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Embedding derived categories of Enriques surfaces in derived categories of Fano varieties

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Abstract. We show that the bounded derived category of coherent sheaves on a general Enriques surface can be realized as a semi-orthogonal component in the derived category of a smooth Fano variety with diagonal Hodge diamond.

Keywords: derived category of coherent sheaves, Fano variety, Enriques surface.

If a smooth projective variety X over the field $\mathbb{C}$ of complex numbers has a full exceptional collection, then its Hodge diamond is *diagonal*, that is,

$$h^{p,q}(X)=0 \quad \text{when} \quad p\neq q.$$

It is natural to ask whether the converse is true. A simple counterexample to this naive question is provided by an Enriques surface S. Its Hodge diamond is of the form

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix},$$

so it is diagonal. On the other hand, its Grothendieck group $K_0(S)$ contains a 2-torsion class (see, for example, [1], Lemma 2.2) and, therefore, its derived category cannot be generated by a full exceptional collection because of the following simple lemma.

Lemma 1 (compare with [2], § 3, [1], Proposition 2.1(5)). *Let $\mathscr{T}$ be a triangulated category whose Grothendieck group $K_0(\mathscr{T})$ contains a torsion class. Then $\mathscr{T}$ does not admit a full exceptional collection.*

Proof. Assume that $\mathscr{T}$ is generated by an exceptional collection of length n. Since the Grothendieck group is additive with respect to semi-orthogonal decompositions, we have $K_0(\mathscr{T})\cong\mathbb{Z}^n$. In particular, $K_0(\mathscr{T})$ is torsion-free. □

A slightly less naive question is whether a *Fano variety* with diagonal Hodge diamond necessarily has a full exceptional collection. This was asked by Alexey Bondal in 1989 and raised again in a recent paper [3]. The main goal of this

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note is to show that the answer is still negative, and counterexamples can again be constructed using Enriques surfaces.

More precisely, we construct a smooth Fano variety X such that its bounded derived category $\mathbf{D}(X)$ of coherent sheaves has a semi-orthogonal decomposition whose components are several exceptional objects along with $\mathbf{D}(S)$, where S is an Enriques surface. Thus the Hodge diamond of X is diagonal, but the Grothendieck group $K_0(X)$ contains a 2-torsion class (coming from $K_0(S)$) and, therefore, $\mathbf{D}(X)$ has no full exceptional collections by Lemma 1.

In fact, we describe two such constructions.

In the first, S is a general Enriques surface belonging to a certain divisorial family in the moduli space of Enriques surfaces. Such surfaces S are called 'nodal Enriques surfaces' or 'Reye congruences'. By Theorem 3.2.2 in [4], an Enriques surface S of this type can be embedded in the Grassmannian $\mathrm{Gr}(2,4)$, and Lemma 5.1 in [5] describes a resolution of its structure sheaf.

Consider the blow-up

$$M = \mathrm{Bl}_S(\mathrm{Gr}(2,4)).$$

Theorem 2. *The variety M is a Fano 4-fold with a semi-orthogonal decomposition*

$$\mathbf{D}(M) = \langle \mathbf{D}(S), E_1, \ldots, E_6 \rangle,$$

where $E_1, \ldots, E_6$ are exceptional bundles. The Hodge diamond of M is diagonal, but $K_0(M)$ contains a 2-torsion class. In particular, $\mathbf{D}(M)$ has no full exceptional collections.

Proof. By Lemmas 5.2, 5.3 in [5], M can be embedded in the product $\mathrm{Gr}(2,4) \times \mathbb{P}^3$ as the zero locus of a regular section of the rank-3 vector bundle $\mathrm{S}^2\mathscr{U}^\vee \boxtimes \mathscr{O}(1)$, where $\mathscr{U}$ is the tautological bundle on the Grassmannian. The determinant of this vector bundle is isomorphic to $\mathscr{O}(3) \boxtimes \mathscr{O}(3)$. Hence, by the adjunction formula, $\omega_M^{-1} \cong (\mathscr{O}(1) \boxtimes \mathscr{O}(1))|_M$ is the restriction of an ample line bundle. Thus M is a Fano 4-fold.

