Gushel–Mukai varieties: Linear spaces and periods

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Abstract Beauville and Donagi proved in 1985 that the primitive middle cohomology of a smooth complex cubic 4-fold and the primitive second cohomology of its variety of lines, a smooth hyper-Kähler 4-fold, are isomorphic as polarized integral Hodge structures. We prove analogous statements for smooth complex Gushel–Mukai varieties of dimension 4 (resp., 6), that is, smooth dimensionally transverse intersections of the cone over the Grassmannian $\mathsf{Gr}(2,5)$, a quadric, and two hyperplanes (resp., of the cone over $\mathsf{Gr}(2,5)$ and a quadric). The associated hyper-Kähler 4-fold is in both cases a smooth double cover of a hypersurface in $\mathbf{P}^5$ called an *Eisenbud–Popescu–Walter sextic.*

1. Introduction

We continue in this article our investigation of Gushel–Mukai (GM) varieties started in [5]. We discuss linear subspaces contained in smooth complex GM varieties and their relation to Eisenbud–Popescu–Walter (EPW) stratifications. These results are applied to the computation of the period map for GM varieties of dimension 4 or 6.

We work over the field of complex numbers. A smooth *Gushel–Mukai variety* is (see [5, Definition 2.1]) a smooth dimensionally transverse intersection

$$\mathsf{CGr}(2,V_5)\cap\mathbf{P}(W)\cap Q$$

of the cone over the Grassmannian $\mathsf{Gr}(2,V_5)$ of 2-dimensional subspaces in a fixed 5-dimensional vector space V_5, with a linear subspace $\mathbf{P}(W)$ and a quadric Q. This class of varieties includes all smooth prime Fano varieties X of dimension $n\geq 3$, coindex 3, and degree 10 (i.e., such that there is an ample class H with $\operatorname{Pic}(X)=\mathbf{Z}H$, $K_X=-(n-2)H$, and $H^n=10$; see [5, Theorem 2.16]).

One can naturally associate with any smooth GM variety of dimension n a triple (V_6,V_5,A), called a *Lagrangian data set*, where V_6 is a 6-dimensional vector space containing V_5 as a hyperplane and the subspace $A\subset\bigwedge^3 V_6$ is Lagrangian

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with respect to the symplectic structure on $\bigwedge^3 V_6$ given by wedge product. Moreover, $\mathbf{P}(A) \cap \mathsf{Gr}(3, V_6) = \varnothing$ in $\mathbf{P}(\bigwedge^3 V_6)$ when $n \geq 3$. (We say that A *has no decomposable vectors*.)

Conversely, given a Lagrangian data set (V_6, V_5, A) with no decomposable vectors in A, one can construct two smooth GM varieties of respective dimensions $n = 5 - \ell$ and $n = 6 - \ell$ (where $\ell := \dim(A \cap \bigwedge^3 V_5) \leq 3$), with associated Lagrangian data set (V_6, V_5, A) (see [5, Theorem 3.10 and Proposition 3.13]; see Section 2.1 for more details).

Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$, we define three chains of subschemes

$$Y_A^{\geq 3} \subset Y_A^{\geq 2} \subset Y_A^{\geq 1} \subset \mathbf{P}(V_6), \qquad Y_{A^\perp}^{\geq 3} \subset Y_{A^\perp}^{\geq 2} \subset Y_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6^\vee),$$
$$Z_A^{\geq 4} \subset Z_A^{\geq 3} \subset Z_A^{\geq 2} \subset Z_A^{\geq 1} \subset \mathsf{Gr}(3, V_6)$$

called *Eisenbud–Popescu–Walter (EPW) stratifications* (see Section 2.2). The first two were extensively studied by O'Grady (see [21]–[26]) and the third was studied in [12]. If A has no decomposable vectors, then the strata

$$Y_A := Y_A^{\geq 1} \subset \mathbf{P}(V_6), \qquad Y_{A^\perp} := Y_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6^\vee), \qquad \text{and}$$
$$Z_A := Z_A^{\geq 1} \subset \mathsf{Gr}(3, V_6)$$

are hypersurfaces of respective degrees 6, 6, and 4, called the *EPW sextic*, the *dual EPW sextic*, and the *EPW quartic* associated with A. Moreover, there are canonical double coverings

$$\widetilde{Y}_A \to Y_A, \qquad \widetilde{Y}_{A^\perp} \to Y_{A^\perp}, \qquad \text{and} \qquad \widetilde{Z}_A^{\geq 2} \to Z_A^{\geq 2},$$

called the *double EPW sextic*, the *double dual EPW sextic*, and the *EPW cube* associated with A, respectively (see [7] for a uniform construction). In general (more precisely, when $Y_A^{\geq 3} = \varnothing$, $Y_{A^\perp}^{\geq 3} = \varnothing$, and $Z_A^{\geq 4} = \varnothing$), these are hyper-Kähler manifolds which are deformation equivalent to the Hilbert square or cube of a K3 surface.

We showed in [5] that these EPW stratifications control many geometrical properties of GM varieties. For instance, smooth GM varieties of dimension 3 or 4 are birationally isomorphic if their associated EPW sextics are isomorphic (see [5, Theorems 4.7 and 4.15]). In this article, we describe the Hilbert schemes of linear spaces contained in smooth GM varieties in terms of their EPW stratifications and relate the Hodge structures of smooth GM varieties of dimension 4 or 6 to those of their associated double EPW sextics.

Let X be a smooth GM variety. We denote by $F_k(X)$ the Hilbert scheme of linearly embedded projective k-spaces in X. The scheme $F_2(X)$ has two connected components $F_2^\sigma(X)$ and $F_2^\tau(X)$ corresponding to the two types of projective planes in $\mathsf{Gr}(2, V_5)$. We construct maps

$$F_1(X) \to \mathbf{P}(V_5), \qquad F_2^\sigma(X) \to \mathbf{P}(V_5), \qquad F_3(X) \to \mathbf{P}(V_5),$$
$$F_2^\tau(X) \to \mathsf{Gr}(3, V_5)$$

and describe them in terms of the EPW varieties defined by the Lagrangian A associated with X (Theorems 4.2, 4.3, 4.5, and 4.7). We prove, in particular, the following results.

If X is a smooth GM 6-fold with associated Lagrangian A such that $Y_A^{\geq 3} = \varnothing$, then the scheme $F_2^\sigma(X)$ has dimension 4 and the above map $F_2^\sigma(X) \to \mathbf{P}(V_5)$ factors as

$$F_2^\sigma(X) \to \widetilde{Y}_A \times_{\mathbf{P}(V_6)} \mathbf{P}(V_5) \to \mathbf{P}(V_5),$$

where the first map is a locally trivial (in the étale topology) $\mathbf{P}^1$-bundle (see Theorem 4.3(a)).

If X is a smooth general (with explicit generality assumptions) GM 4-fold, then $F_1(X)$ has dimension 3 and the map $F_1(X) \to \mathbf{P}(V_5)$ factors as

$$F_1(X) \to \widetilde{Y}_A \times_{\mathbf{P}(V_6)} \mathbf{P}(V_5) \to \mathbf{P}(V_5),$$

where the first map is a small resolution of singularities (a contraction of two rational curves; see Theorem 4.7(c)).

Consequently, the universal plane $\mathscr{L}_2^\sigma(X)$ in the 6-fold case and the universal line $\mathscr{L}_1(X)$ in the 4-fold case give correspondences

$$X \longleftarrow \mathscr{L}_2^\sigma(X) \longrightarrow \widetilde{Y}_A \qquad \text{and} \qquad X \longleftarrow \mathscr{L}_1(X) \longrightarrow \widetilde{Y}_A$$

between X and its associated double EPW sextic $\widetilde{Y}_A$. We use them to construct, in dimensions $n = 4$ or 6, isomorphisms

$$H^n(X;\mathbf{Z})_{00} \simeq H^2(\widetilde{Y}_A;\mathbf{Z})_0$$

of polarized integral Hodge structures (up to Tate twists; see Theorem 5.1 for precise statements) between the vanishing middle cohomology of X (defined in (5)) and the primitive second cohomology of $\widetilde{Y}_A$ (defined in (6)).

This isomorphism is the main result of this article. It implies that the period point of a smooth GM variety of dimension 4 or 6 (defined as the class of its vanishing cohomology Hodge structure in the appropriate period space) with associated Lagrangian data set (V_6, V_5, A) depends only on the $\mathrm{PGL}(V_6)$-orbit of A and not on V_5.

More precisely, the period maps from the moduli stacks of GM varieties of dimension 4 or 6 factor through the period map from the moduli stack of double EPW sextics—the period spaces being the same. The first map in this factorization is a fibration with well-understood fibers (see [5, Theorem 3.25]). Since double EPW sextics, when smooth, are hyper-Kähler manifolds, the second map is an open embedding by Verbitsky's Torelli theorem.

The article is organized as follows. In Section 2, we recall some of the results from [5] about the geometry of smooth GM varieties and their relation to EPW varieties. In Section 3, we discuss the singular cohomology of GM varieties. In

Section 4, we describe the Hilbert schemes $F_k(X)$ for smooth GM varieties X. In Section 5, we prove an isomorphism between the vanishing Hodge structure of a general GM variety of dimension 4 or 6 and the primitive Hodge structure of the associated double EPW sextic. We also define the period point of a GM variety and show that it coincides with the period point of the associated double EPW sextic. In Appendix A, we discuss the natural double coverings arising from the Stein factorizations of relative Hilbert schemes of quadric fibrations. In Appendix B, we discuss a resolution of the structure sheaf of an EPW surface $Y_A^{\geq 2}$ in $\mathbf{P}(V_6)$ and compute some cohomology spaces related to its ideal sheaf.

2. Geometry of Gushel–Mukai varieties

2.1. Gushel–Mukai varieties

We work over the field of complex numbers. A smooth Gushel–Mukai (GM) variety of dimension n is (see [5, Definition 2.1 and Proposition 2.28]) a smooth dimensionally transverse intersection

$$X = \mathsf{CGr}(2, V_5) \cap \mathbf{P}(W) \cap Q, \tag{1}$$

where V_5 is a vector space of dimension 5, $\mathsf{CGr}(2, V_5) \subset \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ is the cone (with vertex $\nu := [\mathbf{C}]$) over the Grassmannian of 2-dimensional subspaces in V_5, $W \subset \mathbf{C} \oplus \bigwedge^2 V_5$ is a vector subspace of dimension $n+5$, and $Q \subset \mathbf{P}(W)$ is a quadratic hypersurface.

Being smooth, X does not contain the vertex ν; hence, the linear projection from ν defines a regular map

$$\gamma_X \colon X \to \mathsf{Gr}(2, V_5)$$

called the *Gushel map* of X. We denote by $\mathscr{U}_X$ the pullback to X of the tautological rank-2 subbundle on the Grassmannian. It comes with an embedding $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathscr{O}_X$.

Following [5], we associate with every smooth GM variety X as in (1) the intersection

$$M_X := \mathsf{CGr}(2, V_5) \cap \mathbf{P}(W).$$

This is a variety of dimension $n+1$ with finite singular locus (see [5, Proposition 2.22]).

If the linear space $\mathbf{P}(W)$ does not contain the vertex ν, then the variety M_X is itself a dimensionally transverse section of $\mathsf{Gr}(2, V_5)$ by the image of the linear projection $\mathbf{P}(W) \to \mathbf{P}(\bigwedge^2 V_5)$ from ν. It is smooth if $n \geq 3$ and X is its intersection with a quadratic hypersurface. These GM varieties are called *ordinary*.

If $\mathbf{P}(W)$ contains ν, then the variety M_X is itself a cone with vertex ν over the smooth dimensionally transverse linear section

$$M'_X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W') \subset \mathbf{P}\left(\bigwedge^2 V_5\right),$$

where $W' = W/\mathbf{C} \subset \bigwedge^2 V_5$, and X is a double cover of M'_X branched along the smooth GM variety $X' = M'_X \cap Q$ of dimension $n-1$. These GM varieties are called *special*.

A GM variety $X \subset \mathbf{P}(W)$ is an intersection of quadrics. Following [5], we denote by V_6 the 6-dimensional space of quadratic equations of X. The space V_5 can be naturally identified with the space of Plücker quadrics cutting out $\mathsf{CGr}(2, V_5)$ in $\mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$ and, hence, also with the space of quadrics in $\mathbf{P}(W)$ cutting out the subvariety M_X. This gives a canonical embedding $V_5 \subset V_6$ which identifies V_5 with a hyperplane in V_6 called the *Plücker hyperplane*. The corresponding point $\mathbf{p}_X \in \mathbf{P}(V_6^\vee)$ in the dual projective space is called the *Plücker point*.

2.2. EPW sextics and quartics

Let X be a smooth GM variety of dimension n. As explained in [5, Theorem 3.10], one can associate with X a subspace $A \subset \bigwedge^3 V_6$ which is Lagrangian for the $\det(V_6)$-valued symplectic form given by wedge product. Together with the pair $V_5 \subset V_6$ defined above, it forms a triple (V_6, V_5, A) called the *Lagrangian data set* of X.

The Lagrangian subspace $A \subset \bigwedge^3 V_6$ has no decomposable vectors (i.e., $\mathbf{P}(A) \cap \mathsf{Gr}(3, V_6) = \varnothing$) when $n \geq 3$ (see [5, Theorem 3.16]) and the vector space $A \cap \bigwedge^3 V_5$ has dimension $5 - n$ if X is ordinary and dimension $6 - n$ if X is special (see [5, Proposition 3.13]).

Conversely, given a Lagrangian data set (V_6, V_5, A) such that A has no decomposable vectors, we have $\ell := \dim(A \cap \bigwedge^3 V_5) \leq 3$ and there are

- an ordinary smooth GM variety $X_{\mathrm{ord}}(V_6, V_5, A)$ of dimension $5 - \ell$, and
- a special smooth GM variety $X_{\mathrm{spe}}(V_6, V_5, A)$ of dimension $6 - \ell$,

unique up to isomorphism, whose associated Lagrangian data set is (V_6, V_5, A).

Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$, one can construct interesting varieties that play an important role for the geometry of the associated GM varieties. Following O'Grady, one defines for all integers $\ell \geq 0$ closed subschemes

$$Y_A^{\geq \ell} = \Big\{ [U_1] \in \mathbf{P}(V_6) \ \Big|\ \dim\Big(A \cap \Big(U_1 \wedge \bigwedge^2 V_6\Big)\Big) \geq \ell \Big\} \subset \mathbf{P}(V_6),$$
$$Y_{A^\perp}^{\geq \ell} = \Big\{ [U_5] \in \mathbf{P}(V_6^\vee) \ \Big|\ \dim\Big(A \cap \bigwedge^3 U_5\Big) \geq \ell \Big\} \subset \mathbf{P}(V_6^\vee),$$

and sets

$$Y_A^\ell := Y_A^{\geq \ell} \smallsetminus Y_A^{\geq \ell+1} \qquad \text{and} \qquad Y_{A^\perp}^\ell := Y_{A^\perp}^{\geq \ell} \smallsetminus Y_{A^\perp}^{\geq \ell+1}.$$

Assume that A has no decomposable vectors. Then,

$$Y_A := Y_A^{\geq 1} \subset \mathbf{P}(V_6) \qquad \text{and} \qquad Y_{A^\perp} := Y_{A^\perp}^{\geq 1} \subset \mathbf{P}(V_6^\vee)$$

are normal integral sextic hypersurfaces, called *EPW sextics*; the singular locus of Y_A is the integral surface $Y_A^{\geq 2}$, the singular locus of $Y_A^{\geq 2}$ is the finite set $Y_A^{\geq 3}$

(empty for A general), $Y_A^{\geq 4} = \varnothing$ (see [5, Proposition B.2]), and analogous properties hold for $Y_{A^\perp}^{\geq \ell}$. One can rewrite the dimensions of the GM varieties X associated with a Lagrangian data set (V_6, V_5, A) as follows: if the Plücker point $\mathbf{p}_X$ is in $Y_{A^\perp}^{\ell}$, then we have

$$\dim\bigl(X_{\mathrm{ord}}(V_6, V_5, A)\bigr) = 5 - \ell \qquad \text{and} \qquad \dim\bigl(X_{\mathrm{spe}}(V_6, V_5, A)\bigr) = 6 - \ell.$$

Still under the assumption that A contains no decomposable vectors, O'Grady constructs in [25, Section 1.2] a canonical double cover

$$f_A \colon \widetilde{Y}_A \longrightarrow Y_A \tag{2}$$

branched over the integral surface $Y_A^{\geq 2}$. When the finite set $Y_A^{\geq 3}$ is empty, $\widetilde{Y}_A$ is a smooth hyper-Kähler 4-fold (see [22, Theorem 1.1(2)]).

The hypersurfaces Y_A and $Y_{A^\perp}$ are mutually projectively dual, and the duality is realized, inside the flag variety

$$\mathsf{Fl}(1, 5; V_6) := \{(U_1, U_5) \in \mathbf{P}(V_6) \times \mathbf{P}(V_6^\vee) \mid U_1 \subset U_5 \subset V_6\},$$

by the correspondence (see [5, Proposition B.3])

$$\widehat{Y}_A := \Bigl\{(U_1, U_5) \in \mathsf{Fl}(1, 5; V_6) \Bigm| A \cap \Bigl(U_1 \wedge \textstyle\bigwedge^2 U_5\Bigr) \neq 0\Bigr\}$$

with its birational projections

$$\begin{array}{ccc} & \widehat{Y}_A & \\ {\scriptstyle \mathrm{pr}_{Y,1}} \swarrow & & \searrow {\scriptstyle \mathrm{pr}_{Y,2}} \\ \mathbf{P}(V_6) \supset Y_A & & Y_{A^\perp} \subset \mathbf{P}(V_6^\vee). \end{array}$$

(These projections were denoted by p and q in [5, Proposition B.3]; we change the notation to $\mathrm{pr}_{Y,1}$ and $\mathrm{pr}_{Y,2}$ in this article, but we will switch back to p and q in Appendix B.)

We will need the following result.

LEMMA 2.1

Assume that A has no decomposable vectors. If $E \subset \widehat{Y}_A$ is the exceptional divisor of the map $\mathrm{pr}_{Y,1}$, then we have two inclusions

$$Y_{A^\perp}^{\geq 2} \subset \mathrm{pr}_{Y,2}(E) \subset (Y_A^{\geq 2})^\vee \cap Y_{A^\perp},$$

where $(Y_A^{\geq 2})^\vee \subset \mathbf{P}(V_6^\vee)$ is the projective dual of $Y_A^{\geq 2}$.

Proof

Since $Y_A^{\geq 2}$ is smooth at points of Y_A^2, its projective dual is

$$(Y_A^{\geq 2})^\vee = \overline{\bigcup_{v \in Y_A^2} \langle v \wedge \xi_1 \wedge \xi_1, v \wedge \xi_1 \wedge \xi_2, v \wedge \xi_2 \wedge \xi_2 \rangle},$$

where we write $A \cap (v \wedge \bigwedge^2 V_6) = \langle v \wedge \xi_1, v \wedge \xi_2 \rangle$ for some $\xi_1, \xi_2 \in \bigwedge^2 V_6$, and identify $\bigwedge^5 V_6$ with $V_6^\vee$. Indeed, a vector $v' \in V_6$ is tangent to Y_A^2 at v if one has

$$(v + tv') \wedge (\xi_i + t\xi_i') = a_i + ta_i' \pmod{t^2}$$

for some $\xi_i' \in \bigwedge^2 V_6$ and $a_i' \in A$, for $i \in \{1,2\}$. Since A is Lagrangian, this implies, for $i,j \in \{1,2\}$, that

$$0 = a_i \wedge a_j' = (v \wedge \xi_i) \wedge (v \wedge \xi_j' + v' \wedge \xi_j) = -v' \wedge (v \wedge \xi_i \wedge \xi_j).$$

This means that the embedded tangent space to Y_A^2 at v is contained in the orthogonal to the subspace of $V_6^\vee$ generated by $v \wedge \xi_i \wedge \xi_j$. Since the former, modulo v, is 2-dimensional and the latter is 3-dimensional, the tangent space coincides with this orthogonal and, hence, the above description of the dual variety.

On the other hand, by the argument in the proof of [5, Proposition B.3], one has

$$\mathrm{pr}_{Y,2}(E) = \bigcup_{v \in Y_A^{\geq 2}} \bigcup_{\xi} v \wedge \xi \wedge \xi,$$

where the second union is taken over all $v \wedge \xi \in A \cap (v \wedge \bigwedge^2 V_6)$. In particular, we obtain the inclusion $\mathrm{pr}_{Y,2}(E) \subset (Y_A^2)^\vee$.

For the second inclusion, since $Y_{A^\perp}^{\geq 2}$ is an integral surface (see [5, Theorem B.2]), it is enough to show that E intersects the general fiber C of the map $E' = \mathrm{pr}_{Y,2}^{-1}(Y_{A^\perp}^{\geq 2}) \to Y_{A^\perp}^{\geq 2}$. This fiber is mapped by $\mathrm{pr}_{Y,1}$ to a conic in Y_A (see [5, Proposition B.3]); hence, $H \cdot C = 2$, where H is the pullback of the hyperplane class of Y_A. On the other hand, if H' is the hyperplane class of $Y_{A^\perp}$, then $H' \cdot C = 0$. But E is linearly equivalent to $5H - H'$ (see [5, proof of Lemma B.5]); hence, $E \cdot C = 10$, so that E intersects C nontrivially. This finishes the proof of the lemma. □

Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$, one can also define the closed subschemes

$$Z_A^{\geq \ell} := \left\{ U_3 \subset V_6 \;\middle|\; \dim\left(A \cap \left(\bigwedge^2 U_3 \wedge V_6 \right) \right) \geq \ell \right\} \subset \mathsf{Gr}(3, V_6).$$

The complements $Z_A^\ell := Z_A^{\geq \ell} \smallsetminus Z_A^{\geq \ell+1}$ form a stratification of $\mathsf{Gr}(3,V_6)$. If A has no decomposable vectors, then $Z_A := Z_A^{\geq 1}$ is a normal integral hypersurface in $\mathsf{Gr}(3,V_6)$ cut out by a quartic hypersurface in $\mathbf{P}(\bigwedge^3 V_6)$. We call Z_A an *EPW quartic*. The singular locus of Z_A is then the integral variety $Z_A^{\geq 2}$ of dimension 6, the singular locus of $Z_A^{\geq 2}$ is the integral variety $Z_A^{\geq 3}$ of dimension 3, the singular locus of $Z_A^{\geq 3}$ is the finite set $Z_A^{\geq 4}$ (empty for A general), and $Z_A^{\geq 5} = \varnothing$ (see [12, Proposition 2.6] or [7, Theorem 5.6]).

Moreover, there is a canonical double cover $\widetilde{Z}_A^{\geq 2} \to Z_A^{\geq 2}$ branched over $Z_A^{\geq 3}$, and when $Z_A^{\geq 4}$ is empty, $\widetilde{Z}_A^{\geq 2}$ is a smooth hyper-Kähler 6-fold (see [12, Theorem 1.1]).

The hypersurfaces $Z_A \subset \mathsf{Gr}(3,V_6)$ and $Z_{A^\perp} \subset \mathsf{Gr}(3,V_6^\vee)$ coincide under the natural identification $\mathsf{Gr}(3,V_6) \simeq \mathsf{Gr}(3,V_6^\vee)$. They are related to the EPW sextics via the correspondence

$$\widehat{Z}_A := \left\{ (U_3, U_5) \in \mathsf{Fl}(3,5;V_6) \;\middle|\; A \cap \left(\bigwedge^2 U_3 \wedge U_5 \right) \neq 0 \right\}$$

with its projections

$$\begin{array}{ccc} & \widehat{Z}_A & \\ {}^{\mathrm{pr}_{Z,1}}\swarrow & & \searrow^{\mathrm{pr}_{Z,2}} \\ Z_A & & Y_{A^\perp} \end{array}$$

LEMMA 2.2

Assume that the Lagrangian A contains no decomposable vectors. The map $\mathrm{pr}_{Z,2}$ is dominant; over $Y^1_{A^\perp}$, it is smooth and its fibers are 3-dimensional quadrics. The map $\mathrm{pr}_{Z,1}$ is birational onto a divisor in Z_A containing $Z^{\ge 2}_A$.

Proof

Let $[U_5]$ be a point of $Y^1_{A^\perp}$, and let a be a generator of the 1-dimensional space $A \cap \bigwedge^3 U_5$. The 2-form on U_5 corresponding to $a \in \bigwedge^3 U_5$ via the isomorphism $\bigwedge^3 U_5 \simeq \bigwedge^2 U_5^\vee$ has rank 4 (because a is not decomposable). The fiber $\mathrm{pr}^{-1}_{Z,2}([U_5])$ parameterizes all 3-dimensional subspaces U_3 of U_5 which are isotropic for the 2-form a. Since a has rank 4, it is a smooth 3-dimensional quadric.

Analogously, let $[U_3]$ be a point of Z^1_A, and let a be a generator of the 1-dimensional space $A \cap (\bigwedge^2 U_3 \wedge V_6)$. Let $\bar{a}$ be the image of a in the quotient space $\bigwedge^2 U_3 \otimes (V_6/U_3) \simeq \mathrm{Hom}(U_3^\perp, \bigwedge^2 U_3)$. Over Z^1_A, this defines a morphism of rank-3 vector bundles with fibers $U_3^\perp$ and $\bigwedge^2 U_3$. Over its degeneracy locus (which is a divisor in Z_A), the projection $\mathrm{pr}_{Z,1}$ is an isomorphism (and $\mathrm{pr}^{-1}_{Z,1}([U_3])$ is the unique hyperplane $U_5 \subset V_6$ such that $U_5^\perp \subset U_3^\perp$ is the kernel of $\bar{a}$).

To prove $Z^{\ge 2}_A \subset \mathrm{pr}_{Z,1}(\widehat{Z}_A)$, note that if $\dim(A \cap (\bigwedge^2 U_3 \wedge V_6)) \ge 2$, then we have a pencil of maps $U_3^\perp \to \bigwedge^2 U_3$; some maps in this pencil are degenerate, and their kernels give points in the preimage of $[U_3]$. □

For any hyperplane $V_5 \subset V_6$, we set

$$\begin{aligned} Y^\ell_{A,V_5} &:= Y^\ell_A \times_{\mathbf{P}(V_6)} \mathbf{P}(V_5) && = Y^\ell_A \cap \mathbf{P}(V_5), \\ Z^\ell_{A,V_5} &:= Z^\ell_A \times_{\mathsf{Gr}(3,V_6)} \mathsf{Gr}(3,V_5) && = Z^\ell_A \cap \mathsf{Gr}(3,V_5), \\ \widetilde{Y}_{A,V_5} &:= \widetilde{Y}_A \times_{\mathbf{P}(V_6)} \mathbf{P}(V_5) && = f_A^{-1}(Y_{A,V_5}), \end{aligned}$$

and similarly for $Y^{\ge \ell}_{A,V_5}$ and others. These varieties will play an important role in the geometry of the associated GM varieties. We let

$$f_{A,V_5} \colon \widetilde{Y}_{A,V_5} \to Y_{A,V_5}$$

be the morphism induced by restriction of the double cover $f_A \colon \widetilde{Y}_A \to Y_A$.

We will need the following simple observation.

LEMMA 2.3

Let (V_6, V_5, A) be a Lagrangian data set with no decomposable vectors in A.

