Gr}(2,\mathbb{S}^\vee)$ (see Definition 6.14) with the Grassmannian $\mathrm{Gr}(2,K) \subset \mathrm{Gr}(2,\mathbb{S}^\vee)$. The geometric meaning of the scheme R_K is straightforward: it parametrizes those oversections of X_K (that is, subvarieties $X_{K_2} \subset X$ with $K_2 \subset K$) of codimension 2 that are special or singular.

The following simple lemma shows that R_K is an invariant of the isomorphism class of X_K.

Lemma 7.1. *If X_{K_1} and X_{K_2} are dimensionally transverse linear sections of X and there is an isomorphism $X_{K_1} \cong X_{K_2}$, then $R_{K_1} \cong R_{K_2}$.*

Proof. By Corollary 4.5, an isomorphism $X_{K_1} \cong X_{K_2}$ can be realized by the action of an appropriate element $g \in \mathrm{Spin}(\mathrm{V})$ which takes K_1 to K_2. Since the divisor $R \subset \mathrm{Gr}(2, \mathbb{S}^\vee)$ is $\mathrm{Spin}(\mathrm{V})$-invariant, we conclude that g induces an isomorphism of R_{K_1} and R_{K_2}. □

Lemma 7.2. *Let $K \subset \mathbb{S}^\vee$ be a general subspace of codimension k, $2 \leqslant k \leqslant 5$. Then the subscheme $R_K \subset \mathrm{Gr}(2, K)$ is a smooth quadratic divisor.*

Proof. Consider the universal family of schemes R_K, where K ranges over the Grassmannian $\mathrm{Gr}(k, \mathbb{S}^\vee)$ of all subspaces $K \subset \mathbb{S}^\vee$ of a fixed dimension k. This universal family may be written as the relative Grassmannian

$$R \times_{\mathrm{Gr}(2,\mathbb{S}^\vee)} \mathrm{Fl}(2, k; \mathbb{S}^\vee) \cong \mathrm{Gr}_R(k-2, \mathbb{S}^\vee/\mathcal{K}_2),$$

where $\mathcal{K}_2$ is the restriction of the tautological vector bundle from $\mathrm{Gr}(2, \mathbb{S}^\vee)$ to R. Its dimension is equal to $\dim R + \dim \mathrm{Gr}(k-2, 14) = 27 + (k-2)(16-k)$ and, by Corollary 6.16, its singular locus has dimension $20 + (k-2)(16-k)$. The image of the singular locus in $\mathrm{Gr}(k, \mathbb{S}^\vee)$ has codimension

$$k(16-k) - 20 - (k-2)(16-k) = 12 - 2k.$$

This number is positive when $k \leqslant 5$. Hence the general fibre of the map

$$\mathrm{Gr}_R(k-2, \mathbb{S}^\vee/\mathcal{K}_2) \to \mathrm{Gr}(k, \mathbb{S}^\vee)$$

is smooth. It remains to note that the fibre over a point $[K] \in \mathrm{Gr}(k, \mathbb{S}^\vee)$ is R_K. □

7.2. Birational constructions and rationality. Two birational descriptions of the spinor tenfold X (Propositions 5.1 and 5.8), which are obtained by projections from a 4-space and a 6-quadric respectively, can also be used to give descriptions of the linear sections of X. The first of them is very effective for the sections X_K containing a 4-space (see Corollary 5.7 and Proposition 6.1 above and Proposition 7.7 below), but not very useful otherwise. By contrast, the second description is very useful for all linear sections.

We recall the quadric $\mathscr{Q}_{\kappa,K} \subset \mathscr{Q}_v$, where $v = \gamma(\kappa)$; see (6.7). It is described in the following lemma.

Lemma 7.3. *Let $K \subset \mathbb{S}^\vee$ be a subspace of dimension $k \leqslant 5$. If X_K is a smooth linear section of the spinor tenfold X, then the quadric $\mathscr{Q}_{\kappa,K}$ has dimension $7-k$. It is smooth if and only if*

$$R_{\kappa,K} := \mathbb{P}(K/\kappa) \cap R_K \subset \mathrm{Gr}(2, K) \tag{7.2}$$

is a smooth quadric.