The semi-orthogonal decomposition is obtained from Orlov's blow-up formula since $\mathbf{D}(\mathrm{Gr}(2,4))$ is generated by an exceptional collection of length 6. The Hodge diamond of M is of the form

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&0&&0&&0& \\ 0&&0&&2&&0&&0 \\ &0&&0&&0&&0& \\ &&0&&1&&0&& \\ &&&0&&0&&& \\ &&&&1&&&& \end{array} \; + \; \begin{array}{ccccc} &&1&& \\ &0&&0& \\ 0&&10&&0 \\ &0&&0& \\ &&1&& \end{array} \; = \; \begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&2&&0&& \\ &0&&0&&0&&0& \\ 0&&0&&12&&0&&0 \\ &0&&0&&0&&0& \\ &&0&&2&&0&& \\ &&&0&&0&&& \\ &&&&1&&&& \end{array},$$

a combination of the Hodge diamonds of $\mathrm{Gr}(2,4)$ and S, again thanks to the blow-up representation. Since the Grothendieck group is additive with respect to semi-orthogonal decompositions, we have

$$K_0(M) = K_0(S) \oplus \mathbb{Z}^6.$$

In particular, the 2-torsion class in S gives a 2-torsion class in M. We complete the proof using Lemma 1. □

The second construction works for general Enriques surfaces (that is, for all points of an open subset in the moduli space of Enriques surfaces) at the cost of the corresponding Fano variety being 6-dimensional.

Let V_1 and V_2 be 3-dimensional vector spaces. Consider the Veronese embeddings

$$\mathbb{P}(V_1) \hookrightarrow \mathbb{P}(\mathrm{S}^2 V_1) \hookrightarrow \mathbb{P}(\mathrm{S}^2 V_1 \oplus \mathrm{S}^2 V_2), \qquad \mathbb{P}(V_2) \hookrightarrow \mathbb{P}(\mathrm{S}^2 V_2) \hookrightarrow \mathbb{P}(\mathrm{S}^2 V_1 \oplus \mathrm{S}^2 V_2)$$

and their join $\mathrm{J}(\mathbb{P}(V_1), \mathbb{P}(V_2)) \subset \mathbb{P}(\mathrm{S}^2 V_1 \oplus \mathrm{S}^2 V_2)$. This is a singular 5-dimensional variety whose singularities are resolved by the projective bundle

$$\mathbf{J} := \mathbb{P}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(\mathscr{O}(-2,0) \oplus \mathscr{O}(0,-2)).$$

Indeed, let H_1 and H_2 be the pullbacks to $\mathbf{J}$ of the hyperplane classes of the factors $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$, H the Grothendieck relative class of the projectivization, and $\pi\colon \mathbf{J} \to \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ the projection. Then the natural embedding

$$\mathscr{O}_{\mathbf{J}}(-H) \hookrightarrow \pi^*(\mathscr{O}(-2,0) \oplus \mathscr{O}(0,-2)) \hookrightarrow (\mathrm{S}^2 V_1 \otimes \mathscr{O}) \oplus (\mathrm{S}^2 V_2 \otimes \mathscr{O})$$

induces a morphism $\mathbf{J} \to \mathbb{P}(\mathrm{S}^2 V_1 \oplus \mathrm{S}^2 V_2)$ which contracts the divisors

$$\mathbb{P}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(\mathscr{O}(-2,0)) \subset \mathbf{J}, \qquad \mathbb{P}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(\mathscr{O}(0,-2)) \subset \mathbf{J}$$

onto the Veronese surfaces $\mathbb{P}(V_1) \hookrightarrow \mathbb{P}(\mathrm{S}^2 V_1 \oplus \mathrm{S}^2 V_2)$ and $\mathbb{P}(V_2) \hookrightarrow \mathbb{P}(\mathrm{S}^2 V_1 \oplus \mathrm{S}^2 V_2)$ and takes the fibres of π to the lines joining the corresponding points of them.

In what follows we consider a global section of the vector bundle $\mathscr{O}_{\mathbf{J}}(H)^{\oplus 3}$ on $\mathbf{J}$. Note that

$$H^0(\mathbf{J}, \mathscr{O}_{\mathbf{J}}(H)) \cong H^0(\mathbb{P}(V_1) \times \mathbb{P}(V_2), \mathscr{O}(2,0) \oplus \mathscr{O}(0,2)) \cong \mathrm{S}^2 V_1^\vee \oplus \mathrm{S}^2 V_2^\vee.$$

Therefore such a section is given by a linear map

$$\phi\colon W \to \mathrm{S}^2 V_1^\vee \oplus \mathrm{S}^2 V_2^\vee$$

from a 3-dimensional vector space W. We denote the corresponding section again by ϕ.