If $[U_3] \in Z^4_{A,V_5}$*, then* $(U_3, V_5) \in \widehat{Z}_A$*. In particular, if* $A \cap \bigwedge^3 V_5 = 0$*, then we have* $Z^{\geq 4}_{A,V_5} = \varnothing$*.*

Proof
Assume that $U_3 \subset V_6$ defines a point of Z^4_{A,V_5}. In other words, assume that $\dim(A \cap (\bigwedge^2 U_3 \wedge V_6)) \geq 4$ and $U_3 \subset V_5$. Since $\bigwedge^2 U_3 \wedge V_5$ has codimension 3 in $\bigwedge^2 U_3 \wedge V_6$, we have $A \cap (\bigwedge^2 U_3 \wedge V_5) \neq 0$. This means that $(U_3, V_5) \in \widehat{Z}_A$. Since $\bigwedge^2 U_3 \wedge V_5 \subset \bigwedge^3 V_5$, this contradicts $A \cap \bigwedge^3 V_5 = 0$. □

2.3. The quadric fibrations

In [5, Section 4], we defined two quadric fibrations associated with a smooth GM variety X of dimension n. The first quadric fibration is the map

$$\rho_1 \colon \mathbf{P}_X(\mathscr{U}_X) \to \mathbf{P}(V_5)$$

induced by the tautological embedding $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathscr{O}_X$. It is flat over the complement of the union $Y^{\geq n-1}_{A,V_5} \cup \Sigma_1(X)$, where $\Sigma_1(X)$ is the *kernel locus*

$$\Sigma_1(X) := \mathrm{pr}_{Y,1}\big(\mathrm{pr}^{-1}_{Y,2}(\mathbf{p}_X)\big) \subset Y_{A,V_5} \subset \mathbf{P}(V_5). \tag{3}$$

If A has no decomposable vectors, then the map $\mathrm{pr}_{Y,1} \colon \mathrm{pr}^{-1}_{Y,2}(\mathbf{p}_X) \to Y_{A,V_5}$ is a closed embedding (see [5, Proposition B.3]). So, if $\mathbf{p}_X \in Y^{\ell}_{A^\perp}$, then the variety $\Sigma_1(X)$ is isomorphic to $\mathbf{P}^{\ell-1}$ embedded via the second Veronese embedding.

The fibers of ρ_1 can be described as follows.

LEMMA 2.4 ([5, Proposition 4.5])
Let X *be a smooth GM variety of dimension* $n \geq 3$*, with associated Lagrangian data set* (V_6, V_5, A)*. For every* $v \in \mathbf{P}(V_5)$*, we have*

(a) *if* $v \in Y^{\ell}_{A,V_5} \smallsetminus \Sigma_1(X)$*, then the fiber* $\rho_1^{-1}(v)$ *is a quadric in* $\mathbf{P}^{n-2}$ *of corank* ℓ*;*

(b) *if* $v \in Y^{\ell}_{A,V_5} \cap \Sigma_1(X)$*, then the fiber* $\rho_1^{-1}(v)$ *is a quadric in* $\mathbf{P}^{n-1}$ *of corank* $\ell - 1$*.*

Since the corank of a quadric does not exceed the linear dimension of its span, we have $\ell \leq n-1$ for $v \notin \Sigma_1(X)$, and $\ell \leq n+1$ for $v \in \Sigma_1(X)$. This implies that $Y^3_{A,V_5} \subset \Sigma_1(X)$ for $n = 3$.

The second quadric fibration is the map

$$\rho_2 \colon \mathbf{P}_X(V_5/\mathscr{U}_X) \to \mathsf{Gr}(3, V_5)$$

induced by the natural embedding $(V_5/\mathscr{U}_X) \otimes \bigwedge^2 \mathscr{U}_X \hookrightarrow \bigwedge^3 V_5 \otimes \mathscr{O}_X$. It is flat over the complement of the union $Z^{\geq n-2}_{A,V_5} \cup \Sigma_2(X)$, where $\Sigma_2(X)$ is the *isotropic locus*

$$\Sigma_2(X) := \mathrm{pr}_{Z,1}\big(\mathrm{pr}^{-1}_{Z,2}(\mathbf{p}_X)\big) \subset Z_{A,V_5} \subset \mathsf{Gr}(3, V_5). \tag{4}$$

By Lemma 2.3, we have $Z^4_{A,V_5} \subset \Sigma_2(X)$.

In contrast with the case of the kernel locus, the map

$$\mathrm{pr}_{Z,1}\colon\ \mathrm{pr}_{Z,2}^{-1}(\mathbf{p}_X) \to Z_{A,V_5}$$

is no longer an embedding: its fiber over a point U_3 is the projective space $\mathbf{P}(A \cap (\bigwedge^2 U_3 \wedge V_5))$, and we set

$$\Sigma_2^{\geq k}(X) := \Big\{ U_3 \in \Sigma_2(X) \ \Big|\ \dim\Big(A \cap \Big(\bigwedge^2 U_3 \wedge V_5\Big)\Big) \geq k \Big\} \qquad \text{and}$$
$$\Sigma_2^{k}(X) := \Sigma_2^{\geq k}(X) \smallsetminus \Sigma_2^{\geq k+1}(X),$$

so that $\Sigma_2(X) = \Sigma_2^{\geq 1}(X)$. Note that $\Sigma_2^{\geq 3}(X)$ is empty if A has no decomposable vectors.

The fibers of ρ_2 can be described as follows.

LEMMA 2.5 ([5, Proposition 4.10])

Let X be a smooth GM variety of dimension $n \geq 3$, with associated Lagrangian data set (V_6, V_5, A). For every $U_3 \in \mathsf{Gr}(3, V_5)$, we have

(a) *if $U_3 \in Z_{A,V_5}^{\ell} \smallsetminus \Sigma_2(X)$, then the fiber $\rho_2^{-1}(U_3)$ is a quadric in $\mathbf{P}^{n-3}$ of corank ℓ;*

(b) *if $U_3 \in Z_{A,V_5}^{\ell} \cap \Sigma_2^1(X)$, then the fiber $\rho_2^{-1}(U_3)$ is a quadric in $\mathbf{P}^{n-2}$ of corank $\ell - 1$;*

(c) *if $U_3 \in Z_{A,V_5}^{\ell} \cap \Sigma_2^2(X)$, then the fiber $\rho_2^{-1}(U_3)$ is a quadric in $\mathbf{P}^{n-1}$ of corank $\ell - 2$.*

Lemmas 2.5 and 2.6 will be essential for the descriptions of the schemes of linear spaces contained in GM varieties.

LEMMA 2.6

Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors, let $V_5 \subset V_6$ be a hyperplane, and let $X = X_{\mathrm{ord}}(V_6, V_5, A)$ be the corresponding ordinary GM variety, of dimension $n := 5 - \dim(A \cap \bigwedge^3 V_5)$. If $n \geq 3$, then

(a) *$Y_{A,V_5}^{\geq 2}$ is a curve which is smooth if and only if $Y_{A,V_5}^{3} = \varnothing$ and the Plücker point $\mathbf{p}_X$ does not lie on the projective dual variety of $Y_A^{\geq 2}$;*

(b) *Y_{A,V_5} is a normal integral 3-fold and*

$$\mathrm{Sing}(Y_{A,V_5}) = Y_{A,V_5}^{\geq 2} \cup \Sigma_1(X), \qquad \mathrm{Sing}(\widetilde{Y}_{A,V_5}) = \mathrm{Sing}(Y_{A,V_5}^{\geq 2}) \cup f_{A,V_5}^{-1}\big(\Sigma_1(X)\big).$$

Proof

(a) The integral surface $Y_A^{\geq 2}$ is not contained in a hyperplane (see [5, Lemma B.6]); hence, its hyperplane section $Y_{A,V_5}^{\geq 2}$ is a curve. The statement about smoothness follows from the definition of projective duality.

(b) Since Y_A is an integral sextic hypersurface, we have $\dim(Y_{A,V_5}) = 3$. If a point $P \in Y_{A,V_5} \smallsetminus Y_{A,V_5}^{\geq 2}$ is singular, then the tangent space to Y_A at P coincides with $\mathbf{P}(V_5)$. Therefore, $(P, \mathbf{p}_X) \in \widehat{Y}_A$; hence, $P \in \Sigma_1(X) = \mathrm{pr}_{Y,1}(\mathrm{pr}_{Y,2}^{-1}(\mathbf{p}_X))$. On the other hand, all points of $Y_{A,V_5}^{\geq 2}$ are singular on Y_{A,V_5}, since $Y_A^{\geq 2} = \mathrm{Sing}(Y_A)$. This gives the required description of $\mathrm{Sing}(Y_{A,V_5})$.

Over Y^1_{A,V_5}, the map f_{A,V_5} is étale; hence, the singular locus of $\widetilde{Y}_{A,V_5}$ over Y^1_{A,V_5} is equal to $f^{-1}_{A,V_5}(\Sigma_1(X))$. On the other hand, one checks that, along the ramification locus $f^{-1}_{A,V_5}(Y^{\geq 2}_{A,V_5})$, the double sextic $\widetilde{Y}_{A,V_5}$ is smooth if and only if $Y^{\geq 2}_{A,V_5}$ is. This gives the required description of $\mathrm{Sing}(\widetilde{Y}_{A,V_5})$.

Finally, Y_{A,V_5} is normal and integral, because it is a hypersurface in $\mathbf{P}^4$ with 1-dimensional singular locus (indeed, $Y^{\geq 2}_{A,V_5}$ has dimension 1 by part (a) and $\dim(\Sigma_1(X)) = (5-n) - 1 = 4 - n \leq 1$ when $n \geq 3$ by (3) and the discussion after it). $\square$

3. Cohomology of smooth GM varieties

3.1. Hodge numbers

Recall that the Hodge diamond of $\mathsf{Gr}(2, V_5)$ is

$$\begin{array}{c}
1\\
0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 0\quad 2\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\quad 0\quad 0\\
0\quad 0\quad 0\quad 2\quad 0\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\quad 0\quad 0\\
0\quad 0\quad 2\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\\
1
\end{array}$$

The abelian group $H^\bullet(\mathsf{Gr}(2,V_5);\mathbf{Z})$ is free with basis the Schubert classes

$$\boldsymbol{\sigma}_{i,j} \in H^{2(i+j)}\big(\mathsf{Gr}(2,V_5),\mathbf{Z}\big), \quad 3 \geq i \geq j \geq 0.$$

We write $\boldsymbol{\sigma}_i$ for $\boldsymbol{\sigma}_{i,0}$; thus, $\boldsymbol{\sigma}_1$ is the hyperplane class, $\boldsymbol{\sigma}_{1,1} = c_2(\mathscr{U})$, and $\boldsymbol{\sigma}_i = c_i(V_5/\mathscr{U})$, where $\mathscr{U}$ is the tautological rank-2 subbundle and $V_5/\mathscr{U}$ is the tautological rank-3 quotient bundle.

We compute the Hodge numbers of smooth GM varieties.

PROPOSITION 3.1

The Hodge diamond of a smooth complex GM variety of dimension n is

$(n=1)$

$$\begin{array}{c}
1\\
6\quad 6\\
1
\end{array}$$

$(n=3)$

$$\begin{array}{c}
1\\
0\quad 0\\
0\quad 1\quad 0\\
0\quad 10\quad 10\quad 0\\
0\quad 1\quad 0\\
0\quad 0\\
1
\end{array}$$

$(n=5)$

$$\begin{array}{c}
1\\
0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 0\quad 2\quad 0\quad 0\\
0\quad 0\quad 10\quad 10\quad 0\quad 0\\
0\quad 0\quad 2\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\\
1
\end{array}$$

$(n=2)$

$$\begin{array}{c}
1\\
0\quad 0\\
1\quad 20\quad 1\\
0\quad 0\\
1
\end{array}$$

$(n=4)$

$$\begin{array}{c}
1\\
0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 1\quad 22\quad 1\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\\
1
\end{array}$$

$(n=6)$

$$\begin{array}{c}
1\\
0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 0\quad 2\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\quad 0\quad 0\\
0\quad 0\quad 1\quad 22\quad 1\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\quad 0\quad 0\\
0\quad 0\quad 2\quad 0\quad 0\\
0\quad 0\quad 0\quad 0\\
0\quad 1\quad 0\\
0\quad 0\\
1
\end{array}$$

Proof
When $n = 1$, the Hodge numbers are those of a curve of genus 6. When $n = 2$, the Hodge numbers are those of a K3 surface.

Assume that $3 \le n \le 5$. Since the Hodge numbers of smooth complex varieties are deformation invariant, we may assume that the GM variety X is ordinary. It is then a smooth dimensionally transverse intersection of (ample) hypersurfaces in $G := \mathsf{Gr}(2, V_5)$, and the Lefschetz hyperplane theorem (see Lemma 3.2) implies that the Hodge numbers of X of degree less than n are those of G. Moreover, $h^{n,0}(X) = 0$ because X is a Fano variety.

When $n = 3$, the missing Hodge number $h^{1,2}(X)$ was computed in [17]. When $n = 4$, the Hodge diamond was computed in [13, Lemma 4.1]. When $n = 6$, it was computed in [6, Corollary 4.5]. When $n = 5$, the missing Hodge numbers $h^{1,4}(X)$ and $h^{2,3}(X)$ were obtained by Nagel using a computer (see [19, Introduction]). We now present our own computation.

To compute $h^{1,4}(X)$, we assume that X is an ordinary 5-fold. Consider the exact sequences

$$0 \to \mathscr{O}_X(-2) \to \Omega^1_G|_X \to \Omega^1_X \to 0 \qquad \text{and} \qquad 0 \to \Omega^1_G(-2) \to \Omega^1_G \to \Omega^1_G|_X \to 0.$$

The sheaf $\Omega^1_G(-2)$ is acyclic (by Bott's theorem) and so is $\mathscr{O}_X(-2)$ (by Kodaira vanishing); hence, $h^i(X, \Omega^1_X) = h^i(X, \Omega^1_G|_X) = h^i(G, \Omega^1_G)$ and $h^{1,i}(X) = h^{1,i}(G)$. In particular, we obtain $h^{1,4}(X) = 0$.

To compute $h^{2,3}(X)$, we assume that X is a special 5-fold, that is, is a double covering of a smooth hyperplane section M'_X of G branched along a smooth GM 4-fold X'. Using this double covering, we compute Euler characteristics

$$\chi_{\mathrm{top}}(X) = 2\chi_{\mathrm{top}}(M'_X) - \chi_{\mathrm{top}}(X').$$

Since X' is a GM 4-fold, we have $\chi_{\mathrm{top}}(X') = 1 + 1 + 24 + 1 + 1 = 28$. On the other hand, the inclusion $M'_X \subset G$ induces isomorphisms $H^k(G; \mathbf{Z}) \simeq H^k(M'_X; \mathbf{Z})$ for all $k \in \{0, \dots, 5\}$. In particular, $\chi_{\mathrm{top}}(M'_X) = 8$; hence, $\chi_{\mathrm{top}}(X) = -12$. Since $\chi_{\mathrm{top}}(X) = 1 + 1 + 2 - 2h^{2,3}(X) + 2 + 1 + 1$, we obtain $h^{2,3}(X) = 10$. This finishes the proof of the proposition. □

3.2. Integral cohomology

We now prove that the integral cohomology groups of smooth GM varieties are torsion-free. We start with a classical lemma.

LEMMA 3.2 (Lefschetz)
Let X be a dimensionally transverse intersection of dimension n of ample hypersurfaces in a smooth projective variety M.

(a) *The induced map $H^k(M; \mathbf{Z}) \xrightarrow{\sim} H^k(X; \mathbf{Z})$ is bijective for $k < n$ and injective for $k = n$.*

(b) *The induced map $H_k(X; \mathbf{Z}) \xrightarrow{\sim} H_k(M; \mathbf{Z})$ is bijective for $k < n$ and surjective for $k = n$.*

(c) *If X is moreover smooth and $H^\bullet(M;\mathbf{Z})$ is torsion-free, then so is $H^\bullet(X;\mathbf{Z})$.*

Proof
Parts (a) and (b) are the Lefschetz hyperplane theorem and follow from the fact that $M \smallsetminus X$ is the union of $\dim(M) - n$ smooth affine open subsets (see [9, Chapter 5, Theorem (2.6)]).

For (c), since X is smooth, the Poincaré duality isomorphisms

$$H^k(X;\mathbf{Z}) \simeq H_{2n-k}(X;\mathbf{Z})$$

and (a) together with (b) imply that the integral homology and cohomology groups of X are torsion-free in all degrees except perhaps n. By the universal coefficient theorem, the torsion subgroup of $H^n(X;\mathbf{Z})$ is isomorphic to the torsion subgroup of $H_{n-1}(X;\mathbf{Z})$, which is zero by (b); hence, all integral cohomology groups of X are torsion-free. □

A similar result holds for cyclic covers. (This is the main theorem of [3]; see also the remarks at the very end of the article.)

LEMMA 3.3
Let $\gamma\colon X \to M$ be a cyclic cover between smooth projective varieties of dimension n whose branch locus is a smooth ample divisor on M.

(a) *The induced map $\gamma^*\colon H^k(M;\mathbf{Z}) \xrightarrow{\sim} H^k(X;\mathbf{Z})$ is bijective for $k<n$ and injective for $k=n$.*

(b) *The induced map $\gamma_*\colon H_k(X;\mathbf{Z}) \xrightarrow{\sim} H_k(M;\mathbf{Z})$ is bijective for $k<n$ and surjective for $k=n$.*

(c) *If $H^\bullet(M;\mathbf{Z})$ is torsion-free, then so is $H^\bullet(X;\mathbf{Z})$.*

We now describe the integral cohomology groups of smooth GM varieties.

PROPOSITION 3.4
Let X be a smooth GM variety of dimension n.

(a) *The group $H^\bullet(X;\mathbf{Z})$ is torsion-free.*

(b) *The map $\gamma_X^{*,k}\colon H^k(\mathsf{Gr}(2,V_5);\mathbf{Z}) \to H^k(X;\mathbf{Z})$ is bijective for $k<n$ and injective for $k=n$.*

Proof
When X is ordinary, it is a dimensionally transverse intersection of (ample) hypersurfaces in $\mathsf{Gr}(2,V_5)$; hence, Lemma 3.2 implies both parts (a) and (b) of the proposition.

When X is special, its Gushel map factors as $\gamma_X\colon X \xrightarrow{\gamma} M'_X \hookrightarrow \mathsf{Gr}(2,V_5)$, where γ is a double cover branched along an ample divisor, and M'_X is a dimensionally transverse intersection of (ample) hypersurfaces in $\mathsf{Gr}(2,V_5)$. Both parts (a) and (b) are then consequences of Lemma 3.2 (applied to $M'_X \subset \mathsf{Gr}(2, V_5)$) and Lemma 3.3 (applied to the double cover γ). ☐

COROLLARY 3.5

Let X be a smooth GM variety of dimension n. If $n \geq 3$, then the degree of any hypersurface in X is divisible by 10. If $n \geq 5$, the degree of any subvariety of codimension 2 in X is even.

Proof

We use Proposition 3.4(b). Let $Y \subset X$ be a subvariety of codimension c. If $c = 1$, then the class of Y in $H^2(X;\mathbf{Z})$ is a multiple of the class $\gamma_X^*\boldsymbol{\sigma}_1$, which has degree 10.

If $c = 2$ (and $n \geq 5$), then the class of Y in $H^4(X;\mathbf{Z})$ is an integral combination of $\gamma_X^*\boldsymbol{\sigma}_2$, which has degree 6, and $\gamma_X^*\boldsymbol{\sigma}_{1,1}$, which has degree 4. The degree of Y is therefore even. ☐

We will need the following computation, which was already used in [4].

LEMMA 3.6

Let X be a smooth ordinary GM 4-fold, and let $Q_0 \subset X$ be its σ-quadric, that is, the intersection of X with the 3-space $\Pi := \mathbf{P}(v_0 \wedge V_5) \subset M_X$, where $v_0 \in \mathbf{P}(V_5)$ is the unique point in the kernel locus $\Sigma_1(X)$ defined by (3). Then

$$[Q_0] = \gamma_X^*(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_{1,1}) \in H^4(X;\mathbf{Z}).$$

Proof

Let γ_{M_X} be the inclusion $M_X \hookrightarrow \mathsf{Gr}(2, V_5)$. By the Lefschetz theorem (Lemma 3.2), the map $\gamma_{M_X}^* : H^4(\mathsf{Gr}(2, V_5);\mathbf{Z}) \to H^4(M_X;\mathbf{Z})$ is an isomorphism. Therefore, there exist integers a and b such that $[\Pi] = \gamma_{M_X}^*(a\boldsymbol{\sigma}_2 + b\boldsymbol{\sigma}_{1,1})$; hence, $[Q_0] = \gamma_X^*(a\boldsymbol{\sigma}_2 + b\boldsymbol{\sigma}_{1,1})$. Since the class of Π in $H^6(\mathsf{Gr}(2, V_5);\mathbf{Z})$ is $\boldsymbol{\sigma}_3$, Gysin's formula and Schubert calculus give

$$\begin{aligned}\boldsymbol{\sigma}_3 = \gamma_{M_X *}\big([\Pi]\big) &= \gamma_{M_X *}\gamma_{M_X}^*(a\boldsymbol{\sigma}_2 + b\boldsymbol{\sigma}_{1,1}) = (a\boldsymbol{\sigma}_2 + b\boldsymbol{\sigma}_{1,1}) \cdot \boldsymbol{\sigma}_1 \\ &= a(\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_{2,1}) + b\boldsymbol{\sigma}_{2,1}\end{aligned}$$

in $H^6(\mathsf{Gr}(2, V_5);\mathbf{Z})$. This implies $a = 1$ and $a + b = 0$, hence the lemma. ☐

The following lemma is also useful; we keep the notation of Lemma 3.6.

LEMMA 3.7

Let X be a smooth ordinary GM 4-fold. The restriction of the bundle $\mathscr{U}_X$ to the quadric Q_0 splits as $\mathscr{O}_{Q_0} \oplus \mathscr{O}_{Q_0}(-1)$.

Proof

Denote as before by $\mathscr{U}$ the tautological rank-2 vector bundle on the Grassmannian $\mathsf{Gr}(2, V_5)$. Since $\mathscr{U}_X = \mathscr{U}|_X$ and $Q_0 \subset \Pi \subset \mathsf{Gr}(2, V_5)$, it is enough to show $\mathscr{U}|_\Pi \simeq \mathscr{O}_\Pi \oplus \mathscr{O}_\Pi(-1)$. Recall that $\Pi = \mathbf{P}(v_0 \wedge V_5)$ parameterizes all 2-dimensional subspaces in V_5 that contain v_0. Consequently, we have an injection of vector bundles $\mathscr{O}_\Pi \hookrightarrow \mathscr{U}|_\Pi$ given by the vector v_0. Its cokernel is a line bundle isomorphic to $\det(\mathscr{U}|_\Pi) \simeq \mathscr{O}_\Pi(-1)$; hence, we have an exact sequence

$$0 \to \mathscr{O}_\Pi \to \mathscr{U}|_\Pi \to \mathscr{O}_\Pi(-1) \to 0.$$

It remains to note that $\operatorname{Ext}^1(\mathscr{O}_\Pi(-1), \mathscr{O}_\Pi) = H^1(\Pi, \mathscr{O}_\Pi(1)) = 0$, since $\Pi \simeq \mathbf{P}^3$. □

3.3. Middle cohomology lattices of smooth GM varieties of dimension 4 or 6

Let X be a smooth GM variety of even dimension n with Gushel map $\gamma_X \colon X \to \mathsf{Gr}(2, V_5)$. The abelian group $H^n(X;\mathbf{Z})$ is torsion-free (see Proposition 3.4) and, endowed with the intersection form, it is, by Poincaré duality, a unimodular lattice. We set $h := \gamma_X^* \boldsymbol{\sigma}_1 \in H^2(X;\mathbf{Z})$ and

$$\begin{aligned} H^n(X;\mathbf{Z})_0 &:= \{x \in H^n(X;\mathbf{Z}) \mid x \cdot h = 0\}, \\ H^n(X;\mathbf{Z})_{00} &:= \{x \in H^n(X;\mathbf{Z}) \mid x \cdot \gamma_X^*(H^n(\mathsf{Gr}(2,V_5);\mathbf{Z})) = 0\}. \end{aligned} \tag{5}$$

These sublattices of $H^n(X;\mathbf{Z})$ are called the *primitive* and the *vanishing* lattices of X.

LEMMA 3.8

For every n, we have an injection $H^n(X;\mathbf{Z})_{00} \subset H^n(X;\mathbf{Z})_0$, and for $n = 2$ and $n = 6$, we have an equality $H^n(X;\mathbf{Z})_{00} = H^n(X;\mathbf{Z})_0$.

Proof

Since h is pulled back from $\mathsf{Gr}(2, V_5)$, we have $(x \cdot h) \cdot \gamma_X^*(H^{n-2}(\mathsf{Gr}(2,V_5);\mathbf{Z})) = 0$ for every $x \in H^n(X;\mathbf{Z})_{00}$. By Lemma 3.2, the map

$$\gamma_X^* \colon H^{n-2}(\mathsf{Gr}(2,V_5);\mathbf{Z}) \to H^{n-2}(X;\mathbf{Z})$$

is an isomorphism; hence, $(x \cdot h) \cdot H^{n-2}(X;\mathbf{Z}) = 0$. We conclude that $x \cdot h = 0$ by Poincaré duality.

Since $H^2(\mathsf{Gr}(2,V_5);\mathbf{Z}) = \mathbf{Z}\boldsymbol{\sigma}_1$, the definitions of $H^2(X;\mathbf{Z})_0$ and $H^2(X;\mathbf{Z})_{00}$ are the same for $n = 2$. Furthermore, the product

$$H^4(\mathsf{Gr}(2,V_5);\mathbf{Z}) \xrightarrow{\cdot \boldsymbol{\sigma}_1} H^6(\mathsf{Gr}(2,V_5);\mathbf{Z})$$

is an isomorphism by Schubert calculus; hence for $n = 6$, the definitions are equivalent. □

Given a Lagrangian subspace $A \subset \bigwedge^3 V_6$ with no decomposable vectors and such that $Y_A^{\geq 3} = \varnothing$, the 4-fold $\widetilde{Y}_A$ introduced in (2) is a hyper-Kähler manifold which is a deformation of the symmetric square of a K3 surface (see [22, Theorem 1.1(2)]). In particular, the group $H^\bullet(\widetilde{Y}_A;\mathbf{Z})$ is torsion-free (see [18, Theorem 1]).

We denote by $\tilde{h} \in H^2(\widetilde{Y}_A;\mathbf{Z})$ the pullback by f_A of the hyperplane class on $Y_A \subset \mathbf{P}(V_6)$ and define the *primitive* cohomology

$$H^2(\widetilde{Y}_A;\mathbf{Z})_0 := \{y \in H^2(\widetilde{Y}_A;\mathbf{Z}) \mid y \cdot \tilde{h}^3 = 0\}. \tag{6}$$

We consider $H^n(X,\mathbf{Z})_{00}$ and $H^2(\widetilde{Y}_A,\mathbf{Z})_0$ as polarized Hodge structures via the intersection pairing on the first and the Beauville–Bogomolov quadratic form q_B on the second. Recall that q_B can be defined by (see [1, Théorèm 5(c)])

$$\forall y \in H^2(\widetilde{Y}_A;\mathbf{Z})_0 \quad q_B(y) = \frac{1}{2} y^2 \cdot \tilde{h}^2. \tag{7}$$

This form makes $H^2(\widetilde{Y}_A;\mathbf{Z})_0$ into a lattice of rank 22.