Proof. Put $v = \gamma(\kappa)$. The inclusion $\mathscr{Q}_v \subset \mathbb{P}(\kappa^\perp)$ proved in Lemma 5.10 shows that $\dim \mathscr{Q}_{\kappa,K} \geqslant 6 - (k-1) = 7 - k$. If the inequality is strict, then there is a point $\kappa' \in K$ distinct from κ such that $\mathscr{Q}_v \subset \mathbb{P}({\kappa'}^\perp)$. By Lemma 5.10 this means that $\gamma(\kappa') = v$. Therefore the map $\gamma\colon \mathbb{P}(K) \to \mathrm{Q}$ is not injective, contrary to Corollary 4.11. Thus $\dim \mathscr{Q}_{\kappa,K} = 7 - k$.

When K_2 ranges over the linear space $\mathbb{P}(K/\kappa)$ of all 2-dimensional subspaces K_2 with $\kappa \subset K_2 \subset K$, the quadrics $\mathscr{Q}_{\kappa,K_2}$ run over the linear system of those hyperplane sections of the smooth quadric $\mathscr{Q}_v$ that contain $\mathscr{Q}_{\kappa,K}$. Furthermore, Corollary 6.11 yields that the quadric $\mathscr{Q}_{\kappa,K_2}$ is singular if and only if $[K_2] \in R_{\kappa,K}$. Thus $R_{\kappa,K}$ is a linear section of the quadric $\mathscr{Q}_v^\vee$ (which is projectively dual to $\mathscr{Q}_v$) by the subspace orthogonal to the linear span of the quadric $\mathscr{Q}_{\kappa,K}$. In particular, using an analogue of Lemma 3.3, we conclude that it is smooth if and only if $\mathscr{Q}_{\kappa,K}$ is smooth. $\square$

Remark 7.4. The same argument shows that the corank of $\mathscr{Q}_{\kappa,K}$ is equal to the corank of $R_{\kappa,K}$. This observation is especially useful when $\dim K = 5$ and R_K is smooth. In this case, the corank stratification of the family of quadrics $R_{\kappa,K}$ is controlled by the corresponding EPW-sextic, see Proposition 4.5 in [11].

The following proposition describes the blow-up of X_K along the quadric $\mathscr{Q}_{\kappa,K}$. We also recall the quadric $\mathscr{Q}_{\kappa,-}$ defined in (5.22).

Proposition 7.5. *Let X_K be a smooth dimensionally transverse linear section of the spinor tenfold X of codimension $k \leqslant 5$. Then there is a piecewise-Zariski locally trivial fibration*

$$\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K) \to \mathscr{Q}_{\kappa,-}$$

whose generic fibre is $\mathbb{P}^{5-k}$ and whose special fibres are projective spaces of higher dimensions.

Proof. We repeat the argument of Proposition 6.8. Consider the isomorphism in Corollary 5.11. Clearly, the pre-image of the linear section $X_K = X_\kappa \cap \mathbb{P}(K^\perp)$ in $\mathrm{Bl}_{\mathscr{Q}_v}(X_\kappa)$ is isomorphic to the blow-up of X_K along the quadric $\mathscr{Q}_{\kappa,K}$. By Corollary 5.11 it may also be described as a relative linear section of codimension $k-1$ in the $\mathbb{P}^4$-bundle $\mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4)$. This linear section corresponds to the composition

$$\mathscr{O}(-1) \oplus \mathcal{S}_4 \hookrightarrow (\mathbb{S}^\vee/\kappa)^\vee \otimes \mathscr{O} \twoheadrightarrow (K/\kappa)^\vee \otimes \mathscr{O} \tag{7.3}$$

and it is generically surjective since otherwise the general fibre of the map

$$\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K) \hookrightarrow \mathbb{P}_{\mathscr{Q}_{\kappa,-}}(\mathscr{O}(-1) \oplus \mathcal{S}_4) \to \mathscr{Q}_{\kappa,-}$$

has dimension at least $6-k$ and, therefore, $\dim(\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K)) \geqslant 5+6-k = 11-k$, contrary to the hypothesis that X_K has dimension $10-k$. We now consider the corank stratification for the morphism (7.3); it follows from the above that the map $\mathrm{Bl}_{\mathscr{Q}_{\kappa,K}}(X_K) \to \mathscr{Q}_{\kappa,-}$ is locally trivial with fibre $\mathbb{P}^{5-k-c}$ over the corank-c stratum. $\square$

Using Proposition 7.5, we can easily deduce that X_K is rational.