Lemma 3. *The zero locus $S \subset \mathbf{J}$ of a general section ϕ of the bundle $\mathscr{O}_{\mathbf{J}}(H)^{\oplus 3}$ on $\mathbf{J}$ is an Enriques surface. A general Enriques surface can be obtained in this way.*

Proof. Consider another projective bundle

$$\widetilde{\mathbf{J}} \cong \mathbb{P}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(\mathscr{O}(-1,0) \oplus \mathscr{O}(0,-1)).$$

It is isomorphic to the blow-up of the union $\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)$ of two skew planes in $\mathbb{P}(V_1 \oplus V_2)$ with exceptional divisors

$$E_1 = \mathbb{P}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(\mathscr{O}(-1,0)) \subset \widetilde{\mathbf{J}}, \qquad E_2 = \mathbb{P}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(\mathscr{O}(0,-1)) \subset \widetilde{\mathbf{J}}.$$

Let $\widetilde{H}_1$ and $\widetilde{H}_2$ be the pullbacks to $\widetilde{\mathbf{J}}$ of the hyperplane classes of the factors $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$, and let $\widetilde{H}$ be the Grothendieck relative class of the projectivization. Then $E_1 \equiv \widetilde{H} - \widetilde{H}_2$ and $E_2 \equiv \widetilde{H} - \widetilde{H}_1$.

Consider the involution of the vector bundle $\mathscr{O}(-1,0) \oplus \mathscr{O}(0,-1)$ acting with weight -1 on the first summand and with weight 1 on the second, and let τ be the corresponding involution of $\widetilde{\mathbf{J}}$. The fixed-point locus of τ is the union $E_1 \sqcup E_2$ of the exceptional divisors, and the quotient $\widetilde{\mathbf{J}}/\tau$ is isomorphic to $\mathbf{J}$ with the quotient map $f\colon \widetilde{\mathbf{J}} \to \mathbf{J}$ induced by the projection

$$\mathrm{S}^2(\mathscr{O}(-1,0)\oplus\mathscr{O}(0,-1)) = \mathscr{O}(-2,0)\oplus\mathscr{O}(-1,-1)\oplus\mathscr{O}(0,-2) \twoheadrightarrow \mathscr{O}(-2,0)\oplus\mathscr{O}(0,-2).$$

We also note that $\mathscr{O}_{\widetilde{\mathbf{J}}}(2\widetilde{H}) \cong f^*(\mathscr{O}_{\mathbf{J}}(H))$, and this induces an isomorphism

$$H^0(\widetilde{\mathbf{J}}, \mathscr{O}_{\widetilde{\mathbf{J}}}(2\widetilde{H}))^\tau \cong H^0(\mathbf{J}, \mathscr{O}_{\mathbf{J}}(H)) \cong \mathrm{S}^2V_1^\vee \oplus \mathrm{S}^2V_2^\vee$$

between the space of τ-invariant global sections of $\mathscr{O}_{\widetilde{\mathbf{J}}}(2\widetilde{H})$ and the space of global sections of $\mathscr{O}_{\mathbf{J}}(H)$. Therefore, the pre-image

$$\widetilde{S} := f^{-1}(S) \subset \widetilde{\mathbf{J}}$$

is the zero locus of a general τ-invariant section of the vector bundle $\mathscr{O}_{\mathbf{J}}(2\widetilde{H})^{\oplus 3}$. We have

$$\begin{aligned} K_{\widetilde{S}} &\equiv K_{\widetilde{\mathbf{J}}} + 6\widetilde{H} \equiv (-3\widetilde{H}_1 - 3\widetilde{H}_2) + (\widetilde{H}_1 + \widetilde{H}_2 - 2\widetilde{H}) + 6\widetilde{H} \\ &\equiv 4\widetilde{H} - 2\widetilde{H}_1 - 2\widetilde{H}_2 \equiv 2E_1 + 2E_2. \end{aligned}$$

Recall that S is defined by a map $\phi\colon W \to \mathrm{S}^2V_1^\vee \oplus \mathrm{S}^2V_2^\vee$. Clearly, $\widetilde{S} \cap E_i$ is equal to the intersection of the three conics in $\mathbb{P}(V_i)$ corresponding to the induced map $\phi_i\colon W \to \mathrm{S}^2V_i^\vee$. Hence this set is empty for a general choice of ϕ. This shows that the surface $\widetilde{S}$ is disjoint from E_1 and E_2 (and, therefore, $K_{\widetilde{S}} \equiv 0$) for general S.