Given a lattice L and a nonzero integer m, we denote by $L(m)$ the lattice L with the bilinear form multiplied by m. The *discriminant* of L is the finite abelian group

$$D(L) := L^\vee / L.$$

As usual, we denote by

- I_1 the odd lattice $\mathbf{Z}$ with intersection form (1),
- $I_{r,s}$ the odd lattice $I_1^{\oplus r} \oplus I_1(-1)^{\oplus s}$,
- U the even hyperbolic lattice $\mathbf{Z}^2$ with intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$,
- E_8 the unique positive definite, even, unimodular lattice of rank 8.

The following three lattices are important in this article:

$$\Gamma_4 := I_{22,2}, \qquad \Gamma_6 := E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}, \qquad \Lambda := E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{2,0}(2). \tag{8}$$

PROPOSITION 3.9

Let X be a smooth GM variety of dimension $n = 4$ or 6. There are isomorphisms of lattices $H^n(X;\mathbf{Z}) \simeq \Gamma_n$ and $H^n(X;\mathbf{Z})_{00} \simeq \Lambda((-1)^{n/2})$.

Proof

When $n = 4$, the lattices $H^4(X;\mathbf{Z})$ and $H^4(X;\mathbf{Z})_{00}$ are described in [4, Proposition 5.1] (although the proof that these groups are torsion-free is missing). When $n = 6$, the class $c_1(X) = 4\gamma_X^* \boldsymbol{\sigma}_1$ is divisible by 2; hence, the Stiefel–Whitney class $w_2(X)$, which is its image in $H^2(X;\mathbf{Z}/2\mathbf{Z})$, vanishes. Since $w_1(X) = 0$ (as for any complex compact manifold) and we are in dimension 6, the (unimodular) lattice $H^6(X;\mathbf{Z})$ is *even* (see [11, p. 115]). Since its signature is $(4,20)$ by Proposition 3.1, it is therefore isomorphic to Γ_6.

The intersection form on the sublattice $\gamma_X^*(H^6(\mathsf{Gr}(2,V_5);\mathbf{Z})) \subset H^6(X;\mathbf{Z})$ has matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ in the Schubert basis $(\gamma_X^*\boldsymbol{\sigma}_{2,1}, \gamma_X^*\boldsymbol{\sigma}_3)$. This sublattice is moreover primitive: if not, then its saturation is unimodular, even, and positive definite of rank 2, which is absurd. By [20, Proposition 1.6.1], the discriminant group of its orthogonal $H^6(X;\mathbf{Z})_{00}$ is therefore isomorphic to the discriminant group of $\gamma_X^*(H^6(\mathsf{Gr}(2,V_5);\mathbf{Z}))$, which is $(\mathbf{Z}/2\mathbf{Z})^2$. The lattice $H^6(X;\mathbf{Z})_{00}$ is moreover even and has signature $(2,20)$. As noted in [4, Section 5.1], there is only one lattice with these characteristics, to wit $\Lambda(-1)$. ☐

As a lattice, $H^2(\widetilde{Y}_A,\mathbf{Z})_0$ is also isomorphic to $\Lambda(-1)$ (see [26, (4.1.3)]). In Section 5, we will show that the polarized Hodge structures on $H^n(X,\mathbf{Z})_{00}$

and $H^2(\widetilde{Y}_A, \mathbf{Z})_0$ are isomorphic (up to a twist). The isomorphism will be given by a correspondence constructed in the next section.

4. Linear spaces on Gushel–Mukai varieties

4.1. Linear spaces and their types

Let X be a smooth GM variety with its canonical embedding $X \subset \mathbf{P}(W)$. We let $F_k(X)$ be the Hilbert scheme which parameterizes linearly embedded $\mathbf{P}^k$ in X, that is, the closed subscheme of $\mathsf{Gr}(k+1, W)$ of linear subspaces $W_{k+1} \subset W$ such that $\mathbf{P}(W_{k+1}) \subset X$.

The composition of the Gushel map $\gamma_X : X \to \mathsf{Gr}(2, V_5)$ with the Plücker embedding of the Grassmannian $\mathsf{Gr}(2, V_5) \subset \mathbf{P}(\bigwedge^2 V_5)$ is induced by the linear projection $W \subset \mathbf{C} \oplus \bigwedge^2 V_5 \to \bigwedge^2 V_5$ from the vertex ν of the cone $\mathsf{CGr}(2, V_5)$. Since $\nu \notin X$, the Gushel map embeds $\mathbf{P}(W_{k+1})$ linearly into $\mathsf{Gr}(2, V_5)$.

We recall the description of linear subspaces contained in $\mathsf{Gr}(2, V_5)$. Any such subspace sits in a maximal linear subspace, and there are two types of those. First, for every 1-dimensional subspace $U_1 \subset V_5$, there is a projective 3-space

$$\mathbf{P}(V_5/U_1) \simeq \mathbf{P}(U_1 \wedge V_5) \subset \mathsf{Gr}(2, V_5).$$

Second, for every 3-dimensional vector subspace $U_3 \subset V_5$, there is a projective plane

$$\mathbf{P}\left(\bigwedge^2 U_3\right) \simeq \mathsf{Gr}(2, U_3) \subset \mathsf{Gr}(2, V_5).$$

We will say that a linear subspace $P \subset \mathsf{Gr}(2, V_5)$ is

- a *σ-space* if it is contained in $\mathbf{P}(U_1 \wedge V_5)$ for some $U_1 \subset V_5$;
- a *τ-space* if it is contained in $\mathbf{P}(\bigwedge^2 U_3)$ for some $U_3 \subset V_5$;
- a *mixed space* if it is both a σ- and a τ-space.

In $\mathsf{Gr}(2, V_5)$, there are no projective 4-spaces and every projective 3-space is a σ-space. For any distinct $U_1', U_1'' \subset V_5$, the intersection $\mathbf{P}(U_1' \wedge V_5) \cap \mathbf{P}(U_1'' \wedge V_5)$ is the point $[U_1' \wedge U_1'']$. Hence, for every projective 3-space $P \subset \mathsf{Gr}(2, V_5)$, there is a unique $U_1 \subset V_5$ such that $P = \mathbf{P}(U_1 \wedge V_5)$. This defines a map

$$\sigma : F_3(X) \to \mathbf{P}(V_5).$$

If $U_1 \not\subset U_3$, then we have $\mathbf{P}(U_1 \wedge V_5) \cap \mathbf{P}(\bigwedge^2 U_3) = \varnothing$. If instead $U_1 \subset U_3$, then the intersection $\mathbf{P}(U_1 \wedge V_5) \cap \mathbf{P}(\bigwedge^2 U_3) = \mathbf{P}(U_1 \wedge U_3)$ is a line. Therefore, projective planes in $\mathsf{Gr}(2, V_5)$ are never of mixed type, and we have a decomposition into connected components

$$F_2(X) = F_2^\sigma(X) \sqcup F_2^\tau(X),$$

where $F_2^\sigma(X)$ is the subscheme of σ-planes and $F_2^\tau(X)$ is the subscheme of τ-planes. Again, there is a map

$$\sigma : F_2^\sigma(X) \to \mathbf{P}(V_5)$$

taking a σ-plane P to the unique $U_1 \subset V_5$ such that $P \subset \mathbf{P}(U_1 \wedge V_5)$.

Analogously, for any distinct subspaces $U_3', U_3'' \subset V_5$, the intersection $\mathsf{Gr}(2,U_3') \cap \mathsf{Gr}(2,U_3'') = \mathsf{Gr}(2,U_3' \cap U_3'')$ is either empty (if $\dim(U_3' \cap U_3'') = 1$) or a point (if $\dim(U_3' \cap U_3'') = 2$). Therefore, for any τ-plane $P \subset \mathsf{Gr}(2,V_5)$, there is a unique subspace $U_3 \subset V_5$ such that $P = \mathbf{P}(\bigwedge^2 U_3)$. This defines a map

$$\tau\colon F_2^\tau(X) \to \mathsf{Gr}(3,V_5).$$

Finally, any line on $\mathsf{Gr}(2,V_5)$ is a mixed space, and there are maps

$$\sigma\colon F_1(X) \to \mathbf{P}(V_5) \qquad \text{and} \qquad \tau\colon F_1(X) \to \mathsf{Gr}(3,V_5).$$

The following proposition is crucial for our study of the schemes $F_k(X)$. It describes $F_k(X)$ in terms of the relative Hilbert schemes $\mathrm{Hilb}^{\mathbf{P}^k}$ which parameterize linearly embedded $\mathbf{P}^k$ in the fibers of the first and second quadratic fibrations (defined in Section 2.3).

PROPOSITION 4.1

Let X be a smooth GM variety of dimension $n \geq 3$, with associated Lagrangian data set (V_6, V_5, A). The maps

$$\sigma\colon F_1(X) \to \mathbf{P}(V_5), \qquad \sigma\colon F_2^\sigma(X) \to \mathbf{P}(V_5), \qquad \sigma\colon F_3(X) \to \mathbf{P}(V_5)$$

lift to isomorphisms with the following relative Hilbert schemes for the first quadric fibration

$$F_1(X) \simeq \mathrm{Hilb}^{\mathbf{P}^1}\big(\mathbf{P}_X(\mathscr{U}_X)/\mathbf{P}(V_5)\big),$$
$$F_2^\sigma(X) \simeq \mathrm{Hilb}^{\mathbf{P}^2}\big(\mathbf{P}_X(\mathscr{U}_X)/\mathbf{P}(V_5)\big),$$
$$F_3(X) \simeq \mathrm{Hilb}^{\mathbf{P}^3}\big(\mathbf{P}_X(\mathscr{U}_X)/\mathbf{P}(V_5)\big).$$

Analogously, the maps $\tau\colon F_1(X) \to \mathsf{Gr}(3,V_5)$ and $\tau\colon F_2^\tau(X) \to \mathsf{Gr}(3,V_5)$ lift to isomorphisms with the following relative Hilbert schemes for the second quadric fibration

$$F_1(X) \simeq \mathrm{Hilb}^{\mathbf{P}^1}\big(\mathbf{P}_X(V_5/\mathscr{U}_X)/\,\mathsf{Gr}(3,V_5)\big),$$
$$F_2^\tau(X) \simeq \mathrm{Hilb}^{\mathbf{P}^2}\big(\mathbf{P}_X(V_5/\mathscr{U}_X)/\,\mathsf{Gr}(3,V_5)\big).$$

Proof

Let $\mathscr{L}_k^\sigma(X) \subset X \times F_k^\sigma(X)$ be the universal family of k-dimensional σ-spaces. The map $\sigma\colon F_k^\sigma(X) \to \mathbf{P}(V_5)$ induces a map $\mathscr{L}_k^\sigma(X) \to X \times \mathbf{P}(V_5)$ that takes a pair (x,P), where $P \subset X$ is a σ-space of dimension k and $x \in P$ is a point, to $(x,\sigma(P))$. By the definition of a σ-space, if $\sigma(P) = U_1 \subset V_5$, then the space P parameterizes 2-dimensional subspaces $U_2 \subset V_5$ such that $U_1 \subset U_2$. Therefore, U_1 is contained in the 2-space corresponding to the point x. In other words, $(x,\sigma(P)) \in \mathbf{P}_X(\mathscr{U}_X)$. This means that we have a commutative diagram

$$
\begin{array}{ccccc}
X & \xleftarrow{\;q\;} & \mathscr{L}_k^\sigma(X) & \xrightarrow{\;p\;} & F_k^\sigma(X) \\
\Big\| & & \downarrow & & \big\downarrow \sigma \\
X & \xleftarrow{\;\pi\;} & \mathbf{P}_X(\mathscr{U}_X) & \xrightarrow{\;\rho_1\;} & \mathbf{P}(V_5)
\end{array} \tag{9}
$$

Thus, the fibers of $\mathscr{L}_k^\sigma(X)$ over $F_k^\sigma(X)$ embed into the fibers of ρ_1, and this embedding is linear on the fibers. This means that the map σ lifts to a map $F_k^\sigma(X) \to \mathrm{Hilb}^{\mathbf{P}^k}(\mathbf{P}_X(\mathscr{U}_X)/\mathbf{P}(V_5))$.

Similarly, the projection via the morphism $\pi : \mathbf{P}_X(\mathscr{U}_X) \to X$ of any projective space $\mathbf{P}^k$ contained in the fiber of ρ_1 is a $\mathbf{P}^k$ contained in X. This defines a map $\mathrm{Hilb}^{\mathbf{P}^k}(\mathbf{P}_X(\mathscr{U}_X)/\mathbf{P}(V_5)) \to F_k^\sigma(X)$.

It is straightforward to see that the maps we constructed are mutually inverse and thus give the isomorphisms of the first part of the proposition. The second part is proved analogously. □

In the next sections, we use this proposition and Lemmas 2.4 and 2.5 to describe the Hilbert schemes $F_k(X)$.

4.2. Projective 3-spaces on GM varieties

As we already mentioned, smooth GM varieties contain no linear spaces of dimension 4 and higher. The situation with projective 3-spaces is also quite simple.

THEOREM 4.2

Let X be a smooth GM variety of dimension n, with associated Lagrangian data set (V_6, V_5, A). If $n \le 5$, then we have $F_3(X) = \varnothing$. If $n = 6$, then there is an étale double covering $F_3(X) \to Y^3_{A,V_5}$; in particular, $F_3(X)$ is finite and is empty for X general.

Proof

By Proposition 4.1, $F_3(X)$ is the Hilbert scheme of projective 3-spaces in the fibers of the first quadric fibration $\rho_1 \colon \mathbf{P}_X(\mathscr{U}_X) \to \mathbf{P}(V_5)$. On the other hand, by Lemma 2.4, the fiber $Q_v = \rho_1^{-1}(v)$ over a point $v \in \mathbf{P}(V_5)$ is either a quadric in $\mathbf{P}^{n-2}$ if $v \notin \Sigma_1(X)$ or a quadric in $\mathbf{P}^{n-1}$ if $v \in \Sigma_1(X)$. Such a quadric contains a $\mathbf{P}^3$ only in the following cases:

- $n = 6$, $v \notin \Sigma_1(X)$, and $Q_v \subset \mathbf{P}^4$ is a quadric of corank at least 3;
- $n = 6$, $v \in \Sigma_1(X)$, and $Q_v \subset \mathbf{P}^5$ is a quadric of corank at least 2;
- $n = 5$, $v \notin \Sigma_1(X)$, and $Q_v \subset \mathbf{P}^3$ is a quadric of corank at least 4;
- $n = 5$, $v \in \Sigma_1(X)$, and $Q_v \subset \mathbf{P}^4$ is a quadric of corank at least 3;
- $n = 4$, $v \in \Sigma_1(X)$, and $Q_v \subset \mathbf{P}^3$ is a quadric of corank at least 4.

The last three cases do not occur by Lemma 2.4, because $Y_A^4 = \varnothing$ (we could also invoke Corollary 3.5), and neither does the second case, because $\Sigma_1(X) = \varnothing$ for $n = 6$.

In the first case, we have $v \in Y^{\geq 3}_{A,V_5}$. Since $Y^4_A = \varnothing$, the quadric Q_v has rank 2 and, hence, is a union of two distinct 3-spaces, and the Hilbert scheme of 3-spaces in Q_v consists of two reduced points and, hence, is smooth. Therefore, the map $F_3(X) \to Y^3_{A,V_5}$ is an étale double cover. The finiteness of $F_3(X)$ and its emptiness for general X follow from the same properties of Y^3_A. $\square$

4.3. Planes on GM varieties

Similar arguments provide descriptions of the schemes of planes. We start with σ-planes and consider the map $\sigma\colon F^\sigma_2(X) \to \mathbf{P}(V_5)$.

THEOREM 4.3

Let X be a smooth GM variety of dimension n, with associated Lagrangian data set (V_6, V_5, A).

(a) *If $n = 6$ and $Y^3_{A,V_5} = \varnothing$, then the map σ factors as*

$$F^\sigma_2(X) \xrightarrow{\tilde{\sigma}} \widetilde{Y}_{A,V_5} \xrightarrow{f_{A,V_5}} Y_{A,V_5} \hookrightarrow \mathbf{P}(V_5),$$

where $\tilde{\sigma}$ is a $\mathbf{P}^1$-bundle. If $n = 6$ and $Y^3_{A,V_5} \neq \varnothing$, then the scheme $F^\sigma_2(X)$ has one component isomorphic to a generically $\mathbf{P}^1$-fibration over $\widetilde{Y}_{A,V_5}$ and, for each point of Y^3_{A,V_5}, one pair of irreducible components isomorphic to $\mathbf{P}^3$. In particular, $F^\sigma_2(X)$ is smooth if and only if the Plücker point $\mathbf{p}_X$ lies away from the projective dual $(Y^{\geq 2}_A)^\vee \subset \mathbf{P}(V^\vee_6)$ and $Y^3_{A,V_5} = \varnothing$.

(b) *If $n = 5$ and X is ordinary, or special with $\mathbf{p}_X \notin \mathrm{pr}_{Y,2}(E)$, then the map σ factors as*

$$F^\sigma_2(X) \xrightarrow{\sim} \widetilde{Y}^{\geq 2}_{A,V_5} \longrightarrow Y^{\geq 2}_{A,V_5} \hookrightarrow \mathbf{P}(V_5),$$

where $\widetilde{Y}^{\geq 2}_{A,V_5} \to Y^{\geq 2}_{A,V_5}$ is a double covering of the curve $Y^{\geq 2}_{A,V_5}$ branched along Y^3_{A,V_5}. If $n = 5$ and X is special with $\mathbf{p}_X \in \mathrm{pr}_{Y,2}(E)$, then the scheme $F^\sigma_2(X)$ is the union of a double cover $\widetilde{Y}^{\geq 2}_{A,V_5}$ and one double component or two reduced components (depending on whether the kernel point $\Sigma_1(X)$ is in Y^3_{A,V_5} or in Y^2_{A,V_5}) isomorphic to $\mathbf{P}^1$ and contracted by the map σ onto $\Sigma_1(X)$.

(c) *If $n = 4$, then the map σ factors as*

$$F^\sigma_2(X) \xrightarrow{\tilde{\sigma}} Y^3_{A,V_5} \hookrightarrow \mathbf{P}(V_5),$$

where the morphism $\tilde{\sigma}$ is an isomorphism over $Y^3_{A,V_5} \smallsetminus \Sigma_1(X)$ and a double cover over $Y^3_{A,V_5} \cap \Sigma_1(X)$. In particular, $F^\sigma_2(X)$ is finite and is empty if and only if $Y^3_{A,V_5} = \varnothing$.

(d) *If $n \leq 3$, then we have $F^\sigma_2(X) = \varnothing$.*

REMARK 4.4

The double cover $\widetilde{Y}^{\geq 2}_{A,V_5} \to Y^{\geq 2}_{A,V_5}$ appearing in part (b) of the theorem is described in Proposition A.2. We expect it to be the base change to $\mathbf{P}(V_5)$ of a natural double cover of $Y^{\geq 2}_A$ branched along Y^3_A and analogous to O'Grady's double cover $f_A : \widetilde{Y}_A \to Y_A$.

Proof of Theorem 4.3
In each case, by Proposition 4.1, the scheme $F_2^\sigma(X)$ is isomorphic to the relative Hilbert scheme of planes in the fibers of the first quadric fibration $\rho_1\colon \mathbf{P}_X(\mathscr{U}_X) \to \mathbf{P}(V_5)$. The rest follows from the description of its fibers $Q_v := \rho_1^{-1}(v)$ in Lemma 2.4.

(a) We assume that $n = 6$. The locus $\Sigma_1(X)$ is then empty; hence, for any point $v \in \mathbf{P}(V_5)$, the fiber Q_v is a quadric in $\mathbf{P}^4$. If it is nondegenerate, then it contains no planes; hence, the map σ factors through Y_{A,V_5}.

If the corank of Q_v is 1, then there are two families of planes on Q_v, each parameterized by $\mathbf{P}^1$. Hence, over Y^1_{A,V_5}, the map σ factors as a $\mathbf{P}^1$-fibration followed by a double covering.

If the corank of Q_v is 2, then there is only one family of planes on Q_v parameterized by $\mathbf{P}^1$. Over Y^2_{A,V_5}, the map σ is therefore a $\mathbf{P}^1$-fibration.

Finally, if the corank of Q_v is 3, then we have $Q_v = \mathbf{P}^3 \cup \mathbf{P}^3$ (intersecting along a plane); hence, planes on Q_v are parameterized by $\mathbf{P}^3 \cup \mathbf{P}^3$ (dual spaces, intersecting in a point). It follows that $F_2^\sigma(X)$ has two irreducible components isomorphic to $\mathbf{P}^3$ over each point of Y^3_{A,V_5} and a component that dominates Y_{A,V_5}.

Considering the Stein factorization of the map σ restricted to this (main) component of $F_2^\sigma(X)$, we see that it is the composition of a $\mathbf{P}^1$-bundle (away from the preimage of Y^3_{A,V_5}) and a double cover of Y_{A,V_5} branched along $Y^{\geq 2}_{A,V_5}$. This $\mathbf{P}^1$-bundle is étale locally trivial, and its Brauer class is given by the sheaf of even parts of Clifford algebras (see [15, Lemma 4.2]).

To show that this double cover is isomorphic to $\widetilde{Y}_{A,V_5}$ (the base change to $\mathbf{P}(V_5)$ of the double EPW sextic), we compute, using Proposition A.2, the push-forward of the structure sheaf $\mathscr{O}_{F_2^\sigma(X)}$ to $\mathbf{P}(V_5)$. In the notation of Appendix A, we have $S = \mathbf{P}(V_5)$, $m = 5$, $\mathscr{E} = \mathscr{O}_S \oplus \mathscr{O}_S(-1) \otimes (V_5/\mathscr{O}_S(-1)) \simeq \mathscr{O}_S \oplus T_S(-2)$ (where T_S is the tangent bundle), and $\mathscr{L} = \mathscr{O}_S$. By Lemma 2.4, the degeneracy loci of the first quadratic fibration are $D_c = Y^{\geq c}_{A,V_5}$. Moreover, by [5, Proposition 4.5 and Lemma C.6], the cokernel sheaf $\mathscr{C}$ for the family of quadrics is isomorphic to the sheaf $\operatorname{Coker}(\mathscr{O}(-1) \otimes \bigwedge^2(V_6/\mathscr{O}(-1)) \to A^\vee \otimes \mathscr{O}_S)$; hence, the double cover can be written as

$$\operatorname{Spec}\big(\mathscr{O}_{Y_{A,V_5}} \oplus \mathscr{C}(-3)\big).$$

But this is precisely the base change to $\mathbf{P}(V_5)$ of the double EPW sextic (see [22, Section 4]). The statement about smoothness follows from the above description and Lemma 2.6(b).

(b) We assume that $n = 5$. If $v \notin \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^3$. It contains a plane if and only if its corank is at least 2. Therefore, we have two planes over each point of $Y^2_{A,V_5} \smallsetminus \Sigma_1(X)$ and one double plane over each point of $Y^3_{A,V_5} \smallsetminus \Sigma_1(X)$. On the other hand, if $v \in \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^4$. It contains planes if and only if it is degenerate, and these planes are then parameterized by $\mathbf{P}^1 \sqcup \mathbf{P}^1$ if the corank is 1 (i.e., if $v \in Y^2_{A,V_5}$) or by a double $\mathbf{P}^1$ if the corank is 2 (i.e., if $v \in Y^3_{A,V_5}$).

It follows that if X is ordinary (hence, $\Sigma_1(X) = \varnothing$) or if X is special and $\mathbf{p}_X \notin \mathrm{pr}_{Y,2}(E)$ (so that, by (3), then the kernel point $\Sigma_1(X)$ is not on $Y^{\ge 2}_{A,V_5}$), then there is a double cover $F^\sigma_2(X) \to Y^{\ge 2}_{A,V_5}$ branched along Y^3_{A,V_5}, while if X is special and $\mathbf{p}_X \in \mathrm{pr}_{Y,2}(E)$, then we have extra component(s) in $F^\sigma_2(X)$ as described in the statement of the theorem.

(c) We assume that $n = 4$. If $v \notin \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^2$. It contains a plane (and is then equal to it) if and only if its corank is 3. Hence, $F^\sigma_2(X)$ contains $Y^3_{A,V_5} \smallsetminus \Sigma_1(X)$. On the other hand, if $v \in \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^3$. It contains a plane if and only if its corank is 2 (and then it contains two planes). Hence, $F^\sigma_2(X)$ contains two points for each point of $Y^3_{A,V_5} \cap \Sigma_1(X)$. We conclude using the fact that Y^3_A is finite (see Section 2.2).

(d) This follows from Corollary 3.5. □

Using the second quadric fibration, we describe the scheme $F^\tau_2(X)$.

THEOREM 4.5

Let X be a smooth GM variety of dimension n, with associated Lagrangian data set (V_6, V_5, A).

(a) *If $n = 6$, then the map τ factors as*

$$F^\tau_2(X) \xrightarrow{\tilde\tau} Z^{\ge 2}_{A,V_5} \hookrightarrow \mathsf{Gr}(3, V_5),$$

where $\tilde\tau$ is a double covering branched along $Z^{\ge 3}_{A,V_5}$.

(b) *If $n = 5$, then the map $\tau\colon F^\tau_2(X) \to \mathsf{Gr}(3, V_5)$ factors as*

$$F^\tau_2(X) \xrightarrow{\tilde\tau} Z^{\ge 3}_{A,V_5} \hookrightarrow \mathsf{Gr}(3, V_5),$$

where the morphism $\tilde\tau$ is an isomorphism over $Z^{\ge 3}_{A,V_5} \smallsetminus \Sigma_2(X)$ and a double cover over $Z^{\ge 3}_{A,V_5} \cap \Sigma_2(X)$, branched along Z^4_{A,V_5}.

(c) *If $n = 4$, then the map τ factors as*

$$F^\tau_2(X) \xrightarrow{\tilde\tau} Z^4_{A,V_5} \hookrightarrow \mathsf{Gr}(3, V_5),$$

where $\tilde\tau$ is étale, is an isomorphism over $Z^4_{A,V_5} \smallsetminus \Sigma^2_2(X)$, and is a double cover over $Z^4_{A,V_5} \cap \Sigma^2_2(X)$. In particular, $F^\tau_2(X)$ is finite and it is empty if and only if $Z^4_{A,V_5} = \varnothing$.

(d) *If $n \le 3$, then we have $F^\tau_2(X) = \varnothing$.*

REMARK 4.6

The double cover $F^\tau_2(X) \to Z^{\ge 2}_{A,V_5}$ which appears in part (a) of the theorem is described in Proposition A.2. We expect it to be the base change to $\mathsf{Gr}(3, V_5)$ of the double cover $\widetilde{Z}^{\ge 2}_A \to Z^{\ge 2}_A$ constructed for general A in [12].

Proof of Theorem 4.5

In all cases, by Proposition 4.1, the scheme $F^\tau_2(X)$ is isomorphic to the relative Hilbert scheme of planes in the fibers of the second quadric fibration $\rho_2\colon \mathbf{P}_X(V_5/\mathscr{U}_X) \to \mathsf{Gr}(3,V_5)$. The rest follows from the description of its fibers $Q_{U_3} := \rho_2^{-1}(U_3)$ in Lemma 2.5.