Corollary 7.6. *If $1 \leqslant k \leqslant 5$, then every smooth linear section X_K of codimension k of the spinor tenfold X is rational.*

Proof. By Proposition 7.5, the variety X_K is birational to a locally trivial $\mathbb{P}^{5-k}$-fibration over an open subset of the 5-dimensional quadric $\mathscr{Q}_{\kappa,-}$. Hence it is rational. $\square$

One can show by another argument that all smooth linear sections of X of codimensions 6 and 7 are also rational.

7.3. Linear sections containing 4-spaces. In this subsection we discuss those linear sections of X of codimension higher than 2 that contain a 4-space. We first consider sections of codimension 3. The birational transformation in the following proposition is another example of a special birational transformation of type $(2,1)$ in Proposition 2.12 of [7].

Proposition 7.7. *Let X_K be a smooth dimensionally transverse linear section of X of codimension 3. Then the following conditions are equivalent.*

1) *The Hilbert scheme $F_4(X_K)$ of linear 4-spaces on X_K is non-empty.*

2) *For every 2-dimensional subspace $K_2 \subset K$, the oversection X_{K_2} of X is special, that is, $R_K = \mathrm{Gr}(2,K)$.*

3) *The linear span of the Veronese surface $\gamma(\mathbb{P}(K)) \subset \mathrm{Q} \subset \mathbb{P}(\mathrm{V})$ is a $\mathbb{P}^4$ and is contained in Q.*

If all these conditions hold, then $F_4(X_K) \cong \mathrm{Spec}(\Bbbk)$. Moreover, if $\Pi^4_{U_{5,-}}$ is the 4-space on X_K, then the linear span of $\gamma(\mathbb{P}(K))$ is equal to $\mathbb{P}(U_{5,-}) \subset \mathrm{Q}$ and we have an isomorphism $\mathrm{Bl}_{\Pi^4_{U_{5,-}}}(X_K) \cong \mathrm{Bl}_{Z_K}(\mathbb{P}(W_K))$ and a commutative diagram (6.2), where $W_K \subset W$ is the subspace of codimension 3 corresponding to K, and $Z_K = \mathrm{Gr}(2,U_{5,-}) \cap \mathbb{P}(W_K)$ is a smooth linear section of the Grassmannian of codimension 3.

Proof. We first prove the equivalence of the conditions.

1) $\Rightarrow$ 2). Since $X_K \subset X_{K_2}$, the condition $F_4(X_K) \neq \varnothing$ implies that $F_4(X_{K_2}) \neq \varnothing$ and, therefore, X_{K_2} is special for all $K_2 \subset K$.

2) $\Rightarrow$ 3). The linear span of the surface $\gamma(\mathbb{P}(K))$ is the projectivization of the image of the map $\mathrm{Sym}^2 K \to \mathrm{V}$. Hence its dimension is at most 5. On the other hand, if its dimension is 3 or less, then γ is not a closed embedding, contrary to Corollary 4.11. Thus the linear span of $\gamma(\mathbb{P}(K))$ is either a $\mathbb{P}^5$ or a $\mathbb{P}^4$.

Assume that the linear span of $\gamma(\mathbb{P}(K))$ is a $\mathbb{P}^5$. Then the union of the linear spans of the conics $\gamma(\mathbb{P}(K_2))$, where K_2 ranges over the set of all hyperplanes in K, is the secant variety of the Veronese surface $\gamma(\mathbb{P}(K))$, that is, a symmetric determinantal cubic in $\mathbb{P}^5$. By Proposition 6.1 it is contained in the quadric Q. But then $\mathbb{P}^5 \subset \mathrm{Q}$, which is impossible since Q is smooth of dimension 8.