Furthermore, it is easy to see (for example, using the Koszul resolution of $\mathscr{O}_{\widetilde{S}}$ on $\widetilde{\mathbf{J}}$) that $H^1(\widetilde{S}, \mathscr{O}_{\widetilde{S}}) = 0$ and, therefore, $\widetilde{S}$ is a K3-surface. Since $\widetilde{S}$ is disjoint from the fixed-point locus $E_1 \sqcup E_2$ of τ, the involution τ acts freely on $\widetilde{S}$. Hence,

$$S \cong \widetilde{S}/\tau \subset \widetilde{\mathbf{J}}/\tau = \mathbf{J}$$

is an Enriques surface.

Finally, we note that the surface $\widetilde{S}$ defined above coincides with the surface X in [6], Exercise VIII.18, and the involution τ on $\widetilde{S}$ coincides with the involution σ described there. Hence the quotient $S = \widetilde{S}/\tau$ is a general Enriques surface. □

We now consider the product $\mathbf{J} \times \mathbb{P}(W)$ that parametrizes the linear system of sections of $\mathscr{O}_{\mathbf{J}}(H)$ cutting out S in $\mathbf{J}$. Write H' for the hyperplane class of $\mathbb{P}(W)$ and let

$$X \subset \mathbf{J} \times \mathbb{P}(W)$$

be the universal divisor in the linear system of the equations of S, that is, the zero locus on $\mathbf{J} \times \mathbb{P}(W)$ of the global section of the line bundle $\mathscr{O}_{\mathbf{J}}(H) \boxtimes \mathscr{O}(H')$ corresponding to the map ϕ.

Theorem 4. *The variety X is a Fano 6-fold with a semi-orthogonal decomposition*

$$\mathbf{D}(X) = \langle \mathbf{D}(S), F_1, \dots, F_{36} \rangle,$$

where $F_1, \dots, F_{36}$ are exceptional bundles. The Hodge diamond of X is diagonal, but $K_0(X)$ contains a 2-torsion class. In particular, $\mathbf{D}(X)$ has no full exceptional collections.

Proof. The canonical class of X is equal to

$$\begin{aligned} K_X &\equiv K_{\mathbf{J}} + K_{\mathbb{P}(W)} + (H + H') \\ &\equiv (-3H_1 - 3H_2) + (2H_1 + 2H_2 - 2H) - 3H' + (H + H') \\ &\equiv -H_1 - H_2 - H - 2H'. \end{aligned}$$

We claim that $-K_X$ is ample. Indeed, it suffices to check that the line bundle $H + H_1 + H_2$ is ample on $\mathbf{J}$. By Proposition 3.2 in [7], this is equivalent to the ampleness of its pushforward $\mathscr{O}(3H_1 + H_2) \oplus \mathscr{O}(H_1 + 3H_2)$ on $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. But this ampleness follows from that of the summands by Proposition 2.2 in [7]. We conclude that X is a Fano 6-fold.

The map $X \to \mathbf{J}$ has general fibre $\mathbb{P}^1$, and its fibres jump to $\mathbb{P}^2$ over the Enriques surface $S \subset \mathbf{J}$. Therefore,

$$\mathbf{D}(X) = \langle \mathbf{D}(S), \mathbf{D}(\mathbf{J}), \mathbf{D}(\mathbf{J}) \rangle$$

either by Theorem 8.8 in [8], or by Proposition 2.10 in [9]. Since $\mathbf{J}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2 \times \mathbb{P}^2$, its derived category is generated by $3 \cdot 3 \cdot 2 = 18$ exceptional bundles. Hence we obtain the required semi-orthogonal decomposition for $\mathbf{D}(X)$. Finally, the Hodge diamond of X is of the form

$$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 4 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 8 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 10 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 8 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 4 \quad 0 \\ 0 \quad 0 \\ 1 \end{array} \quad + \quad \begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 10 \quad 0 \\ 0 \quad 0 \\ 1 \end{array} \quad = \quad \begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 4 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 9 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 20 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 9 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 4 \quad 0 \\ 0 \quad 0 \\ 1 \end{array},$$

a combination of the Hodge diamonds of $\mathbf{J} \times \mathbb{P}^1$ and S. Since the Grothendieck group is additive with respect to semi-orthogonal decompositions, we have

$$K_0(X) = K_0(S) \oplus \mathbb{Z}^{36}.$$

In particular, the 2-torsion class in S gives a 2-torsion class in X. We complete the proof using Lemma 1. □

Remark 5. The embedding of the derived category of a general Enriques surface in the derived category of the Fano variety constructed in Theorem 4, solves the so-called 'Fano visitor problem' for such surfaces. This problem was suggested by Alexey Bondal in 2011; see [10]–[14]. We note that a similar embedding of $\mathbf{D}(S)$ in the derived category of a Fano *orbifold* was constructed in [12], 6.2.3.

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