(a) We assume that $n = 6$. We have $\Sigma_2(X) = \varnothing$ and, by Lemma 2.3, $Z^4_{A,V_5} = \varnothing$. For any point $U_3 \in \mathsf{Gr}(3,V_5)$, the fiber Q_{U_3} is a quadric in $\mathbf{P}^3$. It contains a plane only if its corank is at least 2; therefore, the map τ factors through $Z^{\geq 2}_{A,V_5} \subset \mathsf{Gr}(3,V_5)$. The quadric Q_{U_3} is the union of two planes if $U_3 \in Z^2_{A,V_5}$ and a double plane if $U_3 \in Z^3_{A,V_5}$, so the map $\tau\colon F^\tau_2(X) \to Z^{\geq 2}_{A,V_5}$ is a double cover branched along Z^3_{A,V_5}.

(b) We assume that $n = 5$. If $U_3 \notin \Sigma_2(X)$, then the fiber Q_{U_3} is a quadric in $\mathbf{P}^2$. It contains a plane (and is then equal to it) only if its corank is 3; hence, the map τ factors through $Z^{\geq 3}_{A,V_5}$ and is an isomorphism over $Z^{\geq 3}_{A,V_5} \smallsetminus \Sigma_2(X)$. On the other hand, if $U_3 \in \Sigma_2(X)$, then the fiber Q_{U_3} is a quadric in $\mathbf{P}^3$ and it contains a plane only when the quadric has corank at least 2, hence when $U_3 \in Z^{\geq 3}_{A,V_5}$. More precisely, if $U_3 \in Z^3_{A,V_5} \cap \Sigma_2(X)$, then the quadric Q_{U_3} is the union of two planes and the fiber of τ is two points; if $U_3 \in Z^4_{A,V_5}$ (by Lemma 2.3 and (4), then it is also automatically in $\Sigma_2(X)$), the quadric Q_{U_3} is a double plane and the fiber of τ is a point. This proves the required statement.

(c) We assume that $n = 4$. If $U_3 \notin \Sigma_2(X)$, then the fiber Q_{U_3} is a quadric in $\mathbf{P}^1$ and so never contains a plane. If $U_3 \in \Sigma^1_2(X)$, then the fiber Q_{U_3} is a quadric in $\mathbf{P}^2$, so it contains a plane (and is then equal to it) if and only if its corank is 3. This gives one point of $F^\tau_2(X)$ over each point of $Z^4_{A,V_5} \cap \Sigma^1_2(X)$. If $U_3 \in \Sigma^2_2(X)$, the fiber Q_{U_3} is a quadric in $\mathbf{P}^3$, so it contains a plane if and only if its corank is at least 2. This gives two points of $F^\tau_2(X)$ over each point of $Z^4_{A,V_5} \cap \Sigma^2_2(X)$. We conclude using the fact that Z^4_A is finite (Section 2.2).

(d) This follows from Corollary 3.5. □

4.4. Lines on GM varieties

We now consider the scheme $F_1(X)$ of lines contained in X.

THEOREM 4.7

Let X be a smooth GM variety of dimension n, with associated Lagrangian data set (V_6, V_5, A).

(a) *If $n = 6$, then the map $\sigma\colon F_1(X) \to \mathbf{P}(V_5)$ is dominant with general fiber isomorphic to $\mathbf{P}^3$.*

(b) *If $n = 5$, then the map $\sigma\colon F_1(X) \to \mathbf{P}(V_5)$ factors as*

$$F_1(X) \xrightarrow{\tilde\sigma} \widetilde{\mathbf{P}(V_5)} \longrightarrow \mathbf{P}(V_5),$$

where $\widetilde{\mathbf{P}(V_5)} \to \mathbf{P}(V_5)$ is the double cover branched along the sextic hypersurface $Y_{A,V_5} \subset \mathbf{P}(V_5)$ and $\tilde\sigma$ is a $\mathbf{P}^1$-bundle over the complement of the preimage of $Y^{\geq 2}_{A,V_5} \cup \Sigma_1(X)$.

(c) *If $n = 4$, then the map σ factors as*

$$F_1(X) \xrightarrow{\tilde\sigma} \widetilde{Y}_{A,V_5} \xrightarrow{f_{A,V_5}} Y_{A,V_5} \hookrightarrow \mathbf{P}(V_5),$$

where $\tilde{\sigma}$ is birational. The nontrivial fibers of the morphism $\tilde{\sigma}$ are $\mathbf{P}^2$ over each point of $Y^3_{A,V_5} \smallsetminus \Sigma_1(X)$, $\mathbf{P}^2 \cup \mathbf{P}^2$ over each point of $Y^3_{A,V_5} \cap \Sigma_1(X)$, and $\mathbf{P}^1$ over each point of $f^{-1}_{A,V_5}(\Sigma_1(X) \smallsetminus Y^3_{A,V_5})$.

(d) *If $n = 3$, then the map $\sigma\colon F_1(X) \to \mathbf{P}(V_5)$ factors as*

$$F_1(X) \xrightarrow{\tilde{\sigma}} Y^{\geq 2}_{A,V_5} \hookrightarrow \mathbf{P}(V_5),$$

where the morphism $\tilde{\sigma}$ is an isomorphism over $Y^{\geq 2}_{A,V_5} \smallsetminus \Sigma_1(X)$ and a double cover over $Y^{\geq 2}_{A,V_5} \cap \Sigma_1(X)$, branched along $Y^3_{A,V_5} \cap \Sigma_1(X)$.

In Proposition 5.3, we will make part (c) more precise by showing that, for an ordinary GM 4-fold X, the scheme $F_1(X)$ is a small resolution of $\widetilde{Y}_{A,V_5}$, under some explicit generality assumptions.

Proof

In all cases, by Proposition 4.1, the Hilbert scheme $F_1(X)$ is isomorphic to the relative Hilbert scheme of lines in the fibers of the first quadric fibration $\rho_1\colon \mathbf{P}_X(\mathscr{U}_X) \to \mathbf{P}(V_5)$. We now check that the rest follows from the description of its fibers $Q_v := \rho_1^{-1}(v)$ in Lemma 2.4.

(a) We assume that $n = 6$. We have $\Sigma_1(X) = \varnothing$ and all fibers of the morphism ρ_1 are quadrics in $\mathbf{P}^4$. Any such quadric contains a line; hence, the map $\sigma\colon F_1(X) \to \mathbf{P}(V_5)$ is dominant. Moreover, if $v \in \mathbf{P}(V_5) \smallsetminus Y_{A,V_5}$, then the quadric $\rho_1^{-1}(v)$ is smooth; hence, lines on it are parameterized by $\mathbf{P}^3$.

(b) We assume that $n = 5$. If $v \notin \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^3$. If $v \notin Y_{A,V_5}$, then the quadric Q_v is nondegenerate; hence, the family of lines on Q_v is parameterized by the union of two $\mathbf{P}^1$. If $v \in Y^1_{A,V_5}$, the quadric Q_v has corank 1 and lines on Q_v are parameterized by a single $\mathbf{P}^1$. Therefore, the Stein factorization of the map $\sigma\colon F_1(X) \to \mathbf{P}(V_5)$ is a composition of a generically $\mathbf{P}^1$-bundle with a double cover of $\mathbf{P}(V_5)$ branched along Y_{A,V_5}, as claimed. The Brauer class of this $\mathbf{P}^1$-bundle is again given by the sheaf of even parts of Clifford algebras.

(c) We assume that $n = 4$. If $v \notin \Sigma_1(X)$, then the fiber Q_v is a conic in $\mathbf{P}^2$. If $v \notin Y_{A,V_5}$, then the conic Q_v is nondegenerate and, hence, contains no lines. Therefore, the map σ factors through Y_{A,V_5}. If $v \in Y^1_{A,V_5}$, then the conic Q_v is the union of two lines; hence the map σ is étale of degree 2 over $Y^1_{A,V_5} \smallsetminus \Sigma_1(X)$. If $v \in Y^2_{A,V_5}$, then the conic Q_v is a double line; hence, the above double cover is branched along $Y^2_{A,V_5} \smallsetminus \Sigma_1(X)$. Finally, if $v \in Y^3_{A,V_5}$, then the fiber Q_v is the whole plane; hence, lines on Q_v are parameterized by the dual plane.

To show that this double cover is isomorphic to $\widetilde{Y}_{A,V_5}$, we compute, using Proposition A.1, the pushforward of the structure sheaf $\mathscr{O}_{F_1(X)}$ to $\mathbf{P}(V_5)$. We have $S = \mathbf{P}(V_5) \smallsetminus \Sigma_1(X)$, $m = 3$, $\mathscr{L} = \mathscr{O}_S$, and the bundle $\mathscr{E}$ fits into the exact sequence

$$0 \to \mathscr{E} \to \mathscr{O}_S \oplus T_S(-2) \to \mathscr{O}_S^{\oplus 2} \to 0$$

(where T_S is the tangent bundle). By Lemma 2.4, the degeneracy loci of the first quadratic fibration are $D_c = Y^{\geq c}_{A,V_5} \smallsetminus \Sigma_1(X)$. Moreover, by [5, Proposition 4.5 and Lemma C.6], the cokernel sheaf $\mathscr{C}$ for the family of quadrics is isomorphic to $\mathrm{Coker}(\mathscr{O}(-1) \otimes \bigwedge^2(V_6/\mathscr{O}(-1)) \to A^\vee \otimes \mathscr{O}_S)$; hence, again the double cover can be written as $\mathrm{Spec}(\mathscr{O}_{Y_{A,V_5} \smallsetminus \Sigma_1(X)} \oplus \mathscr{C}(-3))$. This is precisely the base change to $\mathbf{P}(V_5) \smallsetminus \Sigma_1(X)$ of the definition of the double EPW sextic (see [22, Section 4]).

If $v \in \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^3$. If $v \in Y^1_{A,X}$, then the quadric Q_v is nondegenerate and lines on Q_v are parameterized by $\mathbf{P}^1 \sqcup \mathbf{P}^1$; over each of the two points of $f^{-1}_{A,V_5}(v)$, the fiber of σ is $\mathbf{P}^1$. If $v \in Y^2_{A,X}$, the quadric has corank 1 and lines on Q_v are parameterized by $\mathbf{P}^1$. Finally, if $v \in Y^3_{A,X}$, then the quadric Q_v has corank 2, $Q_v = \mathbf{P}^2 \cup \mathbf{P}^2$, and lines on Q_v are parameterized by $\mathbf{P}^2 \cup \mathbf{P}^2$.

(d) We assume that $n = 3$. If $v \notin \Sigma_1(X)$, then the fiber Q_v is a quadric in $\mathbf{P}^1$. It contains no lines unless its corank is 2 (in which case it is itself a $\mathbf{P}^1$). Thus, the map σ factors through $Y^{\geq 2}_{A,V_5}$ and is an isomorphism over $Y^2_{A,V_5} \smallsetminus \Sigma_1(X)$. If $v \in \Sigma_1(X)$, the fiber Q_v is a conic in $\mathbf{P}^2$. If $v \in Y^2_{A,V_5}$, then it is a union of two lines; hence, the map $\tilde{\sigma}\colon F_1(X) \to Y^{\geq 2}_{A,V_5}$ is of degree 2 over $Y^2_{A,V_5} \cap \Sigma_1(X)$. Finally, if $v \in Y^3_{A,V_5}$ (it is then automatically in $\Sigma_1(X)$; see the discussion after Lemma 2.4), the fiber Q_v is a double line; hence, the map $\tilde{\sigma}$ is branched over this locus. □

Regarding items (a) and (b) in the theorem, it is possible to describe the fibers of σ over Y_{A,V_5} (resp., over the preimage of $Y^{\geq 2}_{A,V_5} \cup \Sigma_1(X)$). We leave this as an exercise for the interested reader. It is also possible to describe the scheme $F_1(X)$ by using the map τ. Finally, one can use a similar approach to describe the Hilbert schemes of quadrics in GM varieties and even the Hilbert schemes parameterizing cubic subvarieties (twisted cubic curves, cubic scrolls, and so on), but the description becomes more and more involved.

5. Periods of GM varieties

In this section, we relate the periods of GM varieties of dimension 4 or 6 to those of their associated EPW sextic. The following theorem is the main result of this article.

THEOREM 5.1

Let X be a smooth GM variety of dimension $n = 4$ or 6, with associated Lagrangian data set (V_6, V_5, A). Assume that the double EPW sextic $\widetilde{Y}_A$ is smooth (i.e., $Y^{\geq 3}_A = \varnothing$). There is an isomorphism of polarized Hodge structures

$$H^n(X;\mathbf{Z})_{00} \simeq H^2(\widetilde{Y}_A;\mathbf{Z})_0\big((-1)^{n/2-1}\big),$$

where (-1) is the Tate twist.

In Sections 5.1 and 5.2, we prove Theorem 5.1 for X general of dimension 4 (Theorem 5.12) or 6 (Theorem 5.19). In Section 5.4, we define period points and maps and use them to deduce Theorem 5.1 in full generality.

5.1. Periods of GM 4-folds

Our aim in this section is to prove Theorem 5.1 for a smooth GM 4-fold X, with associated Lagrangian data set (V_6, V_5, A). We will construct an explicit isomorphism when X is general. More precisely, we assume that

$$\mathbf{p}_X \notin (Y_A^{\geq 2})^\vee \qquad \text{and} \qquad Y_A^{\geq 3} = \varnothing. \tag{10}$$

Note that $Y_{A^\perp}^{\geq 2} \subset (Y_A^{\geq 2})^\vee$ by Lemma 2.1; hence, for a GM 4-fold X satisfying (10), we have $\mathbf{p}_X \in Y_{A^\perp}^1$, that is, X is ordinary. We will use this observation further on.

Note also that $(Y_A^{\geq 2})^\vee$ does not contain $Y_{A^\perp}$, since $Y_{A^\perp}^\vee = Y_A$ is not equal to $Y_A^{\geq 2}$; therefore, the choice of $\mathbf{p}_X$ satisfying (10) is possible. By Lemma 2.6(a), assumption (10) implies that

$$Y_{A,V_5}^{\geq 2} \text{ is a smooth curve and } Y_{A,V_5}^{\geq 3} = \varnothing. \tag{11}$$

Since X is ordinary, $\Sigma_1(X)$ is a point, which we denote by v_0. Moreover, $\mathbf{p}_X \notin \mathrm{pr}_{Y,2}(E)$ by Lemma 2.1; hence, we have by (3)

$$v_0 \in Y_{A,V_5}^1. \tag{12}$$

We have $\mathrm{Sing}(Y_{A,V_5}) = \{v_0\} \cup Y_{A,V_5}^{\geq 2}$ by Lemma 2.6(b). Furthermore, (12) and Lemma 2.4(b) imply that

$$Q_0 := \pi\big(\rho_1^{-1}(v_0)\big) \tag{13}$$

is a smooth quadric surface contained in X, with span $\Pi := \mathbf{P}(v_0 \wedge V_5) \subset M_X$.

The Hilbert scheme $F_1(X)$ of lines on X was described in Theorem 4.7(c). Under our generality assumption, this description takes the following simple form.

COROLLARY 5.2

Under assumption (10), the map $\tilde{\sigma}\colon F_1(X) \to \widetilde{Y}_{A,V_5}$ is an isomorphism over the complement of the two points of $f_{A,V_5}^{-1}(v_0)$, and the fiber of $\tilde{\sigma}$ over each of these two points is $\mathbf{P}^1$.

We now prove that the 3-fold $F_1(X)$ is smooth.

PROPOSITION 5.3

Let X be a smooth ordinary GM 4-fold, with associated Lagrangian data set (V_6, V_5, A), such that (10) holds. The map $\tilde{\sigma}\colon F_1(X) \to \widetilde{Y}_{A,V_5}$ is then a small resolution of singularities of $\widetilde{Y}_{A,V_5}$. In particular, $F_1(X)$ is a smooth irreducible 3-fold.

Proof

By Lemma 2.6(b) and (11), we have $\mathrm{Sing}(\widetilde{Y}_{A,V_5}) = f_{A,V_5}^{-1}(v_0)$. Since $\tilde{\sigma}$ is an isomorphism over the complement of $f_{A,V_5}^{-1}(v_0)$, it remains to show that $F_1(X)$ is smooth along $\tilde{\sigma}^{-1}(f_{A,V_5}^{-1}(v_0))$. In other words, we have to show that $F_1(X)$ is smooth at points corresponding to lines $L \subset X$ such that $\sigma([L]) = v_0$. By deformation theory, it is enough to prove $H^1(L, \mathscr{N}_{L/X}) = 0$ for any of these lines.

By the definition of the map σ, a line L with $\sigma([L]) = v_0$ lies on the 2-dimensional quadric $Q_0 = \rho_1^{-1}(v_0)$, which is smooth by Lemma 2.4. Therefore, there is an exact sequence

$$0 \to \mathscr{N}_{L/Q_0} \to \mathscr{N}_{L/X} \to \mathscr{N}_{Q_0/X}|_L \to 0.$$

The first term is $\mathscr{O}_L$, since Q_0 is a smooth quadric. It is enough to show that the last term is either $\mathscr{O}_L \oplus \mathscr{O}_L$ or $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-1)$. Since Q_0 is the transversal intersection of Π and a quadric cutting out X in M_X, we have

$$\mathscr{N}_{Q_0/X} \simeq \mathscr{N}_{\Pi/M_X}|_{Q_0}.$$

On the other hand, since M_X is a hyperplane section of $\mathsf{Gr}(2, V_5)$, we have an exact sequence

$$0 \to \mathscr{N}_{\Pi/M_X} \to \mathscr{N}_{\Pi/\,\mathsf{Gr}(2,V_5)} \to \mathscr{O}_\Pi(1) \to 0.$$

Finally, one easily proves the isomorphism $\mathscr{N}_{\Pi/\,\mathsf{Gr}(2,V_5)} \simeq T_\Pi(-1)$. Combining all these, we obtain an exact sequence

$$0 \to \mathscr{N}_{Q_0/X}|_L \to T_\Pi(-1)|_L \to \mathscr{O}_L(1) \to 0.$$

Since $\Pi \simeq \mathbf{P}^3$, the middle term is $\mathscr{O}_L(1) \oplus \mathscr{O}_L \oplus \mathscr{O}_L$. It follows that $\mathscr{N}_{Q_0/X}|_L$ is either $\mathscr{O}_L \oplus \mathscr{O}_L$ or $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-1)$. This completes the proof of the proposition. □

Under our assumptions, the set $f_{A,V_5}^{-1}(v_0)$ consists of two points, which we denote by $\mathbf{p}'$ and $\mathbf{p}''$. We also denote by

$$\mathbf{P}' := \tilde{\sigma}^{-1}(\mathbf{p}') \subset F_1(X) \qquad \text{and} \qquad \mathbf{P}'' := \tilde{\sigma}^{-1}(\mathbf{p}'') \subset F_1(X)$$

the nontrivial fibers of the map $\tilde{\sigma}\colon F_1(X) \to \widetilde{Y}_{A,V_5}$. (Each of them is isomorphic to $\mathbf{P}^1$.)

REMARK 5.4

The involution τ_A of the double cover $f_A\colon \widetilde{Y}_A \to Y_A$ restricts to the involution of the double cover $f_{A,V_5}\colon \widetilde{Y}_{A,V_5} \to Y_{A,V_5}$. However, it does not extend to a regular involution of $F_1(X)$: the small resolutions of the two singular points $\mathbf{p}'$ and $\mathbf{p}''$ of $\widetilde{Y}_{A,V_5}$ are not compatible with this involution.

Denote by $\iota\colon \widetilde{Y}_{A,V_5} \to \widetilde{Y}_A$ the canonical embedding.

PROPOSITION 5.5

Let X be a smooth ordinary GM 4-fold, with associated Lagrangian data set (V_6, V_5, A), such that (10) holds. The restriction $\iota^*\colon H^2(\widetilde{Y}_A; \mathbf{Z}) \to H^2(\widetilde{Y}_{A,V_5}; \mathbf{Z})$ is an isomorphism and the composition

$$H^2(\widetilde{Y}_A;\mathbf{Z}) \xrightarrow{\iota^*} H^2(\widetilde{Y}_{A,V_5};\mathbf{Z}) \xrightarrow{\tilde{\sigma}^*} H^2\big(F_1(X);\mathbf{Z}\big)$$

induces an isomorphism of integral Hodge structures between $H^2(\widetilde{Y}_A;\mathbf{Z})$ *and* $\langle \mathbf{P}',\mathbf{P}''\rangle^{\perp} \subset H^2(F_1(X);\mathbf{Z})$.

Proof
Since (10) holds, $\widetilde{Y}_A$ is smooth; hence, ι^* is an isomorphism by the Lefschetz theorem (Lemma 3.2). Set $U := \widetilde{Y}_{A,V_5} \smallsetminus \{\mathbf{p}',\mathbf{p}''\} \simeq F_1(X) \smallsetminus (\mathbf{P}' \cup \mathbf{P}'')$. We have a commutative diagram

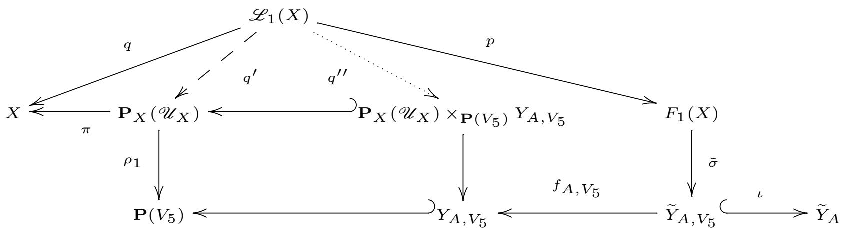
$$\begin{array}{ccccccccccccc}
\cdots & \to & H^1(\{\mathbf{p}',\mathbf{p}''\};\mathbf{Z}) & \to & H^2_c(U;\mathbf{Z}) & \to & H^2(\widetilde{Y}_{A,V_5};\mathbf{Z}) & \to & H^2(\{\mathbf{p}',\mathbf{p}''\};\mathbf{Z}) & \to & H^3_c(U;\mathbf{Z}) & \to & \cdots \\
 & & \downarrow{\scriptstyle \tilde{\sigma}^*} & & \| & & \downarrow{\scriptstyle \tilde{\sigma}^*} & & \downarrow{\scriptstyle \tilde{\sigma}^*} & & \| & & \\
\cdots & \to & H^1(\mathbf{P}' \cup \mathbf{P}'';\mathbf{Z}) & \to & H^2_c(U;\mathbf{Z}) & \to & H^2(F_1(X);\mathbf{Z}) & \to & H^2(\mathbf{P}' \cup \mathbf{P}'';\mathbf{Z}) & \to & H^3_c(U;\mathbf{Z}) & \to & \cdots
\end{array}$$

of exact sequences in cohomology with compact supports. The first column is zero, and the fourth column is the inclusion $0 \to \mathbf{Z} \oplus \mathbf{Z}$. The third column therefore extends to an exact sequence

$$(14) \qquad 0 \longrightarrow H^2(\widetilde{Y}_{A,V_5};\mathbf{Z}) \xrightarrow{\tilde{\sigma}^*} H^2(F_1(X);\mathbf{Z}) \xrightarrow{r} \mathbf{Z} \oplus \mathbf{Z},$$

where $r(x) := (x \cdot [\mathbf{P}'], x \cdot [\mathbf{P}''])$. This proves the proposition. □

Let $p\colon \mathscr{L}_1(X) \to F_1(X)$ be the universal line, and let $q\colon \mathscr{L}_1(X) \to X$ be the natural morphism. These two morphisms define a correspondence between X and $F_1(X)$ and, hence, a map between $H^\bullet(X;\mathbf{Z})$ and $H^\bullet(F_1(X);\mathbf{Z})$, which we investigate. We extend diagram (9) to a commutative diagram

(15)


where the map q' is defined in the same way as the dashed arrow in (9) and the map q'' is constructed by the universal property of the fiber product.

Denote by h the hyperplane class of X and by $\tilde{h}$ the hyperplane class of $\mathbf{P}(V_6)$ and its restrictions to $\mathbf{P}(V_5)$ and $\widetilde{Y}_A$.

LEMMA 5.6
The map q'' *is finite and birational, and* $q'_*([\mathscr{L}_1(X)]) = 6\rho_1^*\tilde{h}$ *in* $H^2(\mathbf{P}_X(\mathscr{U}_X);\mathbf{Z})$. *In particular, the map* q *is generically finite of degree* 6.

Proof
Let $(x,v) \in \mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} Y_{A,V_5} \subset \mathbf{P}_X(\mathscr{U}_X) \subset X \times \mathbf{P}(V_5)$. If $\gamma_X(x) = [U_2]$, then this means that v is in $\mathbf{P}(U_2) \cap Y_{A,V_5}$ and that $(q'')^{-1}(x,v)$ is the Hilbert scheme

of lines in $\rho_1^{-1}(v)$ passing through x. But $\rho_1^{-1}(v)$ is either a reducible conic or a double line if $v \neq v_0$, or the quadric Q_0 defined in (13) if $v = v_0$. Therefore, there is a unique line through x, unless x is a singular point of a singular conic or $v = v_0$, in which case there are two lines through x. Thus, q'' is finite and birational. It follows that the pushforward $q'_*([\mathscr{L}_1(X)]) \in H^2(\mathbf{P}_X(\mathscr{U}_X);\mathbf{Z})$ is the class of the divisor $\mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} Y_{A,V_5}$ and, hence, the pullback via ρ_1 of the class of Y_{A,V_5}, which is equal to $6\tilde{h}$. □

Geometrically, this means that, for a general point x of an ordinary GM 4-fold X, there are six lines passing through x and contained in X.

COROLLARY 5.7
One has $q_*p^*\tilde{\sigma}^*\iota^*\tilde{h}^2 = 6c_2(V_5/\mathscr{U}_X) = 6\gamma_X^*\boldsymbol{\sigma}_2$.

Proof
We need to compute the pullback of $\tilde{h}^2$ to $\mathscr{L}_1(X)$ and its pushforward via q to X. We can rewrite this as $\pi_*q'_*q'^*\rho_1^*\tilde{h}^2$. By the projection formula and Lemma 5.6, this is equal to $\pi_*(\rho_1^*\tilde{h}^2 \cdot 6\rho_1^*\tilde{h}) = 6\pi_*\rho_1^*\tilde{h}^3$. Since π is the projectivization of $\mathscr{U}_X$ and $\rho_1^*\tilde{h}$ is a relative hyperplane class, we have

$$\rho_1^*\tilde{h}^2 + c_1(\mathscr{U}_X)\rho_1^*\tilde{h} + c_2(\mathscr{U}_X) = 0. \tag{16}$$

Therefore, $\rho_1^*\tilde{h}^3 = (c_1(\mathscr{U}_X)^2 - c_2(\mathscr{U}_X))\rho_1^*\tilde{h} + c_1(\mathscr{U}_X)c_2(\mathscr{U}_X)$ and

$$\pi_*\rho_1^*\tilde{h}^3 = c_1(\mathscr{U}_X)^2 - c_2(\mathscr{U}_X) = c_2(V_5/\mathscr{U}_X).$$

The corollary follows. □

LEMMA 5.8
We have $q_*p^*(\mathbf{P}') = q_*p^*(\mathbf{P}'') = [Q_0] \in H^4(X;\mathbf{Z})$.