Therefore the linear span of $\gamma(\mathbb{P}(K))$ is a $\mathbb{P}^4$. Then the union of the linear spans of the conics $\gamma(\mathbb{P}(K_2))$ is equal to the whole of $\mathbb{P}^4$, which then lies in Q by the hypothesis and Proposition 6.1.

3) $\Rightarrow$ 1). Assume that the linear span of the Veronese surface $\gamma(\mathbb{P}(K))$ is $\mathbb{P}(V_5) \subset \mathrm{Q}$, where $V_5 \subset V$ is an isotropic subspace (we do not specify whether it corresponds to a point of X or of $X^\vee$, but later we will see that the second option holds). For every subspace $U_4 \subset V_5$, the pre-image $\gamma^{-1}(\mathbb{P}(U_4)) \subset \mathbb{P}(K)$ is a conic. The conics that can be obtained in this way form a 4-dimensional linear system corresponding to the image of the injective map $V_5^\vee \to \mathrm{Sym}^2(K^\vee)$. Since non-reduced conics in $\mathbb{P}(K)$ are parametrized by a 2-dimensional variety while reducible and reduced conics are parametrized by a divisor in $\mathbb{P}(\mathrm{Sym}^2(K^\vee))$, there is a subspace $U_4 \subset V_5$ such that the conic $\gamma^{-1}(\mathbb{P}(U_4))$ is reducible and reduced, that is,

$$\gamma^{-1}(\mathbb{P}(U_4)) = \mathbb{P}(K_2') \cup \mathbb{P}(K_2''),$$

where $K_2', K_2'' \subset K$ are distinct two-dimensional subspaces. Then we have

$$\gamma(\mathbb{P}(K_2')), \gamma(\mathbb{P}(K_2'')) \subset \mathbb{P}(U_4).$$

Let $U_3', U_3'' \subset U_4$ be the linear spans of the conics $\gamma(\mathbb{P}(K_2'))$ and $\gamma(\mathbb{P}(K_2''))$ respectively and let $U_4 \subset U_{5,-}$ be the unique extension of the isotropic subspace $U_4 \subset V_5 \subset \mathrm{V}$ to the subspace corresponding to a point of $X^\vee$. Then

$$[U_{5,-}] \in L^-_{U_3'} \cap L^-_{U_3''} = F_4(X_{K_2'}) \cap F_4(X_{K_2''}) = F_4(X_K);$$

in the last equality we use the identification

$$X_K = X \cap \mathbb{P}(K^\perp) = X \cap \mathbb{P}({K_2'}^\perp) \cap \mathbb{P}({K_2''}^\perp) = X_{K_2'} \cap X_{K_2''},$$

whence a 4-space lies on X_K if and only if it lies both on $X_{K_2'}$ and $X_{K_2''}$. This proves that $F_4(X_K) \neq \varnothing$ and completes the proof of the implication 3) $\Rightarrow$ 1) and, therefore, of the equivalence of all three conditions.

We now assume that all three conditions of the proposition hold and $U_{5,-}$ is a subspace corresponding to a point of $F_4(X_K)$. Given any subspace $K_2 \subset K$, let $\mathbb{P}(U_3)$ be the linear span of the conic $\gamma(\mathbb{P}(K_2))$. Then $[U_{5,-}] \in F_4(X_{K_2}) = L^-_{U_3}$ by Proposition 6.1 and, therefore, $U_3 \subset U_{5,-}$. Thus the linear span of each conic $\gamma(\mathbb{P}(K_2))$ lies in $\mathbb{P}(U_{5,-})$ and, therefore, the linear span of the surface $\gamma(\mathbb{P}(K))$ also lies in $\mathbb{P}(U_{5,-})$. By condition 3), the linear span of $\gamma(\mathbb{P}(K))$ is equal to $\mathbb{P}(U_{5,-})$ (in particular, the subspace V_5 in the proof of the implication 3) $\Rightarrow$ 1) is equal to $U_{5,-}$). This also proves that $F_4(X_K) \cong \mathrm{Spec}(\Bbbk)$.