Proof
The subscheme $\mathbf{P}' \sqcup \mathbf{P}'' \subset F_1(X)$ parameterizes lines on X that are contained in the smooth quadric surface Q_0. The lines in each of the components $\mathbf{P}'$ and $\mathbf{P}''$ sweep out Q_0 once—hence, this proves the lemma. □

Let X be a smooth GM 4-fold. Consider the morphism

$$\alpha\colon H^4(X;\mathbf{Z}) \longrightarrow H^2\big(F_1(X);\mathbf{Z}\big), \quad x \longmapsto p_*(q^*x). \tag{17}$$

The classes (see Lemma 3.6)

$$c_2(\mathscr{U}_X) = \gamma_X^*\boldsymbol{\sigma}_{1,1} \quad \text{and} \quad [Q_0] = \gamma_X^*(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_{1,1}) = h^2 - 2c_2(\mathscr{U}_X) \tag{18}$$

in $H^4(X;\mathbf{Z})$ generate the subgroup $\gamma_X^*(H^4(\mathsf{Gr}(2,V_5);\mathbf{Z}))$. In the following two lemmas, we compute their images by the map α. We assume as before that X satisfies the assumptions (10).

LEMMA 5.9
We have $\alpha(c_2(\mathscr{U}_X)) = \tilde{\sigma}^*\iota^*\tilde{h}$.

Proof
Consider the diagram (15). The pullback of the bundle $\mathscr{U}_X$ to $\mathbf{P}_X(\mathscr{U}_X)$ is an extension $0 \to \mathscr{O}(-\rho_1^*\tilde{h}) \to \pi^*\mathscr{U}_X \to \mathscr{O}(\rho_1^*\tilde{h} - \pi^*h) \to 0$ of line bundles; hence,

$$\pi^*c_2(\mathscr{U}_X) = \rho_1^*\tilde{h}(\pi^*h - \rho_1^*\tilde{h}).$$

Therefore, we have

$$q^*c_2(\mathscr{U}_X) = q'^*\pi^*c_2(\mathscr{U}_X) = q'^*\left(\rho_1^*\tilde{h}(\pi^*h - \rho_1^*\tilde{h})\right) = p^*\tilde{\sigma}^*\iota^*\tilde{h} \cdot q^*h - p^*\tilde{\sigma}^*\iota^*\tilde{h}^2$$

and since p is a $\mathbf{P}^1$-bundle with relative hyperplane class q^*h, the pushforward by p of the right-hand side equals $\tilde{\sigma}^*\iota^*\tilde{h}$. □

Recall that the surface $Q_0 = X \cap \mathbf{P}(v_0 \wedge (V_5/v_0))$ defined in (13) is a smooth quadric. To compute the class $\alpha([Q_0])$ in $H^2(F_1(X);\mathbf{Z})$, we need some preparation.

First, the Hilbert scheme of lines on Q_0 is $F_1(Q_0) = \mathbf{P}' \sqcup \mathbf{P}''$, and the corresponding universal line is $\mathscr{L}_1(Q_0) = Q_0' \sqcup Q_0''$, where the first (resp., second) component corresponds to lines parameterized by $\mathbf{P}'$ (resp., $\mathbf{P}''$) and the map $\mathscr{L}_1(Q_0) \subset \mathscr{L}_1(X) \xrightarrow{q} X$ induces isomorphisms $Q_0' \simeq Q_0$ and $Q_0'' \simeq Q_0$.

Second, we have $\mathscr{U}_{Q_0} := \mathscr{U}_X|_{Q_0} \simeq \mathscr{O}_{Q_0} \oplus \mathscr{O}_{Q_0}(-1)$ by Lemma 3.7; hence, $\rho_1(\mathbf{P}_{Q_0}(\mathscr{U}_{Q_0})) \subset \mathbf{P}(V_5)$ is the quadratic cone $\mathsf{C}_{v_0}Q_0$ over $Q_0 \subset \mathbf{P}(V_5/v_0)$ with vertex v_0. Set

$$S := Y_{A,V_5} \cap \mathsf{C}_{v_0}Q_0 \subset \mathbf{P}(V_5).$$

This is an intersection of two distinct irreducible hypersurfaces in $\mathbf{P}(V_5)$ (see Lemma 2.6(b)), hence a Cohen–Macaulay surface containing v_0.

LEMMA 5.10
There is a surface $R \subset F_1(X)$ *such that*

$$\alpha\big([Q_0]\big) = [R] \in H^2\big(F_1(X);\mathbf{Z}\big),$$

and the map $\sigma = f_A \circ \tilde{\sigma}\colon R \to \mathbf{P}(V_5)$ *is birational onto* S.

Proof
We first describe $q^{-1}(Q_0)$. Since $\mathscr{L}_1(X)$ is smooth of dimension 4 (see Proposition 5.3) and q is dominant (see Lemma 5.6), $q^{-1}(Q_0)$ has everywhere dimension at least 2. We see on the diagram (15) that the map $q\colon \mathscr{L}_1(X) \to X$ factors through the map $q''\colon \mathscr{L}_1(X) \to \mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} Y_{A,V_5}$; moreover, we have

$$q^{-1}(Q_0) = q''^{-1}\big(\mathbf{P}_{Q_0}(\mathscr{U}_{Q_0}) \times_{\mathbf{P}(V_5)} Y_{A,V_5}\big).$$

The situation is summarized in the following Cartesian diagram:

$$\begin{array}{ccccccc}
Q_0 & \xleftarrow{\pi} & \mathbf{P}_{Q_0}(\mathscr{U}_{Q_0}) & \hookleftarrow & \mathbf{P}_{Q_0}(\mathscr{U}_{Q_0}) \times_{\mathbf{P}(V_5)} Y_{A,V_5} & \xleftarrow{q''} & q^{-1}(Q_0) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
X & \xleftarrow{\pi} & \mathbf{P}_X(\mathscr{U}_X) & \hookleftarrow & \mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} Y_{A,V_5} & \xleftarrow{q''} & \mathscr{L}_1(X)
\end{array}$$

The projection $\mathbf{P}_{Q_0}(\mathscr{U}_{Q_0}) \times_{\mathbf{P}(V_5)} Y_{A,V_5} \to \mathbf{P}(V_5)$ factors through the surface S and is an isomorphism over the dense open subscheme $S_0 := S \smallsetminus \{v_0\}$. Let $\tilde{S}_0 \subset \mathbf{P}_{Q_0}(\mathscr{U}_{Q_0}) \times_{\mathbf{P}(V_5)} Y_{A,V_5}$ be the preimage of S_0, and set

$$R_0 := q''^{-1}(\tilde{S}_0) \subset q^{-1}(Q_0) \subset \mathscr{L}_1(X).$$

Let $\tilde{S}_{00} \subset \tilde{S}_0$ be the open subscheme of $\tilde{S}_0$ parameterizing pairs (x, v) such that $v \neq v_0$ and x is not a singular point of the conic $\rho_1^{-1}(v)$. Let S_{00} be its isomorphic image in $S_0 \subset S$, and let R_{00} be its preimage in R_0. By Lemma 5.6, the map $q''|_{R_0} \colon R_0 \to \tilde{S}_0$ induces an isomorphism of schemes $R_{00} \xrightarrow{\sim} \tilde{S}_{00}$. Thus, the map $\sigma \circ p \colon R_0 \to \mathbf{P}(V_5)$ induces an isomorphism $R_{00} \xrightarrow{\sim} S_{00}$; since we show below that S_{00} is dense in S, it follows that R_{00} has pure dimension 2.

Consider now $R_1 := R_0 \smallsetminus R_{00}$. By definition, its image in S is contained in $S_1 := S_0 \smallsetminus S_{00}$. We show below $\dim(S_1) \leq 1$; hence, S_{00} is dense in S, and since the map $R_1 \to S_1$ is finite (since the Hilbert scheme of lines in any conic is finite), we also have $\dim(R_1) \leq 1$.

Finally, consider the scheme $q^{-1}(Q_0) \smallsetminus R_0 = q^{-1}(Q_0) \cap p^{-1}(\sigma^{-1}(v_0))$. Since $\sigma^{-1}(v_0) = F_1(Q_0)$, we have $q^{-1}(Q_0) \smallsetminus R_0 = \mathscr{L}_1(Q_0) \times_X Q_0 = Q_0' \sqcup Q_0''$. All this shows that $q^{-1}(Q_0)$ has pure dimension 2 and that we can write

$$q^*\big([Q_0]\big) = a'[Q_0'] + a''[Q_0''] + [\overline{R_{00}}]$$

for some integers a' and a''. The map p contracts the surfaces Q_0' and Q_0'' onto the curves $\mathbf{P}'$ and $\mathbf{P}''$, respectively; hence, p_* kills their classes, and we obtain

$$\alpha\big([Q_0]\big) = p_*\big(q^*\big([Q_0]\big)\big) = a' p_*\big([Q_0']\big) + a'' p_*\big([Q_0'']\big) + p_*\big([\overline{R_{00}}]\big) = p_*\big([\overline{R_{00}}]\big).$$

Recall that the map $\sigma \circ p$ induces an isomorphism $R_{00} \xrightarrow{\sim} S_{00}$. Setting

$$R := \overline{p(R_{00})} \subset F_1(X),$$

we obtain $\alpha([Q_0]) = [R]$ and the map $\sigma \colon R \to \mathbf{P}(V_5)$ is birational onto $\overline{S_{00}} = S$.

It remains to prove $\dim(S_1) \leq 1$. Let $\mathscr{E} \subset W \otimes \mathscr{O}_{S_1}$ be the rank-3 vector bundle over S_1 with fiber at a point v of S_1 the linear span of the conic $\rho_1^{-1}(v)$ (see the proof of Theorem 4.7(c) for its description). Consider the line subbundle $\mathscr{L} \hookrightarrow \mathscr{E}$ whose fiber over a point $v \in S_1$ which is the image of a point $(x, v) \in \tilde{S}_0$ is given by the point x. (Alternatively, $\mathscr{L} \simeq \pi^*\mathscr{O}_X(-1)|_{S_1}$, where $\pi \colon \mathbf{P}_X(\mathscr{U}_X) \to X$ is the natural projection, we consider here S_1 as a subscheme in $\tilde{S}_0$, and the embedding $\mathscr{L} \hookrightarrow \mathscr{E}$ is induced by the natural isomorphism $\mathscr{E}^\vee \simeq \rho_{1*}(\pi^*\mathscr{O}_X(1))|_{S_1}$.) By the definition of S_1, the subbundle $\mathscr{L} \subset \mathscr{E}$ is contained in the kernel of the quadratic form on $\mathscr{E}$ corresponding to the conic bundle $\mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} S_1 \subset \mathbf{P}_{S_1}(\mathscr{E}) \to S_1$ and, hence, induces a quadratic form on the quotient bundle $\mathscr{E}/\mathscr{L}$. Its discriminant locus is a divisor in S_1 which, by construction, is contained in the corank-2 locus of ρ_1.

Assume by contradiction that $\dim(S_1) \geq 2$. The dimension of this locus is then at least 1. Since, away from v_0, it coincides with the smooth irreducible curve Y^2_{A,V_5}, this curve is contained in S and, hence, in the quadric $\mathsf{C}_{v_0}Q_0$. This contradicts Corollary B.6. ☐

COROLLARY 5.11
One has $\iota_*\tilde{\sigma}_*\alpha(h^2) = 3\tilde{h}^2$.

Proof
By (18), we have

$$\iota_*\tilde{\sigma}_*\alpha(h^2) = \iota_*\tilde{\sigma}_*\alpha([Q_0]) + 2\iota_*\tilde{\sigma}_*\alpha(c_2(\mathscr{U}_X)) = \iota_*\tilde{\sigma}_*([R]) + 2\iota_*\tilde{\sigma}_*\tilde{\sigma}^*\iota^*\tilde{h}.$$

(We use Lemmas 5.9 and 5.10 in the last equality.) By the projection formula, the second summand equals $2\tilde{h}^2$, so it remains to show that the first summand $\iota_*\tilde{\sigma}_*([R])$ equals $\tilde{h}^2$. Since $F_1(X) \times_{\mathbf{P}(V_5)} S$ is birational to the double cover $\widetilde{Y}_{A,V_5} \times_{\mathbf{P}(V_5)} S$ of S and contains a surface R that maps to S birationally, we have

$$F_1(X) \times_{\mathbf{P}(V_5)} S = R \cup \tau_A(R),$$

where τ_A is the birational involution of $F_1(X)$ induced by the involution of $\widetilde{Y}_A$. Since S is cut out in Y_{A,V_5} by a quadric, we have

$$\iota_*\tilde{\sigma}_*\big([R]\big) + \iota_*\tilde{\sigma}_*\big(\big[\tau_A(R)\big]\big) = 2\iota_*\iota^*\tilde{h} = 2\tilde{h}^2.$$

The two summands on the left-hand side are interchanged by the involution τ_A of the double cover $f_A\colon \widetilde{Y}_A \to Y_A$. We would like to show that they are equal. For that, let us first assume that X, hence also A, is very general. The vector space $H^{2,2}(\widetilde{Y}_A) \cap H^4(\widetilde{Y}_A;\mathbf{Q})$ then has rank 2, generated by $\tilde{h}^2$ and $c_2(\widetilde{Y}_A)$ (see [24, Proposition 3.2]), and both of these classes are τ_A-invariant. Since $\iota_*\tilde{\sigma}_*([R])$ and $\iota_*\tilde{\sigma}_*([\tau_A(R)])$ both belong to this space, they are also τ_A-invariant and, hence, equal. Finally, since $H^4(\widetilde{Y}_A;\mathbf{Z})$ is torsion-free (see [18, Theorem 1]), they are both equal to $\tilde{h}^2$.

Going back to the general case where X only satisfies (10), we note that the class $\iota_*\tilde{\sigma}_*([R]) - \tilde{h}^2 \in H^4(\widetilde{Y}_A;\mathbf{Z})$ depends continuously on X and is zero for X very general, as we showed above. Therefore, it is zero for all X. ☐

We are now ready to prove Theorem 5.1 for general GM 4-folds. Recall that, in (15), the map $p\colon \mathscr{L}_1(X) \to F_1(X)$ is a $\mathbf{P}^1$-fibration for which q^*h is a relative hyperplane class. Therefore, $\mathscr{L}_1(X)$ is isomorphic to the projectivization of a rank-2 vector bundle. We denote by c_1 and c_2 its Chern classes, so that

$$q^*h^2 + p^*c_1 \cdot q^*h + p^*c_2 = 0. \tag{19}$$

The primitive and vanishing cohomology subgroups $H^2(\widetilde{Y}_A;\mathbf{Z})_0 \subset H^2(\widetilde{Y}_A;\mathbf{Z})$ and $H^4(X;\mathbf{Z})_{00} \subset H^4(X;\mathbf{Z})_0 \subset H^4(X;\mathbf{Z})$ were defined in (5) and (6), and the map α was defined in (17).

THEOREM 5.12

Let X be a smooth ordinary GM 4-fold, with associated Lagrangian data set (V_6, V_5, A), such that assumption (10) holds. We have

$$\forall x \in H^4(X;\mathbf{Z})_0 \quad \alpha(x)^2 \cdot \alpha(h^2) = -6x^2. \tag{20}$$

In particular, the restriction $\alpha_0\colon H^4(X;\mathbf{Z})_0 \to H^2(F_1(X);\mathbf{Z})$ of α is injective. Furthermore, it induces an isomorphism

$$\beta\colon H^4(X;\mathbf{Z})_{00} \xrightarrow{\sim} H^2(\widetilde{Y}_A;\mathbf{Z})_0$$

compatible with the Beauville–Bogomolov form

$$\forall x \in H^4(X;\mathbf{Z})_{00} \quad q_B\bigl(\beta(x)\bigr) = -x^2$$

and, hence, an isomorphism $H^4(X;\mathbf{Z})_{00}(-1) \xrightarrow{\sim} H^2(\widetilde{Y}_A;\mathbf{Z})_0$ of polarized Hodge structures.

Proof

We follow the argument from [2]. Since p is a $\mathbf{P}^1$-bundle and q^*h is a relative hyperplane class, we may write

$$\forall x \in H^4(X;\mathbf{Z}) \quad q^*x = p^*x_1 \cdot q^*h + p^*x_2,$$

where $x_i \in H^{2i}(F_1(X);\mathbf{Z})$ for $i \in \{1,2\}$. We then have

$$\alpha(x) = x_1.$$

To see what the primitivity of x means, we compute, using (19),

$$q^*(x \cdot h) = p^*x_1 \cdot q^*h^2 + p^*x_2 \cdot q^*h = p^*(x_2 - x_1 \cdot c_1) \cdot q^*h - p^*(x_1 \cdot c_2).$$

Thus, $x \cdot h = 0$ implies that

$$x_1 \cdot c_2 = 0, \qquad x_2 = x_1 \cdot c_1,$$

and we can rewrite as

$$q^*x = p^*\bigl(\alpha(x)\bigr) \cdot (q^*h + p^*c_1).$$

Taking squares and using (19), we obtain

$$\begin{aligned} q^*x^2 &= p^*\bigl(\alpha(x)^2\bigr) \cdot (q^*h^2 + 2p^*c_1 \cdot q^*h + p^*c_1^2) \\ &= p^*\bigl(\alpha(x)^2\bigr) \cdot \bigl(p^*c_1 \cdot q^*h + p^*(c_1^2 - c_2)\bigr) = \alpha(x)^2 \cdot c_1. \end{aligned}$$

On the other hand, since, by Lemma 5.6, the degree of q is 6, we have $q^*x^2 = 6x^2$. We obtain $\alpha(h^2) = -c_1$ from (19) and all this proves (20). This relation implies that $\alpha_0(x_1)\alpha_0(x_2)\alpha(h^2) = -6x_1x_2$ for all $x_1, x_2 \in H^4(X;\mathbf{Z})_0$ and the injectivity of α_0 follows from the nondegeneracy of the intersection pairing on $H^4(X;\mathbf{Z})_0$.

If $x \in H^4(X;\mathbf{Z})_{00}$, then we have, by adjunction and Lemma 5.8,

$$\alpha(x)\cdot[\mathbf{P}'] = x\cdot q_*p^*[\mathbf{P}'] = x\cdot[Q_0] = 0$$

because, by (18), the class $[Q_0]$ is in the subgroup $\gamma_X^* H^4(\mathsf{Gr}(2,V_5);\mathbf{Z})$. Similarly, we have $\alpha(x)\cdot[\mathbf{P}''] = 0$. From (14), we obtain that α maps $H^4(X;\mathbf{Z})_{00}$ into the subgroup $\tilde{\sigma}^*\iota^* H^2(\widetilde{Y}_A;\mathbf{Z})$ of $H^2(F_1(X);\mathbf{Z})$. By Proposition 5.5, this defines an injective map $\beta\colon H^4(X;\mathbf{Z})_{00} \to H^2(\widetilde{Y}_A;\mathbf{Z})$ such that

$$\forall x \in H^4(X;\mathbf{Z})_{00} \quad \alpha(x) = \tilde{\sigma}^*\big(\iota^*\big(\beta(x)\big)\big).$$

It remains to show that the image of β is the primitive cohomology $H^2(\widetilde{Y}_A;\mathbf{Z})_0$ and that β is compatible with the Beauville–Bogomolov form. Keeping the assumption $x \in H^4(X;\mathbf{Z})_{00}$ and using Corollary 5.7, we have

$$\begin{aligned}0 &= x\cdot 6c_2(V_5/\mathscr{U}_X) = x\cdot q_*p^*\tilde{\sigma}^*\iota^*(\tilde{h}^2) = \alpha(x)\cdot\tilde{\sigma}^*\iota^*(\tilde{h}^2)\\ &= \tilde{\sigma}^*\iota^*\big(\beta(x)\big)\cdot\tilde{\sigma}^*\iota^*(\tilde{h}^2) = \iota^*\big(\beta(x)\big)\cdot\iota^*(\tilde{h}^2) = \beta(x)\cdot\iota_*\iota^*(\tilde{h}^2) = \beta(x)\cdot\tilde{h}^3.\end{aligned}$$

This proves $\beta(x) \in H^2(\widetilde{Y}_A;\mathbf{Z})_0$ by the definition (6) of the primitive cohomology group.

For the compatibility with the Beauville–Bogomolov form q_B, we observe

$$\begin{aligned}-x^2 &= \frac{1}{6}\alpha(x)^2\alpha(h^2) = \frac{1}{6}\tilde{\sigma}^*\iota^*\beta(x)^2\alpha(h^2) = \frac{1}{6}\beta(x)^2\cdot\iota_*\tilde{\sigma}_*\alpha(h^2) = \frac{1}{6}\beta(x)^2\cdot 3\tilde{h}^2\\ &= q_B\big(\beta(x)\big).\end{aligned}$$

The first equality is (20), the second is the definition of β, the third follows from adjunction, the fourth is Corollary 5.11, and the last is (7).

Finally, the lattices $H^4(X;\mathbf{Z})_{00}$ and $H^2(\widetilde{Y}_A;\mathbf{Z})_0$ both have rank 22 and the same discriminant group $(\mathbf{Z}/2\mathbf{Z})^2$ (see [4, Proposition 5.1] or Proposition 3.9 for $H^4(X;\mathbf{Z})_{00}$, and [23, (1.0.9)] for $H^2(\widetilde{Y}_A;\mathbf{Z})_0$); hence, the injective anti-isometry β is a bijection. □

5.2. Periods of GM 6-folds

Our aim in this section is to prove Theorem 5.1 for a smooth GM 6-fold X, with associated Lagrangian data set (V_6, V_5, A). Again, we will provide an explicit isomorphism for general X, namely, for those satisfying the same assumption (10)—which implies (11). Since X is a 6-fold, $\Sigma_1(X)$ is empty, and in contrast with the 4-fold case, Lemma 2.6(b) says that Y_{A,V_5} is smooth away from $Y^{\ge 2}_{A,V_5}$, and $\widetilde{Y}_{A,V_5}$ is a smooth 3-fold.

The scheme of σ-planes on X was described in Theorem 4.3(a). Under our generality assumption, this description takes the following simple form.

COROLLARY 5.13

Let X be a smooth GM 6-fold, with associated Lagrangian data set (V_6, V_5, A), such that assumption (10) holds. Then, $F_2^\sigma(X)$ is a smooth 4-fold and the map $\tilde{\sigma}\colon F_2^\sigma(X) \to \widetilde{Y}_{A,V_5}$ is a $\mathbf{P}^1$-fibration.

We denote by $\mathbf{P}_y \simeq \mathbf{P}^1$ the fiber of $\tilde{\sigma}$ over a point $y \in \widetilde{Y}_{A,V_5}$. These fibers all have the same cohomology class which we denote by $[\mathbf{P}] \in H^6(F_2^\sigma(X);\mathbf{Z})$. As before, we let $\iota\colon \widetilde{Y}_{A,V_5} \to \widetilde{Y}_A$ be the canonical embedding.

PROPOSITION 5.14

Let X be a smooth GM 6-fold, with associated Lagrangian data set (V_6, V_5, A), such that (10) holds. The composition

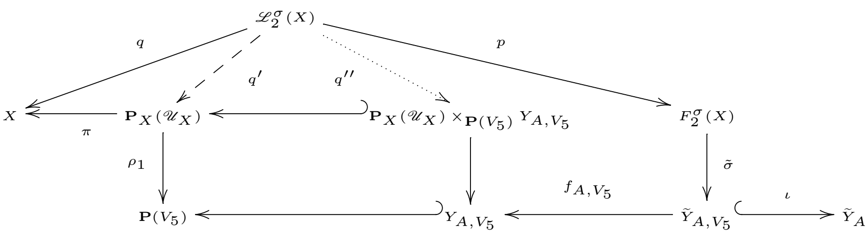
$$H^2(\widetilde{Y}_A;\mathbf{Z}) \xrightarrow{\iota^*} H^2(\widetilde{Y}_{A,V_5};\mathbf{Z}) \xrightarrow{\tilde{\sigma}^*} H^2\big(F_2^\sigma(X);\mathbf{Z}\big)$$

induces an isomorphism of integral Hodge structures between $H^2(\widetilde{Y}_A;\mathbf{Z})$ and $[\mathbf{P}]^\perp \subset H^2(F_2^\sigma(X);\mathbf{Z})$.

Proof

The map ι^* is an isomorphism by Lemma 3.2. We then use the fact that $\tilde{\sigma}$ is a $\mathbf{P}^1$-fibration with fiber class $[\mathbf{P}]$. □

Let $p\colon \mathscr{L}_2^\sigma(X) \to F_2^\sigma(X)$ be the universal σ-plane, and let $q\colon \mathscr{L}_2^\sigma(X) \to X$ be the natural morphism. The analogue of the diagram (15) is

(21) 

where ρ_1 is a fibration in 3-dimensional quadrics, and the dashed and dotted arrows are constructed in the same way.

LEMMA 5.15

The map q'' is generically finite of degree 2 and $q'_*([\mathscr{L}_2^\sigma(X)]) = 12\rho_1^*\tilde{h}$. In particular, the map q is generically finite of degree 12.

Proof

Take a point $(x,v) \in \mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} Y_{A,V_5} \subset X \times \mathbf{P}(V_5)$. If $\gamma_X(x) = [U_2]$, then we have $v \in \mathbf{P}(U_2) \cap Y_{A,V_5}$ and $(q'')^{-1}(x,v)$ is the Hilbert scheme of planes in $\rho_1^{-1}(v)$ passing through x. But $\rho_1^{-1}(v)$ is either a cone over $\mathbf{P}^1 \times \mathbf{P}^1$ or a cone over a conic with vertex a line. In the first case, there is a unique plane of each type through x, unless x is the vertex of the cone, and in the second case, there is a unique plane (with multiplicity 2) through any point of the cone not lying on the vertex. Thus, q'' is generically finite of degree 2.

It follows that the class $q'_*([\mathscr{L}_2^\sigma(X)]) \in H^2(\mathbf{P}_X(\mathscr{U}_X);\mathbf{Z})$ is twice the class of the fiber product $\mathbf{P}_X(\mathscr{U}_X) \times_{\mathbf{P}(V_5)} Y_{A,V_5}$ and, hence, twice the pullback via ρ_1 of the class of Y_{A,V_5}, which is equal to $6\rho_1^*\tilde{h}$. □

Geometrically, this means that, for a general point x of a GM 6-fold X, there are 12 planes passing through x and contained in X.

For every $v \in \mathbf{P}(V_5)$, set

$$Q_v := \pi\big(\rho_1^{-1}(v)\big) = \gamma_X^{-1}\big(\mathbf{P}(v \wedge V_5)\big) \subset X.$$

This is a 3-dimensional quadric.

LEMMA 5.16

Let $y \in \widetilde{Y}_{A,V_5}$, and set $v := f_{A,V_5}(y) \in Y_{A,V_5} \subset \mathbf{P}(V_5)$. We have

$$q_*p^*([\mathbf{P}]) = [Q_v] = \gamma_X^* \boldsymbol{\sigma}_3 \in H^6(X;\mathbf{Z}).$$

Proof

By the proof of Theorem 4.3, the line $\mathbf{P}_y$ parameterizes planes (of one of the two possible types) on the singular quadric Q_v. Since these planes cover the whole quadric and there is a unique such plane through any smooth point of Q_v, the map $q\colon p^{-1}(\mathbf{P}_y) \to Q_v$ is birational; hence, $q_*p^*([\mathbf{P}]) = [Q_v]$. By the definition of Q_v, we have $[Q_v] = \gamma_X^*([\mathbf{P}(v \wedge V_5)]) = \gamma_X^* \boldsymbol{\sigma}_3$. □

COROLLARY 5.17

For any $u \in \mathbf{P}(V_5)$, we have $p_*q^*[Q_u] \cdot [\mathbf{P}] = 2$.

Proof

By adjunction and Lemma 5.16, it is enough to prove $[Q_u] \cdot [Q_v] = 2$ for any distinct $u, v \in \mathbf{P}(V_5)$. Since the quadrics are preimages of the spaces $\mathbf{P}(u \wedge V_5)$ and $\mathbf{P}(v \wedge V_5)$ under the double covering $\gamma_X : X \to \mathsf{Gr}(2, V_5)$, it is enough to show that the intersection of those spaces is 1, that is, that $\boldsymbol{\sigma}_3^2 = 1$, which follows from Schubert calculus. □

LEMMA 5.18

The class $q_*p^*(p_*q^*[Q_u] \cdot \tilde{\sigma}^*\iota^*\tilde{h}^2) \in H^6(X;\mathbf{Z})$ is contained in the subgroup $\gamma_X^*(H^6(\mathsf{Gr}(2,V_5);\mathbf{Z}))$ of $H^6(X;\mathbf{Z})$.

Proof

Using diagram (21), we can rewrite the class in question as

$$q_*p^*\big(p_*q^*[Q_u] \cdot \tilde{\sigma}^*\iota^*\tilde{h}^2\big) = \pi_*\big(q'_*\big(p^*p_*q^*[Q_u]\big) \cdot \rho_1^*\tilde{h}^2\big).$$

The class $q'_*(p^*p_*q^*[Q_u])$ is in $H^4(\mathbf{P}_X(\mathscr{U}_X);\mathbf{Z}) \simeq H^4(X;\mathbf{Z}) \oplus H^2(X;\mathbf{Z}) \cdot \rho_1^*\tilde{h}$; hence, by Proposition 3.4(b), it can be written as a linear combination of the classes π^*h^2, $\pi^*c_2(\mathscr{U}_X)$, and $\pi^*h \cdot \rho_1^*\tilde{h}$. It is thus enough to show that each of these classes multiplied by $\rho_1^*\tilde{h}^2$ and pushed forward to X is in the required subgroup. From (16) and the equality $c_1(\mathscr{U}_X) = -h$, one easily obtains $\pi_*(\pi^*h^2 \cdot \rho_1^*\tilde{h}^2) = h^3$, $\pi_*(\pi^*c_2(\mathscr{U}_X) \cdot \rho_1^*\tilde{h}^2) = h \cdot c_2(\mathscr{U}_X)$, and

$$\pi_*(\pi^*h \cdot \rho_1^*\tilde{h}^3) = h^3 - h \cdot c_2(\mathscr{U}_X).$$

This proves the lemma. □

Consider the morphism

$$\alpha\colon H^6(X;\mathbf{Z}) \longrightarrow H^2\big(F_2^\sigma(X);\mathbf{Z}\big), \quad x \longmapsto p_*(q^*x).$$

The map p is a $\mathbf{P}^2$-fibration for which q^*h is a relative hyperplane class. Therefore, $\mathscr{L}_2^\sigma(X)$ is isomorphic to the projectivization of a rank-3 vector bundle. We denote by c_1, c_2, and c_3 its Chern classes, so that

$$q^*h^3 + p^*c_1 \cdot q^*h^2 + p^*c_2 \cdot q^*h + p^*c_3 = 0. \tag{22}$$

Multiplying by q^*h, one obtains

$$q^*h^4 + p^*(c_2 - c_1^2) \cdot q^*h^2 + p^*(c_3 - c_1c_2) \cdot q^*h - p^*(c_1c_3) = 0. \tag{23}$$

The primitive and vanishing cohomology subgroups $H^2(\widetilde{Y}_A;\mathbf{Z})_0 \subset H^2(\widetilde{Y}_A;\mathbf{Z})$ and $H^6(X;\mathbf{Z})_{00} = H^6(X;\mathbf{Z})_0 \subset H^6(X;\mathbf{Z})$ (they are equal by Lemma 3.8) were defined in (5) and (6).

THEOREM 5.19

Let X be a smooth GM 6-fold, with associated Lagrangian data set (V_6, V_5, A), such that assumption (10) holds. We have

$$\forall x \in H^6(X;\mathbf{Z})_{00} \quad \alpha(x)^2 \cdot c_2 = 12x^2. \tag{24}$$

In particular, the restriction $\alpha_0\colon H^6(X;\mathbf{Z})_{00} \to H^2(F_2^\sigma(X);\mathbf{Z})$ is injective. Furthermore, it induces an isomorphism

$$\beta\colon H^6(X;\mathbf{Z})_{00} \xrightarrow{\sim} H^2(\widetilde{Y}_A;\mathbf{Z})_0$$

compatible with the Beauville–Bogomolov form

$$\forall x \in H^6(X;\mathbf{Z})_{00} \quad q_B\big(\beta(x)\big) = x^2,$$

that is, an isomorphism of polarized Hodge structures.

Proof

We use the same argument as in the proof of Theorem 5.12. Since p is a $\mathbf{P}^2$-bundle and q^*h is a relative hyperplane section, we may write

$$\forall x \in H^6(X;\mathbf{Z}) \quad q^*x = p^*x_1 \cdot q^*h^2 + p^*x_2 \cdot q^*h + p^*x_3,$$

where $x_i \in H^{2i}(F_2^\sigma(X);\mathbf{Z})$ for $i \in \{1,2,3\}$. We then have $\alpha(x) = x_1$. To see what the primitivity of x means, we compute, using (22),

$$\begin{aligned} q^*(x \cdot h) &= p^*x_1 \cdot q^*h^3 + p^*x_2 \cdot q^*h^2 + p^*x_3 \cdot q^*h \\ &= p^*(x_2 - x_1 \cdot c_1) \cdot q^*h^2 + p^*(x_3 - x_1 \cdot c_2) \cdot q^*h - p^*(x_1 \cdot c_3). \end{aligned}$$

Thus, the condition $x \cdot h = 0$ implies that

$$x_1 \cdot c_3 = 0, \qquad x_2 = x_1 \cdot c_1, \qquad x_3 = x_1 \cdot c_2,$$

and we can rewrite as

$$q^*x = p^*\big(\alpha(x)\big) \cdot (q^*h^2 + p^*c_1 \cdot q^*h + p^*c_2).$$

Taking squares and using (22) and (23), we obtain

$$\begin{aligned}(q^*x)^2 &= p^*\big(\alpha(x)^2\big)\cdot\big(q^*h^4 + 2p^*c_1\cdot q^*h^3 + p^*(c_1^2+2c_2)\cdot q^*h^2 \\ &\qquad + 2p^*(c_1c_2)\cdot q^*h + p^*c_2^2\big) \\ &= p^*\big(\alpha(x)^2\big)\cdot\big(p^*c_2\cdot q^*h^2 + p^*(c_1c_2-c_3)\cdot q^*h + p^*(c_2^2-c_1c_3)\big) \\ &= \alpha(x)^2\cdot c_2.\end{aligned}$$

On the other hand, by Lemma 5.15, we have $(q^*x)^2 = 12x^2$. This proves (24). The injectivity of α_0 then follows as in the proof of Theorem 5.12.

Since $x\in H^6(X;\mathbf{Z})_{00}$, we have

$$\alpha(x)\cdot[\mathbf{P}] = x\cdot q_*p^*[\mathbf{P}] = x\cdot[Q_v] = x\cdot\gamma_X^*\boldsymbol{\sigma}_3 = 0.$$

A combination of this equality with Proposition 5.14 shows that there is an injective map $\beta\colon H^6(X;\mathbf{Z})_{00}\to H^2(\widetilde{Y}_A;\mathbf{Z})$ such that

$$\alpha(x) = \tilde{\sigma}^*\big(\iota^*\big(\beta(x)\big)\big).$$

It remains to show that the image of β is in the primitive cohomology $H^2(\widetilde{Y}_A;\mathbf{Z})_0$ and that β is compatible with the Beauville–Bogomolov form.

Let $x\in H^6(X;\mathbf{Z})_{00}$. Set $m := \beta(x)\cdot\tilde{h}^3 = \beta(x)\cdot\iota_*\iota^*\tilde{h}^2 = \iota^*\beta(x)\cdot\iota^*\tilde{h}^2$. Then

$$\alpha(x)\cdot\tilde{\sigma}^*\iota^*\tilde{h}^2 = \tilde{\sigma}^*\iota^*\beta(x)\cdot\tilde{\sigma}^*\iota^*\tilde{h}^2 = m[\mathbf{P}].$$

Multiplying this by $p_*q^*[Q_u]$ and using Corollary 5.17, we obtain

$$\alpha(x)\cdot\big(p_*q^*[Q_u]\cdot\tilde{\sigma}^*\iota^*\tilde{h}^2\big) = 2m.$$

By adjunction, this is equal to $x\cdot q_*p^*(p_*q^*[Q_u]\cdot\tilde{\sigma}^*\iota^*\tilde{h}^2)$, and by Lemma 5.18, the latter is zero since x is in the vanishing cohomology; therefore, $m=0$. This proves $\beta(x)\in H^2(\widetilde{Y}_A;\mathbf{Z})_0$.

To show compatibility with the Beauville–Bogomolov form, we observe

$$12x^2 = \alpha(x)^2c_2 = \tilde{\sigma}^*\iota^*\beta(x)^2c_2 = \beta(x)^2\iota_*\tilde{\sigma}_*c_2.$$

On the other hand, by Proposition 5.23 below, we have $\iota_*\tilde{\sigma}_*c_2 = 6\tilde{h}^2$; hence, by (7),

$$x^2 = \frac{1}{12}\beta(x)^2 6\tilde{h}^2 = \frac{1}{2}\beta(x)^2\tilde{h}^2 = q_B\big(\beta(x)\big).$$

Finally, the lattices $H^6(X;\mathbf{Z})_{00}$ and $H^2(\widetilde{Y}_A;\mathbf{Z})_0$ have the same rank 22 and same discriminant group $(\mathbf{Z}/2\mathbf{Z})^2$ (see Proposition 3.9 for $H^6(X;\mathbf{Z})_{00}$ and [23, (1.0.9)] for $H^2(\widetilde{Y}_A;\mathbf{Z})_0$); hence, the injective isometry β is a bijection. □

5.3. Nearby Lagrangians

Our aim here is to prove Proposition 5.23, which was used in the proof of Theorem 5.19. This section was inspired by [10, Section 6].

We start with some preparation. Let A_1 and A_2 be Lagrangian subspaces in a symplectic vector space $\mathbb{V}$ such that the intersection

$$B := A_1 \cap A_2$$

has codimension 2 in both A_1 and A_2.

LEMMA 5.20
If $\operatorname{codim}_{A_1}(B) = \operatorname{codim}_{A_2}(B) = 2$, *then the Lagrangian subspaces* $A \subset \mathbb{V}$ *such that*

$$\operatorname{codim}_{A_1}(A \cap A_1) = \operatorname{codim}_{A_2}(A \cap A_2) \le 1$$

are parameterized by the line $\mathbf{P}(A_1/B) \simeq \mathbf{P}(A_2/B) \simeq \mathbf{P}^1$. *Moreover, if* $A', A'' \subset \mathbb{V}$ *are two distinct such subspaces, then* $A' \cap A'' = B$.

Proof
Since $B = A_1 \cap A_2$, we have $B^\perp = A_1 + A_2 \subset \mathbb{V}$. The vector space $B^\perp/B$ is symplectic of dimension 4, and for any Lagrangian subspace $A \subset \mathbb{V}$,

$$\bar{A} := (A \cap B^\perp)/(A \cap B)$$

is a Lagrangian subspace in $B^\perp/B$ (called the B-isotropic reduction of A). Since the subspaces $\bar{A}_i = A_i/B$ do not intersect, they give a Lagrangian direct sum decomposition

$$B^\perp/B = \bar{A}_1 \oplus \bar{A}_2.$$

Assume now that $\operatorname{codim}_{A_i}(A \cap A_i) \le 1$. Note that $A \cap A_1 \ne A \cap A_2$ (since otherwise $A_1 \cap A_2$ would be at most 1-codimensional); hence, $A = (A \cap A_1) + (A \cap A_2)$. This implies that $A \subset A_1 + A_2 = B^\perp$; hence, we have $B \subset A^\perp = A$. Thus, $\bar{A} = A/B \subset B^\perp/B$, and in particular, A is determined by the space $\bar{A}$ as its preimage under the linear projection $B^\perp \to B^\perp/B$.

The conditions $\operatorname{codim}_{A_i}(A \cap A_i) \le 1$ imply that the line $\mathbf{P}(\bar{A})$ intersects each skew line $\mathbf{P}(\bar{A}_i)$ in $\mathbf{P}(B^\perp/B)$. Finally, the pairing between $\bar{A}_1$ and $\bar{A}_2$ induced by the symplectic form on $B^\perp/B$ is nondegenerate; hence for every point of $\mathbf{P}(\bar{A}_1)$, there is a unique point in $\mathbf{P}(\bar{A}_2)$ such that the line joining them is Lagrangian. Thus, the set of $\bar{A}$ (and, hence, the set of A as well) is parameterized by either of the lines $\mathbf{P}(\bar{A}_1)$ or $\mathbf{P}(\bar{A}_2)$.

It follows from the above description that the lines $\mathbf{P}(\bar{A})$ form one connected component of the scheme of lines on a smooth quadric in $\mathbf{P}(B^\perp/B)$ (the lines $\mathbf{P}(\bar{A}_1)$ and $\mathbf{P}(\bar{A}_2)$ being in the other component). In particular, two such distinct lines do not intersect; hence, their preimages in $\mathbf{P}(B^\perp)$ intersect along $\mathbf{P}(B)$. This proves the second statement. □

Assume now that the symplectic vector space $\mathbb{V}$ is $\bigwedge^3 V_6$. Let $B \subset \bigwedge^3 V_6$ be an isotropic subspace of dimension 8 (hence, of codimension 2 in any Lagrangian subspace containing it). Set

$$Y_B := \left\{ v \in \mathbf{P}(V_6) \;\middle|\; B \cap \left(v \wedge \textstyle\bigwedge^2 V_6 \right) \ne 0 \right\} \qquad \text{and} \qquad Y_{B,V_5} := Y_B \cap \mathbf{P}(V_5).$$

REMARK 5.21

A parameter count shows $\dim(Y_B) \le 2$ for general B. In fact, this is even true for a general B inside a given Lagrangian subspace which contains no decomposable vectors.

Let $A_1, A_2 \subset \bigwedge^3 V_6$ be Lagrangian subspaces with no decomposable vectors such that $B := A_1 \cap A_2$ has codimension 2 in each of them. Consider the family $\{A_p\}_{p\in \mathbf{P}^1}$ of Lagrangian subspaces discussed in Lemma 5.20, and set

$$\mathscr{Y}^{\ge 2}_{A_1,A_2} := \left\{(v,p) \in \mathbf{P}(V_6) \times \mathbf{P}^1 \;\middle|\; v \in Y^{\ge 2}_{A_p}\right\} \subset \mathbf{P}(V_6) \times \mathbf{P}^1.$$

We denote by $\mathrm{pr}\colon \mathscr{Y}^{\ge 2}_{A_1,A_2} \to \mathbf{P}(V_6)$ the first projection and set

$$\mathscr{Y}^{\ge 2}_{A_1,A_2;V_5} := \mathscr{Y}^{\ge 2}_{A_1,A_2} \times_{\mathbf{P}(V_6)} \mathbf{P}(V_5).$$

LEMMA 5.22

Let $A_1, A_2 \subset \bigwedge^3 V_6$ be Lagrangian subspaces with no decomposable vectors such that $B = A_1 \cap A_2$ has codimension 2 in each of them. If $\dim(Y_B) \le 2$, then we have

$$Y_{A_1} \cap Y_{A_2} = \mathrm{pr}(\mathscr{Y}^{\ge 2}_{A_1,A_2})$$

and the map $\mathrm{pr}\colon \mathscr{Y}^{\ge 2}_{A_1,A_2} \to Y_{A_1} \cap Y_{A_2}$ is an isomorphism over a dense open subset of $Y_{A_1} \cap Y_{A_2}$. Moreover, for any $V_5 \subset V_6$ such that $\dim(Y_{B,V_5}) \le 1$, the map $\mathrm{pr}\colon \mathscr{Y}^{\ge 2}_{A_1,A_2;V_5} \to Y_{A_1,V_5} \cap Y_{A_2,V_5}$ is again an isomorphism over a dense open subset.

Proof

Let us prove $\mathrm{pr}(\mathscr{Y}^{\ge 2}_{A_1,A_2}) \subset Y_{A_1} \cap Y_{A_2}$. If A_p is any of the Lagrangian spaces in the family and $v \in Y^{\ge 2}_{A_p}$, then we have $\dim(A_p \cap (v \wedge \bigwedge^2 V_6)) \ge 2$. But $\mathrm{codim}_{A_i}(A_p \cap A_i) \le 1$; hence, $A_i \cap (v \wedge \bigwedge^2 V_6) \ne 0$, so $v \in Y_{A_i}$ for both $i = 1$ and $i = 2$.

Since $\mathscr{Y}^{\ge 2}_{A_1,A_2}$ is proper, it remains to show that pr is an isomorphism over a dense open subset. Since Y_{A_1} and Y_{A_2} are distinct hypersurfaces in $\mathbf{P}(V_6) = \mathbf{P}^5$, any irreducible component of their intersection has dimension at least 3, so it is enough to show that the map $\mathrm{pr}\colon \mathscr{Y}^{\ge 2}_{A_1,A_2} \to Y_{A_1} \cap Y_{A_2}$ is an isomorphism over the complement of Y_B.

Let $v \in (Y_{A_1} \cap Y_{A_2}) \smallsetminus Y_B$. We first show that v is a smooth point of Y_{A_i}. Assume to the contrary that $v \in Y^{\ge 2}_{A_1}$. Then, $\dim(A_1 \cap (v \wedge \bigwedge^2 V_6)) \ge 2$ but $B \cap (v \wedge \bigwedge^2 V_6) = 0$. It follows that

$$A_1 = B \oplus \left(A_1 \cap \left(v \wedge \textstyle\bigwedge^2 V_6\right)\right)$$

(and, in particular, the second summand is 2-dimensional). On the other hand, take any nonzero $a \in A_2 \cap (v \wedge \bigwedge^2 V_6)$. Then, a is orthogonal to both summands in the above equation (since the first summand is contained in A_2 and the second is contained in $v \wedge \bigwedge^2 V_6$). Therefore, $a \in A_1^\perp = A_1$; hence, $a \in A_1 \cap A_2 = B$ and $v \in Y_B$, a contradiction. The same argument works for A_2 instead of A_1.

We now know that both spaces $A_i \cap (v \wedge \bigwedge^2 V_6)$ are 1-dimensional. If a_1 and a_2 are generators, then their projections to $B^\perp/B$ are linearly independent (otherwise, $v \in Y_B$). Furthermore,

$$A := B \oplus \langle a_1, a_2 \rangle$$

is a Lagrangian subspace in $\bigwedge^3 V_6$ and its intersections with the spaces A_1 and A_2 are both 9-dimensional. Therefore, $A = A_p$ for some $p \in \mathbf{P}^1$, and since $\langle a_1, a_2 \rangle \subset A \cap (v \wedge \bigwedge^2 V_6)$, we obtain $(v,p) \in \mathscr{Y}^{\geq 2}_{A_1,A_2}$ and $v \in \mathrm{pr}(\mathscr{Y}^{\geq 2}_{A_1,A_2})$.

Now let $(v,p) \in \mathscr{Y}^{\geq 2}_{A_1,A_2}$ with $v \notin Y_B$. The space A_p intersects $v \wedge \bigwedge^2 V_6$ away from B; hence, by the second part of Lemma 5.20, p is uniquely determined by v. This means that the map pr is an isomorphism over the complement of Y_B.

Finally, if a hyperplane $V_5 \subset V_6$ satisfies $\dim(Y_{B,V_5}) \leq 1$, then the subset $(Y_{A_1,V_5} \cap Y_{A_2,V_5}) \smallsetminus Y_{B,V_5}$ is dense open in $Y_{A_1,V_5} \cap Y_{A_2,V_5}$ and the map pr is an isomorphism over it. □

Now let X be a smooth special GM 6-fold such that (10) holds, with Lagrangian subspace $A_1 \subset \bigwedge^3 V_6$ and Plücker hyperplane $V_5 \subset V_6$. Since $A_1 \cap \bigwedge^3 V_5 = 0$, the canonical projection $\bigwedge^3 V_6 \to \bigwedge^3 V_6/\bigwedge^3 V_5 \simeq \bigwedge^2 V_5$ induces an isomorphism $A_1 \simeq \bigwedge^2 V_5$.

Recall that $\mathscr{L}_2^\sigma(X) = \mathbf{P}_{F_2^\sigma(X)}(\mathscr{P})$, where $\mathscr{P}$ is a rank-3 vector bundle on $F_2^\sigma(X)$ and the map q in (21) is induced by an embedding of vector bundles $\mathscr{P} \to (\mathbf{C} \oplus \bigwedge^2 V_5) \otimes \mathscr{O}_{F_2^\sigma(X)}$. The composition of the above embedding with the projection to $\bigwedge^2 V_5 \otimes \mathscr{O}_{F_2^\sigma(X)}$ is still a monomorphism of vector bundles (since planes on X do not pass through the vertex of the cone $\mathsf{CGr}(2, V_5)$).

The vector bundle $\mathscr{P}^\vee$ is globally generated by its space $\bigwedge^3 V_5 \simeq \bigwedge^2 V_5^\vee$ of global sections; therefore, for ω general in $\mathbf{P}(\bigwedge^3 V_5)$, the zero-locus in $F_2^\sigma(X)$ of ω viewed as an element of $H^0(F_2^\sigma(X), \mathscr{P}^\vee)$ has dimension 1 and the set of ω such that this dimension jumps has codimension 2 or more. Thus, for a general choice of a line $\mathbf{P}^1 \subset \mathbf{P}(\bigwedge^3 V_5)$, the zero-locus is 1-dimensional for every point $\omega \in \mathbf{P}^1$.

Choose a general codimension 2 subspace $B \subset A_1$ such that

- $X_B := X \times_{\mathbf{P}(\bigwedge^2 V_5)} \mathbf{P}(B)$ is a smooth special 4-fold,
- $\dim(Y_{B,V_5}) \leq 1$,
- for any $\omega \in B^\perp \cap \bigwedge^3 V_5$, the zero-locus of ω in $F_2^\sigma(X)$ is 1-dimensional.

Note that $\dim(B^\perp \cap \bigwedge^3 V_5) = 2$. (The dimension is obviously at least 2, and it is at most 2 since $A_1 \cap \bigwedge^3 V_5 = 0$.) By [5, Proposition 3.14(a)], the Lagrangian subspace of the 4-fold X_B is

$$A_2 := B \oplus \left(B^\perp \cap \bigwedge^3 V_5\right).$$

Each $\omega \in B^\perp \cap \bigwedge^3 V_5$ determines a hyperplane in $\mathbf{P}(\bigwedge^2 V_5)$ containing $\mathbf{P}(B)$. We denote by X_ω the corresponding hyperplane section of X. For all ω, we have inclusions

$$X_B \subset X_\omega \subset X,$$

and every X_ω is a special GM 5-fold which is smooth for general ω. We set

$$\widetilde{D}(B) := \left\{ (\Pi, \omega) \;\middle|\; \Pi \subset X_\omega,\ \omega \in \mathbf{P}\left(B^\perp \cap \bigwedge^3 V_5\right) \right\} \subset F_2^\sigma(X) \times \mathbf{P}^1.$$

Let $\mathrm{pr}\colon \widetilde{D}(B) \to F_2^\sigma(X)$ be the natural projection. It gives a birational map $\widetilde{D}(B) \to D(B)$, where $D(B) \subset F_2^\sigma(X)$ is the degeneracy locus of the morphism of vector bundles $(B^\perp \cap \bigwedge^3 V_5) \otimes \mathscr{O}_{F_2^\sigma(X)} \to \mathscr{P}^\vee$. It satisfies

$$\big[D(B)\big] = c_2(\mathscr{P})$$

in $H^4(F_2^\sigma(X); \mathbf{Z})$.

PROPOSITION 5.23
We have $\iota_* \tilde{\sigma}_* c_2(\mathscr{P}) = 6\tilde{h}^2$ *in* $H^4(\widetilde{Y}_{A_1}; \mathbf{Z})$.

Proof
The above discussion shows that we need to describe $\iota_* \tilde{\sigma}_* [D(B)]$. Let $\mathbf{P}^1_0 \subset \mathbf{P}^1$ be the open subset of those ω such that X_ω is a smooth hyperplane section of X, let $\widetilde{D}_\omega \subset \widetilde{D}(B)$ be the fiber of $\widetilde{D}(B)$ over $\omega \in \mathbf{P}^1$, and let $\widetilde{D}_0 \subset \widetilde{D}(B)$ be the preimage of $\mathbf{P}^1_0$.

Choose any $\omega \in \mathbf{P}^1_0$, and let $A_\omega := A(X_\omega)$ be the Lagrangian subspace associated with X_ω. By [5, Proposition 3.14(a)], we have

$$\dim(A_\omega \cap A_1) = \dim(A_\omega \cap A_2) = 9.$$

This shows that the pencil $\mathbf{P}^1$ is the same as the pencil of Lemma 5.20.

By Theorem 4.3(b), the Stein factorization of the map $\sigma\colon F_2^\sigma(X_\omega) \to \mathbf{P}(V_5)$ is the double covering $\widetilde{Y}^{\ge 2}_{A_\omega, V_5}$ of $Y^{\ge 2}_{A_\omega, V_5} \subset \mathbf{P}(V_5)$. This means that the Stein factorization of the map $\sigma\colon \widetilde{D}_0 \to \mathbf{P}(V_5) \times \mathbf{P}^1_0$ is a double covering of the subscheme

$$\mathscr{Y}^{\ge 2}_{A_1, A_2; V_5} \times_{\mathbf{P}^1} \mathbf{P}^1_0.$$

By Lemma 5.22, its projection to $\mathbf{P}(V_5)$ is birational onto $Y_{A_1, V_5} \cap Y_{A_2, V_5}$.

This means that the composition $\mathrm{pr} \circ \sigma\colon \widetilde{D}(B) \to \mathbf{P}(V_5)$ is generically finite of degree 2 onto $Y_{A_1, V_5} \cap Y_{A_2, V_5}$. Since it factors through $\widetilde{Y}_{A_1, V_5}$, the induced map $\tilde{\sigma}\colon \widetilde{D}(B) \to \widetilde{Y}_{A_1, V_5}$ is either birational onto $\widetilde{Y}_{A_1, V_5} \times_{\mathbf{P}(V_5)} Y_{A_2, V_5}$ or generically surjective of degree 2 onto a section of the double cover

$$\widetilde{Y}_{A_1, V_5} \times_{\mathbf{P}(V_5)} Y_{A_2, V_5} \to Y_{A_1, V_5} \cap Y_{A_2, V_5}.$$

In the first case, we have $\tilde{\sigma}_*[D(B)] = 6\iota^* \tilde{h}$; hence, $\iota_* \tilde{\sigma}_* [D(B)] = 6\tilde{h}^2$. In the second case, we have $\iota_* \tilde{\sigma}_*([D(B)]) + \tau^*_{A_1}(\iota_* \tilde{\sigma}_*([D(B)])) = 12\tilde{h}^2$, where $\tau^*_{A_1}$ is the action on $H^4(\widetilde{Y}_{A_1}; \mathbf{Z})$ of the involution of the double covering f_{A_1}. In the second case, the same arguments used at the end of the proof of Corollary 5.11 show that we also have $\iota_* \tilde{\sigma}_* [D(B)] = 6\tilde{h}^2$. □

5.4. Period points and period maps

In this section, we discuss period points and period maps for smooth GM varieties of dimensions 4 or 6 and for double EPW sextics. We use the notation of

Section 3.3. In particular, we consider the lattices Γ_4, Γ_6, and Λ defined by (8). Consider the automorphism group $O(\Lambda)$ and the *stable orthogonal group*

$$\widetilde{O}(\Lambda) \subset O(\Lambda)$$

of automorphisms of Λ which act trivially on its discriminant group

$$D(\Lambda) = \Lambda^\vee/\Lambda.$$

It has index 2 in $O(\Lambda)$.