The rest of the proof (constructing the diagram (6.2) and describing its properties) is the same as in Proposition 6.1. $\square$

Definition 7.8. A smooth linear section $X_K \subset X$ of codimension 3 in the spinor tenfold X is said to be *very special* if the equivalent conditions of Proposition 7.7 hold for X_K.

Remark 7.9. Since a smooth linear section of $\mathrm{Gr}(2,5)$ of codimension 3 is unique up to projective transformations, it follows that a very special section of X is also unique.

We claim that if the codimension of K is greater than 3, then there are no smooth linear sections of X containing a 4-space.

Lemma 7.10. *If X_K is a smooth linear section of X of codimension 4 or higher, then $F_4(X_K) = \varnothing$.*

Proof. Let $\Pi = \Pi^4_{U_{5,-}}$ be a 4-space on X. A simple computation shows that $\mathscr{N}^\vee_{\Pi/X} \cong \Omega^2_\Pi(2)$. Hence, if K is a 4-dimensional space of hyperplanes that contain Π, there is a morphism

$$K \otimes \mathscr{O} \to I_\Pi(1) \to (I_\Pi/I^2_\Pi)(1) \cong \Omega^2_\Pi(3)$$

of sheaves on X. Since $\mathrm{c}_3(\Omega^2_\Pi(3)) = 5$, every such morphism drops rank to 3 on some non-empty subscheme of Π. Hence the corresponding linear section X_K is singular along this subscheme. $\square$

Note that $\mathrm{c}_4(\Omega^2_\Pi(3)) = 0$. This explains the existence of a smooth linear section of X of codimension 3 containing Π.

We also check that the quadratic line complex R_K in codimensions higher than 3 is always a hypersurface in $\mathrm{Gr}(2, K)$.

Lemma 7.11. *If X_K is a smooth linear section of X of codimension 4, then $R_K \ne \mathrm{Gr}(2, K)$. Moreover, if X_K is general, then there are no very special oversections $X_{K_3} \subset X$ of codimension 3.*

Proof. Assume that $R_K = \mathrm{Gr}(2, K)$. Then, for every $K_2 \subset K$, the linear span of the conic $\gamma(\mathbb{P}(K_2))$ is contained in Q. Hence the secant variety of $\gamma(\mathbb{P}(K))$ is contained in Q. On the other hand, the secant variety of a 3-dimensional Veronese variety is not contained in a quadric and, therefore, the linear span of $\gamma(\mathbb{P}(K))$ is contained in Q. But the dimension of the span is at least 7 while Q contains no linear spaces of dimension higher than 4.

We now assume that $\dim K = 4$ and X_K is general. Then $R_K \subset \mathrm{Gr}(2, K)$ is a smooth quadratic divisor (Lemma 7.2). In particular, R_K contains no planes by Lefschetz' theorem. But if X_{K_3} is very special, then the plane $\mathrm{Gr}(2, K_3)$ is contained in R_K. $\square$

Bibliography

[1] A. G. Kuznetsov, "Hyperplane sections and derived categories", *Izv. Ross. Akad. Nauk Ser. Mat.* **70**:3 (2006), 23–128; English transl., *Izv. Math.* **70**:3 (2006), 447–547.

[2] A. Kuznetsov, "Semiorthogonal decompositions in algebraic geometry", *Proceedings of the International Congress of Mathematicians*, vol. 2 (Seoul 2014), Kyung Moon Sa, Seoul 2014, pp. 635–660.

[3] F. L. Zak, *Tangents and secants of algebraic varieties*, Transl. from the Russian manuscript, Transl. Math. Monogr., vol. 127, Amer. Math. Soc., Providence, RI 1993.

[4] K. Ranestad and F.-O. Schreyer, "Varieties of sums of powers", *J. Reine Angew. Math.* **2000**:525 (2000), 147–181.

[5] B. Pasquier, "On some smooth projective two-orbit varieties with Picard number 1", *Math. Ann.* **344**:4 (2009), 963–987.