Another description of $\widetilde{O}(\Lambda)$ will be important. Consider the even lattice Γ_6. By choosing vectors e_1 and e_2 with square 2 in the first and second copies of U in Γ_6, we obtain a primitive embedding of the lattice $I_{2,0}(2)$ into Γ_6. Furthermore, the group $O(\Gamma_6)$ acts transitively on the set of such embeddings (see [14]). The orthogonal sublattice $\langle e_1, e_2\rangle^\perp \subset \Gamma_6$ is isomorphic to $\Lambda(-1)$. (It is even, of signature $(2, 20)$, with discriminant group $(\mathbf{Z}/2\mathbf{Z})^2$.)

The subgroup $O(\Gamma_6)_{\langle e_1,e_2\rangle} \subset O(\Gamma_6)$ stabilizing the sublattice $\langle e_1, e_2\rangle$ preserves the orthogonal $\Lambda(-1)$. This defines a map $O(\Gamma_6)_{\langle e_1,e_2\rangle} \to O(\Lambda)$ which is surjective, and the stable group $\widetilde{O}(\Lambda)$ is the isomorphic image under this map of the subgroup $O(\Gamma_6)_{e_1,e_2} \subset O(\Gamma_6)_{\langle e_1,e_2\rangle}$ of elements stabilizing both e_1 and e_2.

Analogously, in the lattice Γ_4, there are vectors e_1 and e_2 generating a sublattice isomorphic to $I_{2,0}(2)$ such that $e_1 + e_2$ is characteristic in Γ_4. Again by [14], the group $O(\Gamma_4)$ acts transitively on the set of such embeddings, the orthogonal $\langle e_1, e_2\rangle^\perp$ is isomorphic to Λ, and there are morphisms $O(\Gamma_4)_{\langle e_1,e_2\rangle} \twoheadrightarrow O(\Lambda)$ and $O(\Gamma_4)_{e_1,e_2} \xrightarrow{\sim} \widetilde{O}(\Lambda)$ (see [4, Section 5.1] for details).

The groups $\widetilde{O}(\Lambda)$ and $O(\Lambda)$ act properly and discontinuously on the complex variety

$$\Omega := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid \omega \cdot \omega = 0,\ \omega \cdot \bar{\omega} < 0\}. \tag{25}$$

The quotient

$$\mathscr{D} := \widetilde{O}(\Lambda)\backslash\Omega$$

is a quasiprojective 20-dimensional variety. It has a canonical involution $r_{\mathscr{D}}$, associated with the further degree-2 quotient $\mathscr{D} \to O(\Lambda)\backslash\Omega$.

PROPOSITION 5.24

Let X be a smooth GM variety of dimension $n = 4$ or 6. The 1-dimensional subspace $H^{n/2+1,n/2-1}(X) \subset H^n(X, \mathbf{C})$ gives rise to a well-defined point in $\mathscr{D}$.

This point is called the *period point* of X and will be denoted by $\wp(X)$.

Proof

Assume first that $n = 4$. The abelian group $H^4(\mathsf{Gr}(2, V_5); \mathbf{Z})$ is generated by the Schubert classes $\boldsymbol{\sigma}_{1,1}$ and $\boldsymbol{\sigma}_2$. By [4, Section 5.1], there exists an isometry $\phi\colon H^4(X; \mathbf{Z}) \xrightarrow{\sim} \Gamma_4$, called a *marking* of X, such that

$$\phi^{-1}(e_1) = \gamma_X^* \boldsymbol{\sigma}_{1,1} \qquad \text{and} \qquad \phi^{-1}(e_2) = \gamma_X^* \boldsymbol{\sigma}_2 - \gamma_X^* \boldsymbol{\sigma}_{1,1}, \tag{26}$$

where $e_1, e_2 \in \Gamma_4$ were defined above. Any two markings differ by the action of an element of the group $O(\Gamma_4)_{e_1,e_2} \simeq \widetilde{O}(\Lambda)$. The marking carries the vanishing cohomology lattice $H^4(X;\mathbf{Z})_{00}$ (defined in (5)) onto the orthogonal $\langle e_1, e_2\rangle^\perp \simeq \Lambda$. Its complexification $\phi_{\mathbf{C}}\colon H^4(X;\mathbf{C}) \xrightarrow{\sim} \Gamma_4 \otimes \mathbf{C}$ takes the 1-dimensional subspace $H^{3,1}(X)$ (see Proposition 3.1), which is orthogonal to $\gamma_X^* H^4(\mathsf{Gr}(2,V_5);\mathbf{C})$, to a point in the manifold Ω defined in (25). The equivalence class of this point in the quotient $\mathscr{D} = \widetilde{O}(\Lambda)\backslash\Omega$ is well defined.

The situation when $n = 6$ is similar: the abelian group $H^6(\mathsf{Gr}(2,V_5);\mathbf{Z})$ is generated by $\boldsymbol{\sigma}_{2,1}$ and $\boldsymbol{\sigma}_3$, there exists a marking $\phi\colon H^6(X;\mathbf{Z}) \xrightarrow{\sim} \Gamma_6$ such that

$$\phi^{-1}(e_1) = \gamma_X^* \boldsymbol{\sigma}_{2,1} \qquad \text{and} \qquad \phi^{-1}(e_2) = \gamma_X^* \boldsymbol{\sigma}_3, \tag{27}$$

where again $e_1, e_2 \in \Gamma_6$ were defined above, and any two markings differ by the action of an element of the group $O(\Gamma_6)_{e_1,e_2} \simeq \widetilde{O}(\Lambda)$. The marking carries the vanishing cohomology lattice $H^6(X;\mathbf{Z})_{00}$ (defined in (5)) onto the orthogonal $\langle e_1, e_2\rangle^\perp \simeq \Lambda(-1)$, and its complexification $\phi_{\mathbf{C}}$ takes the 1-dimensional subspace $H^{4,2}(X)$ (see Proposition 3.1) to a point in the same domain Ω (note that the *anti-isometry* property of $\phi_{\mathbf{C}}$ is offset by the change in sign in the Hodge–Riemann relations for a $(4,2)$-class on a 6-fold in comparison with a $(3,1)$-class on a 4-fold) whose equivalence class in $\mathscr{D}$ is well defined. □

REMARK 5.25

If, in the above construction of the period point, we replace the conditions (26) and (27) by similar conditions with e_1 and e_2 exchanged, we obtain a new period point which is $r_{\mathscr{D}}(\wp(X))$. (This is because there is an element of $O(\Gamma_n)_{\langle e_1,e_2\rangle}$ which exchanges e_1 and e_2, and the image of this isometry by the surjection $O(\Gamma_n)_{\langle e_1,e_2\rangle} \twoheadrightarrow O(\Lambda)$ is not in $\widetilde{O}(\Lambda)$.)

An analogous construction can be made in another situation: if $\widetilde{Y}_A$ is a smooth double EPW sextic, the 1-dimensional subspace $H^{2,0}(\widetilde{Y}_A) \subset H^2(\widetilde{Y}_A;\mathbf{C})_0$ gives rise to a period point $\wp^{\mathrm{EPW}}(\widetilde{Y}_A)$ in the *same* variety $\mathscr{D}$ (see [26, Section 4.2]). This period point may also be defined for all Lagrangian subspaces A with no decomposable vectors (i.e., even when $Y_A^{\geq 3} \neq \varnothing$; see [26, Section 5.1]). The main result of [23] is $\wp^{\mathrm{EPW}}(\widetilde{Y}_{A^\perp}) = r_{\mathscr{D}}(\wp^{\mathrm{EPW}}(\widetilde{Y}_A))$.

LEMMA 5.26

For any smooth GM variety X of dimension 4 or 6, with associated Lagrangian subspace $A(X)$ satisfying (10), one has

$$\text{either } \wp(X) = \wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)}) \text{ or } \wp(X) = r_{\mathscr{D}}(\wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)})).$$

Proof

Consider first the case of 4-folds. Choose markings ϕ for X and ψ for $\widetilde{Y}_{A(X)}$, and consider the commutative diagram

$$\begin{array}{ccc} H^4(X;\mathbf{Z})_{00}(-1) & \xrightarrow{\ \beta\ } & H^2(\widetilde{Y}_{A(X)};\mathbf{Z})_0 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Lambda & \xrightarrow{\ \psi\circ\beta\circ\phi^{-1}\ } & \Lambda \end{array}$$

where β is the isomorphism of Theorem 5.12. Since β is compatible with polarizations, the bottom map $g := \psi \circ \beta \circ \phi^{-1}$ is in $O(\Lambda)$. Since β is a morphism of Hodge structures, we have

$$g\big(\phi_{\mathbf{C}}\big(H^{3,1}(X)\big)\big) = \psi_{\mathbf{C}}(H^{2,0}(\widetilde{Y}_{A(X)}).$$

If $g \in \widetilde{O}(\Lambda)$, then we have $\wp(X) = \wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)})$; otherwise $\wp(X) = r_{\mathscr{D}}(\wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)}))$.

For 6-folds, we use the same argument with the isomorphism β of Theorem 5.19. ☐

PROPOSITION 5.27

Either, for any smooth GM variety X of dimension 4 (resp., 6) whose associated double EPW sextic $\widetilde{Y}_{A(X)}$ is smooth, one has $\wp(X) = \wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)})$, or for any such variety X, one has $\wp(X) = r_{\mathscr{D}}(\wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)}))$.

Proof

Let $\mathscr{X} \to S$ be a smooth family of GM varieties of dimension 4 (resp., 6) over an irreducible base S such that every GM variety of dimension 4 (resp., 6) is isomorphic to some fiber of that family (see the proof of [16, Proposition 3.4] for a construction of such a family). It is classical that the period point construction defines a *period map* $\wp_S : S \to \mathscr{D}$ which is algebraic.

By [8], we have a family of Lagrangian data sets $(\mathscr{V}_6, \mathscr{V}_5, \mathscr{A})$, where $\mathscr{V}_6$ is a rank-6 vector bundle on S, $\mathscr{V}_5 \subset \mathscr{V}_6$ is a rank-5 vector subbundle, and $\mathscr{A} \subset \bigwedge^3 \mathscr{V}_6$ is a Lagrangian subbundle. We choose an open covering (S_α) of S such that these vector bundles are all trivial on each S_α. Refining further the covering and applying [25, Proposition 3.1], we construct, for each α, a family $\widetilde{\mathscr{Y}_\alpha} \to S_\alpha$ of (possibly singular) double EPW sextics. These families define period maps which fit together to define an algebraic map $\wp^{\mathrm{EPW}}_{S^0} : S^0 \to \mathscr{D}$, where $S^0 \subset S$ is the dense open subset where the double EPW sextics are smooth.

Since $\mathscr{D}$ is separated, the sets

$$S^0_1 := \big\{ s \in S^0 \;\big|\; \wp_S(s) = \wp^{\mathrm{EPW}}_{S^0}(s) \big\} \qquad \text{and}$$
$$S^0_2 := \big\{ s \in S^0 \;\big|\; \wp_S(s) = r_{\mathscr{D}} \circ \wp^{\mathrm{EPW}}_{S^0}(s) \big\}$$

are closed in S^0. By Lemma 5.26, the dense subset $S^{00} \subset S^0$ corresponding to smooth GM 4-folds (resp., 6-folds) satisfying (10) is the union of its closed subsets $S^0_1 \cap S^{00}$ and $S^0_2 \cap S^{00}$. Since S^{00} is irreducible, one of them, say, S^0_i, is S^{00}. This means that S^0_i contains S^{00} and, hence, its closure S^0, and this proves the proposition. ☐

To go from one of the possibilities of the proposition to the other, it suffices to change the convention defining the period point $\wp(X)$ (and the period map) as explained in Remark 5.25. We may therefore assume that

$$\wp(X) = \wp^{\mathrm{EPW}}(\widetilde{Y}_{A(X)}) \tag{28}$$

holds for any smooth GM 4-fold or 6-fold with smooth $\widetilde{Y}_{A(X)}$. This implies Theorem 5.1 in full generality.

We end this section with some consequences of (28) based on results from [5] and [4].

REMARK 5.28 (Period partners)

In [5, Section 3.6], we said that smooth GM varieties of the same dimension are *period partners* if they are constructed from the same Lagrangian subspace $A \subset \bigwedge^3 V_6$ (with no decomposable vectors) but possibly different hyperplanes $V_5 \subset V_6$. By Theorem 5.1, period partners of dimensions 4 or 6 have the same period point.

Conversely, since double EPW sextics have the same period point if and only if they are isomorphic (see [26, Theorem 1.3]), smooth GM 4-folds (or 6-folds) are period partners if and only if they have the same period point. By [5, Theorem 3.25], isomorphism classes of period partners of a GM 4-fold are parameterized by $Y^1_{A^\perp} \sqcup Y^2_{A^\perp}$, modulo the finite group $\mathrm{Aut}(Y_{A^\perp})$. (For A general, $Y^{\geq 3}_{A^\perp}$ is empty and $Y^1_{A^\perp} \sqcup Y^2_{A^\perp} = Y_{A^\perp}$.) Similarly, isomorphism classes of period partners of a GM 6-fold are parameterized by $\mathbf{P}(V_6^\vee) \smallsetminus Y_{A^\perp}$, modulo $\mathrm{Aut}(Y_{A^\perp})$.

REMARK 5.29 (Hodge-special GM varieties)

Pretending that smooth GM varieties have coarse moduli spaces (see [8]), we go, following [4], through some geometrically defined subvarieties of these moduli spaces and discuss, using the period map, their relation with some natural divisors in the period domain $\mathscr{D}$. We use the notation introduced in [4].

- *Smooth GM 4-folds containing σ-planes* (see [4, Section 7.1]). They form a codimension-2 family $\mathscr{X}_{\sigma\text{-planes}}$ whose period points cover a divisor $\mathscr{D}''_{10} \subset \mathscr{D}$. A smooth GM 4-fold X contains a σ-plane if and only if $Y^{\geq 3}_{A,V_5} \neq \varnothing$ (see Theorem 4.3(c)). In particular, $Y^{\geq 3}_{A} \neq \varnothing$; this means that A is in the O'Grady divisor Δ (see [26, (2.2.3)]) and implies that $\overline{\wp^{\mathrm{EPW}}(\Delta)} = \mathscr{D}''_{10}$.

If A is general in Δ, then the set $Y^{\geq 3}_{A}$ is just one point v (see [21, Section 5.4]). The condition $Y^{\geq 3}_{A,V_5} \neq \varnothing$ is then equivalent to $v \in V_5$, that is, to $\mathbf{p}_X \in v^\perp$. Thus, a general fiber of the period map $\mathscr{X}_{\sigma\text{-planes}} \to \mathscr{D}''_{10}$ is equal to the hyperplane section of $Y_{A^\perp}$ defined by $v^\perp$ (modulo automorphisms). This fiber was also described in [4, Section 7.1] as a $\mathbf{P}^1$-bundle over a degree-10 K3 surface.

- *Smooth GM 4-folds containing τ-quadratic surfaces* (see [4, Section 7.3]). They form a codimension 1 family $\mathscr{X}_{\tau\text{-quadrics}}$ whose period points cover the divisor $\mathscr{D}'_{10} = r_{\mathscr{D}}(\mathscr{D}''_{10}) \subset \mathscr{D}$. A general fiber of the period map $\mathscr{X}_{\tau\text{-quadrics}} \to \mathscr{D}'_{10}$ is, on the one hand, isomorphic to $Y_{A^\perp}$ (modulo automorphisms) and, on the other hand, birationally isomorphic to the quotient by an involution of the symmetric square of a K3 surface (see [4, Section 7.3]). This fits with [25, Corollary 3.12 and Theorem 4.15]: a desingularization of $\widetilde{Y}_{A^\perp}$ is the symmetric square of a K3 surface.

• *Smooth GM* 4*-folds containing a cubic scroll* (see [4, Section 7.4]). They form a codimension 1 family which contains the 3-codimensional family of smooth GM 4-folds containing a τ-plane (called a ρ-plane in [4]), and the period points of both families cover an $r_{\mathscr{D}}$-invariant divisor $\mathscr{D}_{12} \subset \mathscr{D}$. By Theorem 4.5(c), the condition to contain a τ-plane implies that $Z_A^{\geq 4} \neq \varnothing$. The divisor $\mathscr{D}_{12}$ is therefore contained in the closure of the image by the period map $\wp^{\mathrm{EPW}}$ of the locus of Lagrangian subspaces A such that $Z_A^{\geq 4} \neq \varnothing$; since this locus is an irreducible divisor (see [12, Lemma 3.6]), they are equal.

• *Singular GM* 4*-folds* (see [4, Section 7.6]). The O'Grady divisor Σ (see [26]) corresponds to Lagrangian subspaces A containing decomposable vectors. The corresponding period points (under a suitable extension of the period map $\wp^{\mathrm{EPW}}$ discussed in [26]) fill out a divisor $\mathbb{S}_2^\star \subset \mathscr{D}$ (see [26, (4.3.3) and Proposition 4.12]); this is the $r_{\mathscr{D}}$-stable divisor $\mathscr{D}_8$ of [4], which corresponds to periods of nodal GM 4-folds (see [4, Section 7.6]).

• *Smooth GM* 6*-folds containing a* $\mathbf{P}^3$. By Theorem 4.2, a smooth GM 6-fold contains a $\mathbf{P}^3$ if and only if $Y_{A,V_5}^{\geq 3} \neq \varnothing$. In particular, A is in Δ, and the period point is in $\mathscr{D}''_{10}$. As above, when A is general in Δ, one has $Y_A^{\geq 3} = \{v\}$, and the condition $Y_{A,V_5}^{\geq 3} \neq \varnothing$ is equivalent to $\mathbf{p}_X \in v^\perp$. Thus, the fiber of the period map is equal to the hyperplane section of $\mathbf{P}(V_6^\vee) \smallsetminus Y_{A^\perp}$ defined by $v^\perp$ (modulo automorphisms), and the codimension of the family of GM 6-folds containing a $\mathbf{P}^3$ is 2.

Appendix A: Linear spaces on families of quadrics

Let S be a base scheme which we assume to be Cohen–Macaulay and irreducible. Let $\mathscr{E}$ be a vector bundle on S of rank m, and let $\mathscr{L} \subset \mathsf{S}^2\mathscr{E}^\vee$ be a line subbundle. Consider the projectivization $\mathrm{pr}\colon \mathbf{P}_S(\mathscr{E}) \to S$ and the relative line bundle $\mathscr{O}(1)$ on $\mathbf{P}_S(\mathscr{E})$. Let $\mathscr{Q} \subset \mathbf{P}_S(\mathscr{E})$ be the family of quadrics defined as the zero-locus of the section of the line bundle $\mathrm{pr}^*\mathscr{L}^\vee \otimes \mathscr{O}(2)$ corresponding to the morphism $\mathscr{L} \hookrightarrow \mathsf{S}^2\mathscr{E}^\vee$ via the isomorphism

$$H^0\big(\mathbf{P}_S(\mathscr{E}), \mathrm{pr}^*\mathscr{L}^\vee \otimes \mathscr{O}(2)\big) \simeq H^0(S, \mathscr{L}^\vee \otimes \mathsf{S}^2\mathscr{E}^\vee) \simeq \mathrm{Hom}(\mathscr{L}, \mathsf{S}^2\mathscr{E}^\vee).$$

We denote by $D_c(\mathscr{Q}) \subset S$ the corank-c degeneracy locus of the induced map $\mathscr{E} \otimes \mathscr{L} \to \mathscr{E}^\vee$ of vector bundles and denote by $\mathscr{C}$ the cokernel sheaf of this map; it is supported on $D_1(\mathscr{Q})$.

In this appendix, we discuss the relative Hilbert scheme

$$F_k(\mathscr{Q}) := \mathrm{Hilb}^{\mathbf{P}^k}(\mathscr{Q}/S).$$

We concentrate on the cases $k \in \{1, 2\}$ (i.e., on the Hilbert schemes of lines and planes) and describe the Stein factorization of the morphism $\varphi\colon F_k(\mathscr{Q}) \to S$. Note that $F_k(\mathscr{Q})$ is a subscheme in the relative Grassmannian $\pi\colon \mathsf{Gr}_S(k+1,\mathscr{E}) \to S$. We denote by $\mathscr{U}$ the tautological subbundle of rank $k+1$ on $\mathsf{Gr}_S(k+1,\mathscr{E})$.

PROPOSITION A.1

Assume that $D_{m-2}(\mathscr{Q}) \neq S$. We have a resolution

$$0 \to \mathscr{L}^{\otimes 3} \otimes \big(\det(\mathscr{U})\big)^{\otimes 3} \to \mathscr{L}^{\otimes 2} \otimes \mathsf{S}^2\mathscr{U} \otimes \det(\mathscr{U}) \\ \to \mathscr{L} \otimes \mathsf{S}^2\mathscr{U} \to \mathscr{O}_{\mathsf{Gr}_S(2,\mathscr{E})} \to \mathscr{O}_{F_1(\mathscr{Q})} \to 0$$

on $\mathsf{Gr}_S(2,\mathscr{E})$. Moreover, the pushforward to S of $\mathscr{O}_{F_1(\mathscr{Q})}$ is given as

- *if $m=3$, then $\varphi_*\mathscr{O}_{F_1(\mathscr{Q})} \simeq \mathscr{O}_{D_1(\mathscr{Q})} \oplus (\mathscr{C} \otimes \mathscr{L} \otimes \det(\mathscr{E}))$;*
- *if $m=4$, then $\varphi_*\mathscr{O}_{F_1(\mathscr{Q})} \simeq \mathscr{O}_S \oplus (\mathscr{L}^{\otimes 2} \otimes \det(\mathscr{E}))$;*
- *if $m \geq 5$, then $\varphi_*\mathscr{O}_{F_1(\mathscr{Q})} \simeq \mathscr{O}_S$.*

Proof

Since $F_1(\mathscr{Q})$ is the zero-locus of a section of the rank-3 vector bundle $\mathscr{L}^\vee \otimes \mathsf{S}^2\mathscr{U}^\vee$ on $\mathsf{Gr}_S(2,\mathscr{E})$, its codimension is at most 3. On the other hand, for a quadric of rank r in $\mathbf{P}^{m-1}$, the dimension of the Hilbert scheme of lines is equal to $2m-7$ for $r \geq 3$ and to $2m-6$ for $r \leq 2$. Stratifying $F_1(\mathscr{Q})$ by the preimages of the subsets $S \smallsetminus D_{m-2}(\mathscr{Q})$ and $D_{m-2}(\mathscr{Q})$, we see that the codimension of the first stratum is 3 and the codimension of the second stratum is $\operatorname{codim}(D_{m-2}(\mathscr{Q}))+2$. Since $D_{m-2}(\mathscr{Q}) \neq S$, the codimension of $F_1(\mathscr{Q})$ is at least 3. Since $\mathsf{Gr}_S(2,\mathscr{E})$ is Cohen–Macaulay, the section of $\mathscr{L}^\vee \otimes \mathsf{S}^2\mathscr{U}^\vee$ defining $F_1(\mathscr{Q})$ is regular, and the Koszul complex provides a resolution of its structure sheaf. A standard description of the exterior powers of a symmetric square (see [27, Proposition 2.3.9]) gives the above explicit form.

For the second part, we apply the Borel–Bott–Weil theorem to compute the derived pushforwards to S of the terms of the Koszul complex. The result is

$$R^\bullet\pi_*\mathscr{O}_{\mathsf{Gr}_S(2,\mathscr{E})} \simeq \mathscr{O}_S,$$

$$R^i\pi_*\mathsf{S}^2\mathscr{U} \simeq \begin{cases} \det(\mathscr{E}) \otimes \mathscr{E}^\vee & \text{if } m=3 \text{ and } i=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$R^i\pi_*\big(\mathsf{S}^2\mathscr{U} \otimes \det(\mathscr{U})\big) \simeq \begin{cases} \det(\mathscr{E}) \otimes \mathscr{E} & \text{if } m=3 \text{ and } i=1,\\ \det(\mathscr{E}) & \text{if } m=4 \text{ and } i=2,\\ 0 & \text{otherwise,}\end{cases}$$

$$R^i\pi_*\big(\big(\det(\mathscr{U})\big)^{\otimes 3}\big) \simeq \begin{cases} (\det(\mathscr{E}))^{\otimes 2} & \text{if } m=3 \text{ and } i=2,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, when $m=3$, the first page of the spectral sequence for the pushforward of the Koszul complex is

$$\begin{array}{cccc} \mathscr{L}^{\otimes 3}\otimes(\det(\mathscr{E}))^{\otimes 2} & 0 & 0 & 0 \\ 0 & \mathscr{L}^{\otimes 2}\otimes\det(\mathscr{E})\otimes\mathscr{E} & \longrightarrow \mathscr{L}\otimes\det(\mathscr{E})\otimes\mathscr{E}^{\vee} & 0 \\ 0 & 0 & 0 & \mathscr{O}_S \end{array}$$

(the arrow is the unique nontrivial differential d_1) and converges to $\varphi_*\mathscr{O}_{F_1(\mathscr{Q})}$. It follows that there is an exact sequence

$$0 \to \big(\mathscr{L}^{\otimes 3}\otimes\big(\det(\mathscr{E})\big)^{\otimes 2}\big)\oplus\big(\mathscr{L}^{\otimes 2}\otimes\det(\mathscr{E})\otimes\mathscr{E}\big) \xrightarrow{\alpha} \big(\mathscr{L}\otimes\det(\mathscr{E})\otimes\mathscr{E}^{\vee}\big)\oplus\mathscr{O}_S \to \varphi_*\mathscr{O}_{F_1(\mathscr{Q})} \to 0.$$

The map α is the direct sum of a twist of the map $\alpha_1\colon \mathscr{L}\otimes\mathscr{E}\to\mathscr{E}^{\vee}$ and of a twist of its determinant $\alpha_0\colon \mathscr{L}^{\otimes 3}\otimes(\det(\mathscr{E}))^{\otimes 2}\to\mathscr{O}_S$. The cokernel of α_1 is $\mathscr{C}$, and the cokernel of α_0 is the structure sheaf of the degeneracy locus $D_1(\mathscr{Q})$. This gives the result for $m=3$.

For $m=4$, the pushforward of the Koszul complex gives an isomorphism $\varphi_*\mathscr{O}_{F_1(\mathscr{Q})}\simeq\mathscr{O}_S\oplus(\mathscr{L}^{\otimes 2}\otimes\det(\mathscr{E}))$, and for $m\geq 5$, just $\varphi_*\mathscr{O}_{F_1(\mathscr{Q})}\simeq\mathscr{O}_S$. □

For $k=2$, the computation is analogous, but more complicated, since the Koszul complex is longer. We denote by $\Sigma^{a,b,c}\mathscr{U}$ the Schur functor of the rank-3 tautological subbundle $\mathscr{U}$ on $\mathsf{Gr}_S(3,\mathscr{E})$ corresponding to the highest weight (a,b,c) of the group GL_3. We also consider the composition

$$\mathscr{L}\otimes\mathscr{E}\otimes\mathscr{E}^{\vee}\longrightarrow\mathscr{E}^{\vee}\otimes\mathscr{E}^{\vee}\longrightarrow\textstyle\bigwedge^2\mathscr{E}^{\vee},$$

where the first map is given by the family of quadrics and the second is canonical. Denote by $\mathscr{C}_2$ its cokernel sheaf; it is supported on the degeneracy locus $D_2(\mathscr{Q})$.