[6] L. Manivel, *Double spinor Calabi–Yau varieties*, 2017, arXiv: 1709.07736.

[7] Baohua Fu and Jun-Muk Hwang, "Special birational transformations of type (2, 1)", *J. Algebraic Geom.* **27**:1 (2018), 55–89.

[8] S. Gorchinskiy, "Integral Chow motives of threefolds with K-motives of unit type", *Bull. Korean Math. Soc.* **54**:5 (2017), 1827–1849.

[9] S. Galkin, L. Katzarkov, A. Mellit, and E. Shinder, *Minifolds and phantoms*, 2013, arXiv: 1305.4549.

[10] A. Kuznetsov, "Küchle fivefolds of type $c5$", *Math. Z.* **284**:3-4 (2016), 1245–1278.

[11] O. Debarre and A. Kuznetsov, "Gushel–Mukai varieties: classification and birationalities", *Algebr. Geom.* **5**:1 (2018), 15–76.

[12] J. M. Landsberg and L. Manivel, "On the projective geometry of rational homogeneous varieties", *Comment. Math. Helv.* **78**:1 (2003), 65–100.

[13] G. Ottaviani, "Spinor bundles on quadrics", *Trans. Amer. Math. Soc.* **307**:1 (1988), 301–316.

[14] A. Kuznetsov, "Exceptional collections for Grassmannians of isotropic lines", *Proc. Lond. Math. Soc.* (3) **97**:1 (2008), 155–182.

[15] O. Debarre and A. Kuznetsov, "Gushel–Mukai varieties: linear spaces and periods", *Kyoto J. Math.* (to appear); 2016, arXiv: 1605.05648.

[16] A. Kuznetsov and A. Perry, "Derived categories of Gushel–Mukai varieties", *Compos. Math.* **154**:7 (2018), 1362–1406; 2016, arXiv: 1605.06568.

[17] L. Ein and N. Shepherd-Barron, "Some special Cremona transformations", *Amer. J. Math.* **111**:5 (1989), 783–800.

[18] Yu. G. Prokhorov, "On the number of singular points of terminal factorial Fano threefolds", *Mat. Zametki* **101**:6 (2017), 949–954; English transl., *Math. Notes* **101**:6 (2017), 1068–1073.

[19] A. Kuznetsov, "Homological projective duality", *Publ. Math. Inst. Hautes Études Sci.* **105** (2007), 157–220.

[20] S. Mukai, "Curves and symmetric spaces. I", *Amer. J. Math.* **117**:6 (1995), 1627–1644.

[21] A. G. Kuznetsov, "Derived categories of Fano threefolds V_{12}", *Mat. Zametki* **78**:4 (2005), 579–594; English transl., *Math. Notes* **78**:4 (2005), 537–550.

[22] S. Mukai, "Duality of polarized $K3$ surfaces", *New Trends in Algebraic Geometry* (Warwick 1996), London Math. Soc. Lecture Note Ser., vol. 264, Cambridge Univ. Press, Cambridge 1999, pp. 311–326.

[23] M. Marcolli and G. Tabuada, "From exceptional collections to motivic decompositions via noncommutative motives", *J. Reine Angew. Math.* **2015**:701 (2015), 153–167.

[24] Yu. G. Prokhorov, "On G-Fano threefolds", *Izv. Ross. Akad. Nauk Ser. Mat.* **79**:4 (2015), 159–174; English transl., *Izv. Math.* **79**:4 (2015), 795–808.

[25] M. Sato and T. Kimura, "A classification of irreducible prehomogeneous vector spaces and their relative invariants", *Nagoya Math. J.* **65** (1977), 1–155.

[26] W. Kraśkiewicz and J. Weyman, *Geometry of orbit closures for the representations associated to gradings of Lie algebras of types E_6, F_4 and G_2*, 2012, arXiv: 1201.1102.

[27] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Corr. reprint of the 1988 ed., Mod. Birkhäuser Class., Birkhäuser/Springer Basel AG, Basel 2011; Russian transl., Mathematics. New in Foreign Science, vol. 36, Mir, Moscow 1984.

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Received 29/DEC/17
Translated by THE AUTHOR