PROPOSITION A.2

Assume that $D_{m-4}(\mathscr{Q})\neq S$ *and* $\operatorname{codim}(D_{m-2}(\mathscr{Q}))\geq 3$. *There is a resolution*

$$\begin{aligned} 0 &\to \mathscr{L}^{\otimes 6}\otimes\Sigma^{4,4,4}\mathscr{U}\to\mathscr{L}^{\otimes 5}\otimes\Sigma^{4,4,2}\mathscr{U}\to\mathscr{L}^{\otimes 4}\otimes\Sigma^{4,3,1}\mathscr{U} \\ &\to(\mathscr{L}^{\otimes 3}\otimes\Sigma^{4,1,1}\mathscr{U})\oplus(\mathscr{L}^{\otimes 3}\otimes\Sigma^{3,3,0}\mathscr{U}) \\ &\to\mathscr{L}^{\otimes 2}\otimes\Sigma^{3,1,0}\mathscr{U}\to\mathscr{L}\otimes\mathsf{S}^2\mathscr{U}\to\mathscr{O}_{\mathsf{Gr}_S(3,\mathscr{E})}\to\mathscr{O}_{F_2(\mathscr{Q})}\to 0 \end{aligned}$$

on $\mathsf{Gr}_S(3,\mathscr{E})$. *Moreover, the pushforward to* S *of* $\mathscr{O}_{F_2(\mathscr{Q})}$ *is given as*

- *if* $m=4$, *then we have* $\varphi_*\mathscr{O}_{F_2(\mathscr{Q})}\simeq\mathscr{O}_{D_2(\mathscr{Q})}\oplus(\mathscr{C}_2\otimes\mathscr{L}\otimes\det(\mathscr{E}))$;
- *if* $m=5$, *then we have* $\varphi_*\mathscr{O}_{F_2(\mathscr{Q})}\simeq\mathscr{O}_{D_1(\mathscr{Q})}\oplus(\mathscr{C}\otimes\mathscr{L}^{\otimes 2}\otimes\det(\mathscr{E}))$;
- *if* $m=6$, *then we have* $\varphi_*\mathscr{O}_{F_2(\mathscr{Q})}\simeq\mathscr{O}_S\oplus(\mathscr{L}^{\otimes 3}\otimes\det(\mathscr{E}))$;
- *if* $m\geq 7$, *then we have* $\varphi_*\mathscr{O}_{F_2(\mathscr{Q})}\simeq\mathscr{O}_S$.

Proof

By definition, $F_2(\mathscr{Q})$ is the zero-locus of a section of the rank-6 vector bundle $\mathscr{L}^{\vee}\otimes\mathsf{S}^2\mathscr{U}^{\vee}$ on $\mathsf{Gr}_S(3,\mathscr{E})$, so its codimension is at most 6. On the other hand, for a quadric of rank r in $\mathbf{P}^{m-1}$, the dimension of the Hilbert scheme of planes is equal to $3m-15$ for $r \geq 5$, $3m-14$ for $4 \geq r \geq 3$, and $3m-12$ for $r \leq 2$. Thus, stratifying $F_2(\mathscr{Q})$ by the subsets $S \smallsetminus D_{m-4}(\mathscr{Q})$, $D_{m-4}(\mathscr{Q}) \smallsetminus D_{m-2}(\mathscr{Q})$, and $D_{m-2}(\mathscr{Q})$, we see that, under our assumption, the codimension is 6, the section of $\mathscr{L}^\vee \otimes \mathsf{S}^2 \mathscr{U}^\vee$ defining $F_2(\mathscr{Q})$ is regular, and the Koszul complex provides a resolution of its structure sheaf. A standard description of the exterior powers of a symmetric square (see [27, Proposition 2.3.9]) gives the above explicit form.

For the second part, we apply the Borel–Bott–Weil theorem to compute the derived pushforwards to S of the terms of the Koszul complex. The result is

$$R^\bullet \pi_* \mathscr{O}_{\mathsf{Gr}_S(2,\mathscr{E})} \simeq \mathscr{O}_S,$$

$$R^i \pi_* \mathsf{S}^2 \mathscr{U} \simeq \begin{cases} \det(\mathscr{E}) \otimes \bigwedge^2 \mathscr{E}^\vee & \text{if } m=4 \text{ and } i=1, \\ 0 & \text{otherwise,} \end{cases}$$

$$R^i \pi_* \Sigma^{3,1,0} \mathscr{U} \simeq \begin{cases} \det(\mathscr{E}) \otimes ((\mathscr{E} \otimes \mathscr{E}^\vee)/\mathscr{O}_S) & \text{if } m=4 \text{ and } i=1, \\ \det(\mathscr{E}) \otimes \mathscr{E}^\vee & \text{if } m=5 \text{ and } i=2, \\ 0 & \text{otherwise,} \end{cases}$$

$$R^i \pi_* \Sigma^{3,3,0} \mathscr{U} \simeq \begin{cases} (\det(\mathscr{E}))^{\otimes 2} \otimes \mathsf{S}^2 \mathscr{E}^\vee & \text{if } m=4 \text{ and } i=2, \\ 0 & \text{otherwise,} \end{cases}$$

$$R^i \pi_* \Sigma^{4,1,1} \mathscr{U} \simeq \begin{cases} \det(\mathscr{E}) \otimes \mathsf{S}^2 \mathscr{E} & \text{if } m=4 \text{ and } i=1, \\ \det(\mathscr{E}) \otimes \mathscr{E} & \text{if } m=5 \text{ and } i=2, \\ \det(\mathscr{E}) & \text{if } m=6 \text{ and } i=3, \\ 0 & \text{otherwise,} \end{cases}$$

$$R^i \pi_* \Sigma^{4,3,1} \mathscr{U} \simeq \begin{cases} (\det(\mathscr{E}))^{\otimes 2} \otimes ((\mathscr{E} \otimes \mathscr{E}^\vee)/\mathscr{O}_S) & \text{if } m=4 \text{ and } i=2, \\ 0 & \text{otherwise,} \end{cases}$$

$$R^i \pi_* \Sigma^{4,4,2} \mathscr{U} \simeq \begin{cases} (\det(\mathscr{E}))^{\otimes 2} \otimes \bigwedge^2 \mathscr{E} & \text{if } m=4 \text{ and } i=2, \\ (\det(\mathscr{E}))^{\otimes 2} & \text{if } m=5 \text{ and } i=4, \\ 0 & \text{otherwise,} \end{cases}$$

$$R^i \pi_* \Sigma^{4,4,4} \mathscr{U} \simeq \begin{cases} (\det(\mathscr{E}))^{\otimes 3} & \text{if } m=4 \text{ and } i=3, \\ 0 & \text{otherwise.} \end{cases}$$

Writing down the spectral sequence for the pushforward of the Koszul complex as in the proof of Proposition A.1, we obtain, when $m=4$, the exact sequence

$$\begin{aligned} \cdots \to \big(\mathscr{L}^{\otimes 3} \otimes \big(\det(\mathscr{E})\big)^{\otimes 2} \otimes \mathsf{S}^2 \mathscr{E}^\vee\big) \oplus \big(\mathscr{L}^{\otimes 2} \otimes \det(\mathscr{E}) \otimes \big((\mathscr{E} \otimes \mathscr{E}^\vee)/\mathscr{O}_S\big)\big) \\ \xrightarrow{\alpha} \Big(\mathscr{L} \otimes \det(\mathscr{E}) \otimes \textstyle\bigwedge^2 \mathscr{E}^\vee\Big) \oplus \mathscr{O}_S \to \varphi_* \mathscr{O}_{F_2(\mathscr{Q})} \to 0. \end{aligned}$$

The map α is the direct sum of a twist of the map

$$\alpha_1 \colon \mathscr{L} \otimes ((\mathscr{E} \otimes \mathscr{E}^\vee)/\mathscr{O}_S) \to \textstyle\bigwedge^2 \mathscr{E}^\vee$$

and of the exterior cube of the family of quadrics

$$\alpha_0 \colon \mathscr{L}^{\otimes 3} \otimes \mathsf{S}^2(\textstyle\bigwedge^3 \mathscr{E}) \to \mathscr{O}_S.$$

The cokernel of α_1 is $\mathscr{C}_2$, and that of α_0 is $\mathscr{O}_{D_2(\mathscr{Q})}$. This gives the result for $m = 4$.

For $m = 5$, the pushforward of the Koszul complex gives

$$0 \to \left(\mathscr{L}^{\otimes 5} \otimes \left(\det(\mathscr{E})\right)^{\otimes 2}\right) \oplus \left(\mathscr{L}^{\otimes 3} \otimes \det(\mathscr{E}) \otimes \mathscr{E}\right)$$
$$\xrightarrow{\alpha} \left(\mathscr{L}^{\otimes 2} \otimes \det(\mathscr{E}) \otimes \mathscr{E}^\vee\right) \oplus \mathscr{O}_S \to \varphi_* \mathscr{O}_{F_2(\mathscr{Q})} \to 0.$$

The map α is described as in Proposition A.1 and gives the result for $m = 5$. For $m = 6$, the pushforward of the Koszul complex gives an isomorphism $\varphi_* \mathscr{O}_{F_2(\mathscr{Q})} \simeq \mathscr{O}_S \oplus (\mathscr{L}^{\otimes 3} \otimes \det(\mathscr{E}))$, and for $m \geq 7$, just $\varphi_* \mathscr{O}_{F_2(\mathscr{Q})} \simeq \mathscr{O}_S$. □

Appendix B: Resolutions of EPW surfaces

In this appendix, we discuss a resolution of the structure sheaf of an EPW surface $Y_A^{\geq 2}$ in $\mathbf{P}(V_6)$ and compute some cohomology spaces related to its ideal sheaf. We freely use the notation and results of [5, Appendix B], especially those introduced in Proposition B.3. In particular, we set

$$\widehat{Y}_A := \left\{(v, V_5) \in \mathsf{Fl}(1,5;V_6) \;\middle|\; A \cap \left(v \wedge \textstyle\bigwedge^2 V_5\right) \neq 0\right\}$$

and

$$\widehat{Y}'_A := \left\{(a, v, V_5) \in \mathbf{P}(A) \times \mathsf{Fl}(1,5;V_6) \;\middle|\; a \in \mathbf{P}\left(A \cap \left(v \wedge \textstyle\bigwedge^2 V_5\right)\right)\right\}. \tag{29}$$

When A contains no decomposable vectors, the projection $\widehat{Y}'_A \to \mathsf{Fl}(1,5;V_6)$ induces an isomorphism $\widehat{Y}'_A \xrightarrow{\sim} \widehat{Y}_A$. We denote by H and H' the hyperplane classes of $\mathbf{P}(V_6)$ and $\mathbf{P}(V_6^\vee)$ and by $p\colon \widehat{Y}_A \to Y_A$ and $q\colon \widehat{Y}_A \to Y_{A^\perp}$ the projections. (We switch back from the notation $\mathrm{pr}_{Y,1}$ and $\mathrm{pr}_{Y,2}$ used in the main body of the article to the notation used in [5].) We also denote by H_A the pullback of the hyperplane class of $\mathbf{P}(A)$ to $\widehat{Y}_A$ via the map $\widehat{Y}_A \to \mathbf{P}(A)$ provided by the identification of $\widehat{Y}_A$ with $\widehat{Y}'_A$ (when A contains no decomposable vectors). We begin with a simple lemma.

LEMMA B.1

If A contains no decomposable vectors, then there is an isomorphism $p_* \mathscr{O}_{\widehat{Y}_A} \xrightarrow{\sim} \mathscr{O}_{Y_A}$ and $R^{>0} p_* \mathscr{O}_{\widehat{Y}_A} = 0$.

Proof

By definition, $\widehat{Y}_A = \widehat{Y}'_A$ is the zero-locus of the composition

$$\mathscr{O}_{\mathbf{P}(A)}(-H_A) \boxtimes \mathscr{O}_{\mathbf{P}(V_6)} \to A \otimes \mathscr{O}_{\mathbf{P}(A) \times \mathbf{P}(V_6)}$$
$$\to \textstyle\bigwedge^3 V_6 \otimes \mathscr{O}_{\mathbf{P}(A) \times \mathbf{P}(V_6)} \to \mathscr{O}_{\mathbf{P}(A)} \boxtimes \left(\textstyle\bigwedge^3 T_{\mathbf{P}(V_6)}\right)(-3H)$$

on $\mathbf{P}(A) \times \mathbf{P}(V_6)$ and, hence, equals the zero-locus of the corresponding section of the vector bundle $\mathscr{O}_{\mathbf{P}(A)}(H_A) \boxtimes (\bigwedge^3 T_{\mathbf{P}(V_6)})(-3H)$. Since the codimension of $\widehat{Y}_A$ in $\mathbf{P}(A) \times \mathbf{P}(V_6)$ equals the rank of that vector bundle, we have a Koszul resolution

$$\begin{aligned} &\cdots \to \mathscr{O}_{\mathbf{P}(A)}(-2H_A) \boxtimes \left(\bigwedge^2 \left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)\right)(-6H) \\ &\to \mathscr{O}_{\mathbf{P}(A)}(-H_A) \boxtimes \left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)(-3H) \to \mathscr{O}_{\mathbf{P}(A)\times\mathbf{P}(V_6)} \to \mathscr{O}_{\widehat{Y}_A} \to 0. \end{aligned} \tag{30}$$

Pushing it forward to $\mathbf{P}(V_6)$, we obtain an exact sequence

$$0 \to \det\left(\bigwedge^2 T_{\mathbf{P}(V_6)}(-3H)\right) \to \mathscr{O}_{\mathbf{P}(V_6)} \to p_* \mathscr{O}_{\widehat{Y}_A} \to 0$$

and deduce the vanishing of higher pushforwards. The first nonzero term of this sequence is isomorphic to $\mathscr{O}_{\mathbf{P}(V_6)}(-6H)$; hence, $p_*\mathscr{O}_{\widehat{Y}_A}$ is the structure sheaf of a sextic hypersurface, which clearly coincides with Y_A. □

The crucial observation on which the results of this appendix are based is the following.

LEMMA B.2

If A contains no decomposable vectors, then there is a linear equivalence of divisors

$$2H_A \equiv_{\text{lin}} H + H'. \tag{31}$$

Proof

The definition (29) implies that the image of the tautological embedding $\mathscr{O}_{\widehat{Y}_A}(-H_A) \hookrightarrow A \otimes \mathscr{O}_{\widehat{Y}_A}$ is contained in the kernel of the composition

$$A \otimes \mathscr{O}_{\widehat{Y}_A} \to \bigwedge^3 V_6 \otimes \mathscr{O}_{\widehat{Y}_A} \to p^*\left(\left(\bigwedge^3 T_{\mathbf{P}(V_6)}\right)(-3H)\right),$$

itself identified with the kernel of the morphism

$$p^*((\bigwedge^2 T_{\mathbf{P}(V_6)})(-3H)) \to ((\bigwedge^3 V_6)/A) \otimes \mathscr{O}_{\widehat{Y}_A}.$$

Thus, we have a natural embedding

$$\mathscr{O}_{\widehat{Y}_A}(-H_A) \hookrightarrow p^*\left(\left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)(-3H)\right)$$

of vector bundles: over a point (a, v, V_5) of $\widehat{Y}_A = \widehat{Y}'_A$ with $a = v \wedge \eta$, it is given by $\eta \in \bigwedge^2(V_5/v) \subset \bigwedge^2(V_6/v)$. Its wedge square is a map

$$\mathscr{O}_{\widehat{Y}_A}(-2H_A) \to p^*\left(\left(\bigwedge^4 T_{\mathbf{P}(V_6)}\right)(-6H)\right) \simeq p^*(\Omega_{\mathbf{P}(V_6)})$$

that sends a point (a, v, V_5) of $\widehat{Y}_A$, with $a = v \wedge \eta$, to $\eta \wedge \eta \in \bigwedge^4(V_5/v) \subset \bigwedge^4(V_6/v)$, which is nonzero since a is indecomposable. This map is therefore an embedding of vector bundles. Twisting it by $\mathscr{O}(H)$ and composing with the canonical embedding gives a map

$$\mathscr{O}_{\widehat{Y}_A}(H - 2H_A) \hookrightarrow p^*\left(\Omega_{\mathbf{P}(V_6)}(H)\right) \hookrightarrow V_6^\vee \otimes \mathscr{O}_{\widehat{Y}_A}$$

which defines a map $\widehat{Y}_A \to \mathbf{P}(V_6^\vee)$, $(a, v, V_5) \mapsto v \wedge \eta \wedge \eta$. In the proof of [5, Proposition B.3], it was shown that this map is the projection $q\colon \widehat{Y}_A \to Y_{A^\perp}$. Therefore, we have an isomorphism of line bundles $\mathscr{O}_{\widehat{Y}_A}(H - 2H_A) \simeq \mathscr{O}_{\widehat{Y}_A}(-H')$ and, hence, (31). □

This allows us to find a simple resolution of the ideal sheaf $\mathscr{I}_{Y_A^{\ge 2}, Y_A}$ of the EPW surface $Y_A^{\ge 2}$ in the EPW sextic Y_A.

LEMMA B.3

If A contains no decomposable vectors, then there is an exact sequence

$$\begin{aligned} 0 &\to \Big(\textstyle\bigwedge^2 \Big(\bigwedge^2 T_{\mathbf{P}(V_6)}\Big)\Big)(-12H) \to A^\vee \otimes \Big(\textstyle\bigwedge^2 T_{\mathbf{P}(V_6)}\Big)(-9H) \\ &\to \mathsf{S}^2 A^\vee \otimes \mathscr{O}_{\mathbf{P}(V_6)}(-6H) \to \mathscr{I}_{Y_A^{\ge 2}, Y_A} \to 0 \end{aligned} \tag{32}$$

of sheaves on $\mathbf{P}(V_6)$.

Proof

Denote by E the exceptional divisor of the birational morphism $p\colon \widehat{Y}_A \to Y_A$. We show first that E coincides with the scheme-theoretic preimage of the EPW surface $Y_A^{\ge 2}$. For this, recall that $Y_A^{\ge 2}$ is by definition the corank-2 degeneracy locus of the composition

$$A \otimes \mathscr{O}_{\mathbf{P}(V_6)} \to \textstyle\bigwedge^3 V_6 \otimes \mathscr{O}_{\mathbf{P}(V_6)} \to \Big(\bigwedge^3 T_{\mathbf{P}(V_6)}\Big)(-3H).$$

When pulled back to $\widehat{Y}_A$, it extends to a complex

$$\mathscr{O}_{\widehat{Y}_A}(-H_A) \hookrightarrow A \otimes \mathscr{O}_{\widehat{Y}_A} \to p^*\Big(\Big(\textstyle\bigwedge^3 T_{\mathbf{P}(V_6)}\Big)(-3H)\Big) \twoheadrightarrow \mathscr{O}_{\widehat{Y}_A}(H_A);$$

hence, the preimage of $Y_A^{\ge 2}$ is the degeneracy locus of the induced map

$$(A \otimes \mathscr{O}_{\widehat{Y}_A})/\mathscr{O}_{\widehat{Y}_A}(-H_A) \to \operatorname{Ker}\Big(p^*\Big(\Big(\textstyle\bigwedge^3 T_{\mathbf{P}(V_6)}\Big)(-3H)\Big) \twoheadrightarrow \mathscr{O}_{\widehat{Y}_A}(H_A)\Big).$$

This is a morphism between two vector bundles of rank 9; hence, the preimage of $Y_A^{\ge 2}$ is the Cartier divisor in $\widehat{Y}_A$ defined by a section of the line bundle

$$\begin{aligned} &\det\Big(\operatorname{Ker}\Big(p^*\Big(\Big(\textstyle\bigwedge^3 T_{\mathbf{P}(V_6)}\Big)(-3H)\Big) \twoheadrightarrow \mathscr{O}_{\widehat{Y}_A}(H_A)\Big)\Big) \\ &\qquad \otimes \det\big((A \otimes \mathscr{O}_{\widehat{Y}_A})/\mathscr{O}_{\widehat{Y}_A}(-H_A)\big)^\vee \simeq \mathscr{O}_{\widehat{Y}_A}(6H - 2H_A). \end{aligned}$$

But $6H - 2H_A \equiv_{\mathrm{lin}} 6H - H - H' \equiv_{\mathrm{lin}} 5H - H'$, and this is linearly equivalent to E by a computation in [5, a paragraph before Lemma B.6]. All global sections of the line bundle $\mathscr{O}_{\widehat{Y}_A}(E)$ are proportional (since E is the exceptional divisor of a birational morphism); hence, the scheme-theoretic preimage of $Y_A^{\ge 2}$ equals E.

Since $E = p^{-1}(Y_A^{\ge 2})$, there is an embedding of schemes $p(E) \subset Y_A^{\ge 2}$. On the other hand, $p(E)$ and $Y_A^{\ge 2}$ coincide set-theoretically (see [5, Proposition B.3]), and the scheme $Y_A^{\ge 2}$ is reduced and normal (see [5, Theorem B.2]); hence, $p(E) = Y_A^{\ge 2}$. Since the fibers of the map $p\colon E \to Y_A^{\ge 2}$ are connected, it also follows that there is an isomorphism $p_*\mathscr{O}_E \xrightarrow{\sim} \mathscr{O}_{Y_A^{\ge 2}}$.

We now compute $p_*\mathscr{O}_E$. We use the linear equivalence $6H - 2H_A \equiv_{\mathrm{lin}} E$ shown above and compute the derived pushforward of the line bundle

$$\mathscr{O}_{\widehat{Y}_A}(-E) \simeq \mathscr{O}_{\widehat{Y}_A}(2H_A - 6H).$$

Twisting the Koszul resolution (30) by $\mathscr{O}_{\widehat{Y}_A}(2H_A - 6H)$ and pushing forward to $\mathbf{P}(V_6)$, we obtain an exact sequence

$$0 \to \left(\bigwedge^2\left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)\right)(-12H) \to A^\vee \otimes \left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)(-9H)$$
$$\to \mathsf{S}^2 A^\vee \otimes \mathscr{O}_{\mathbf{P}(V_6)}(-6H) \to p_*\mathscr{O}_{\widehat{Y}_A}(-E) \to 0$$

and the vanishing of higher pushforwards. Using this and Lemma B.1 and applying pushforward to the standard exact sequence

$$0 \to \mathscr{O}_{\widehat{Y}_A}(-E) \to \mathscr{O}_{\widehat{Y}_A} \to \mathscr{O}_E \to 0,$$

we obtain an exact sequence

$$0 \to p_*\mathscr{O}_{\widehat{Y}_A}(-E) \to \mathscr{O}_{Y_A} \to p_*\mathscr{O}_E \to 0.$$

The right-hand term is isomorphic to $\mathscr{O}_{Y_A^{\geq 2}}$, as we have shown above; hence, the left-hand term is $\mathscr{I}_{Y_A^{\geq 2}, Y_A}$. This proves the lemma. □

REMARK B.4

The equality $E = p^{-1}(Y_A^{\geq 2})$ shown in the proof means that $\widehat{Y}_A$ is the blowup of Y_A along $Y_A^{\geq 2}$.

The sequence (32) can be merged with the standard resolution of Y_A to give an exact sequence

$$0 \to \left(\bigwedge^2\left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)\right)(-12H) \to A^\vee \otimes \left(\bigwedge^2 T_{\mathbf{P}(V_6)}\right)(-9H)$$
$$\to (\mathsf{S}^2 A^\vee \oplus \mathbf{C}) \otimes \mathscr{O}_{\mathbf{P}(V_6)}(-6H) \to \mathscr{O}_{\mathbf{P}(V_6)} \to \mathscr{O}_{Y_A^{\geq 2}} \to 0, \tag{33}$$

which can be used to compute the cohomology of line bundles on $Y_A^{\geq 2}$.

COROLLARY B.5

If A contains no decomposable vectors, then the following table computes the cohomology spaces for some twists of the sheaf $\mathscr{O}_{Y_A^{\geq 2}}$.

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$
$H^2(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}(tH))$	$\bigwedge^2 A$	0	0	0	0	0	0
$H^1(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}(tH))$	0	0	$\bigwedge^2 V_6^\vee$	$A^\vee$	0	0	0
$H^0(Y_A^{\geq 2}, \mathscr{O}_{Y_A^{\geq 2}}(tH))$	$\mathbf{C}$	$V_6^\vee$	$\mathsf{S}^2 V_6^\vee$	$\mathsf{S}^3 V_6^\vee$	$\mathsf{S}^4 V_6^\vee$	$\mathsf{S}^5 V_6^\vee / V_6$	$\mathsf{S}^6 V_6^\vee / (\mathsf{S}^2 A^\vee \oplus \mathbf{C})$

Moreover, $H^0(\mathbf{P}(V_6), \mathscr{I}_{Y_A^{\geq 2}, \mathbf{P}(V_6)}(2H)) = H^1(\mathbf{P}(V_6), \mathscr{I}_{Y_A^{\geq 2}, \mathbf{P}(V_6)}(H)) = 0.$

Proof
It consists of a straightforward computation using (33) and the Borel–Bott–Weil theorem. □

In Section 5, we used the following simple consequence of these computations.

COROLLARY B.6
If A contains no decomposable vectors, then the curve $Y^{\geq 2}_{A,V_5} \subset \mathbf{P}(V_5)$ is not contained in a quadric.

Proof
We have an exact sequence

$$0 \to \mathscr{I}_{Y^{\geq 2}_A, \mathbf{P}(V_6)}(-H) \to \mathscr{I}_{Y^{\geq 2}_A, \mathbf{P}(V_6)} \to \mathscr{I}_{Y^{\geq 2}_{A,V_5}, \mathbf{P}(V_5)} \to 0$$

of sheaves on $\mathbf{P}(V_6)$. The cohomology sequence of its twist by $\mathscr{O}_{\mathbf{P}(V_6)}(2H)$ gives an exact sequence

$$H^0\big(\mathbf{P}(V_6), \mathscr{I}_{Y^{\geq 2}_A, \mathbf{P}(V_6)}(2H)\big) \to H^0\big(\mathbf{P}(V_5), \mathscr{I}_{Y^{\geq 2}_{A,V_5}, \mathbf{P}(V_5)}(2H)\big) \to H^1\big(\mathbf{P}(V_6), \mathscr{I}_{Y^{\geq 2}_A, \mathbf{P}(V_6)}(H)\big).$$

By Corollary B.5, the spaces at both ends vanish and, hence, so does the middle space. □

Added in proof: After this paper was accepted, we developed in [7] a more general approach to deal with double coverings of various EPW varieties that allow us in particular to answer some of the questions raised in this paper. For instance, we constructed in [7, Theorem 5.2(2)] the double cover $\widetilde{Y}^{\geq 2}_A \to Y^{\geq 2}_A$, and the equivalence of its base change to $\mathbf{P}(V_5)$ with the cover of Theorem 4.3(b) is proved in [7, Corollary 5.5]. Similarly, [7, Corollary 5.5] applies to identify the base change to $\mathbf{P}(V_5)$ of the double EPW sextic with the cover of Theorem 4.7(c).

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