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RATIONALITY OVER NONCLOSED FIELDS OF FANO THREEFOLDS WITH HIGHER GEOMETRIC PICARD RANK

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Abstract We prove rationality criteria over nonclosed fields of characteristic 0 for five out of six types of geometrically rational Fano threefolds of Picard number 1 and geometric Picard number bigger than 1. For the last type of such threefolds, we provide a unirationality criterion and construct examples of unirational but not stably rational varieties of this type.

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1. Introduction

1.1. The results

The goal of this paper is to discuss rationality of smooth Fano threefolds over algebraically nonclosed fields of characteristic 0. In [10], we considered the case of geometrically rational Fano threefolds with geometric Picard number $\uprho(X_{\bar{\Bbbk}}) = 1$, and here, we switch the focus to the case of geometrically rational Fano threefolds X with Picard numbers

$$\uprho(X) = 1 \qquad \text{and} \qquad \uprho(X_{\bar{\Bbbk}}) > 1. \tag{1.1.1}$$

In fact, Fano threefolds satisfying (1.1.1) have been classified in [21], and [1] explains which of these are geometrically rational. A combination of these results gives the following:

Theorem 1.1 ([21, Theorem 1.2], [1]). *There are exactly six families of geometrically rational Fano threefolds satisfying (1.1.1) as listed in Table 1.*

TABLE 1. Geometrically rational Fano threefolds X satisfying (1.1.1)

	$\iota(X_{\bar{\mathsf{k}}})$	$\rho(X_{\bar{\mathsf{k}}})$	$-K_X^3$	$\mathrm{g}(X)$	$\mathrm{h}^{1,2}(X_{\bar{\mathsf{k}}})$	$X_{\bar{\mathsf{k}}}$
$\mathsf{X}_{(3,3)}$	1	2	20	11	3	an intersection of three divisors of bidegree (1,1) in $\mathbb{P}^3_{\bar{\mathsf{k}}} \times \mathbb{P}^3_{\bar{\mathsf{k}}}$;
$\mathsf{X}_{(1,1,1,1)}$	1	4	24	13	1	a divisor of multidegree (1,1,1,1) in $(\mathbb{P}^1_{\bar{\mathsf{k}}})^4$;
$\mathsf{X}_{(4,4)}$	1	2	28	15	0	the blowup of a smooth quadric $Q_{\bar{\mathsf{k}}} \subset \mathbb{P}^4_{\bar{\mathsf{k}}}$ along a linearly normal smooth rational quartic curve;
$\mathsf{X}_{(2,2,2)}$	1	3	30	16	0	an intersection of three divisors of multidegrees (0,1,1), (1,0,1), (1,1,0) in $\mathbb{P}^2_{\bar{\mathsf{k}}} \times \mathbb{P}^2_{\bar{\mathsf{k}}} \times \mathbb{P}^2_{\bar{\mathsf{k}}}$;
$\mathsf{X}_{(2,2)}$	2	2	48	25	0	a divisor of bidegree (1,1) in $\mathbb{P}^2_{\bar{\mathsf{k}}} \times \mathbb{P}^2_{\bar{\mathsf{k}}}$;
$\mathsf{X}_{(1,1,1)}$	2	3	48	25	0	$\mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}$.

The first column of Table 1 contains the name for the family we use in this paper, the next columns contain the index $\iota(X_{\bar{\mathsf{k}}})$, defined as

$$\iota(X_{\bar{\mathsf{k}}}) = \max\left\{ i \;\middle|\; \tfrac{1}{i} K_{X_{\bar{\mathsf{k}}}} \in \operatorname{Pic}(X_{\bar{\mathsf{k}}}) \right\},$$

the geometric Picard number $\rho(X_{\bar{\mathsf{k}}})$, the anticanonical degree $(-K_X)^3$, the genus $\mathrm{g}(X)$, defined by

$$(-K_X)^3 = 2\mathrm{g}(X) - 2$$

and the Hodge number $\mathrm{h}^{1,2}(X_{\bar{\mathsf{k}}})$ of the threefold, while the last column provides a geometric description of these varieties over an algebraic closure $\bar{\mathsf{k}}$ of the base field.

We discuss some geometric properties of threefolds from Table 1 in §2. In particular, we describe their extremal contractions over $\bar{\mathsf{k}}$ and identify their Hilbert schemes of lines and conics, as well as the subschemes of the Hilbert schemes of twisted cubic curves passing through a general point.

However, our main interest is in rationality criteria, and the next theorem is our main result.

Theorem 1.2. *Let X be a Fano threefold from Table 1; in particular, we assume $\rho(X) = 1$.*

(i) *X is unirational if and only if $X(\mathsf{k}) \neq \varnothing$.*

(ii) *If X has type $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(2,2)}$ or $\mathsf{X}_{(1,1,1)}$, then X is k-rational if and only if $X(\mathsf{k}) \neq \varnothing$.*

(iii) *If X has type $\mathsf{X}_{(3,3)}$, then X is never k-rational.*

Note that over an algebraically closed field, threefolds of types $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(2,2)}$ and $\mathsf{X}_{(1,1,1)}$ have $\mathrm{h}^{1,2} = 0$, hence, trivial intermediate Jacobians, while the intermediate Jacobians of threefolds of types $\mathsf{X}_{(3,3)}$ and $\mathsf{X}_{(1,1,1,1)}$ over $\bar{\mathsf{k}}$ are Jacobians of curves of genus 3 and 1, respectively (and k-forms of these over k); this explains the difference in the behavior.

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It is a classical fact that the existence of a k-point is necessary for rationality or unirationality, so the major part of the proof of the theorem consists of proving rationality or unirationality under this assumption. We use for this a case-by-case analysis (see §1.2 for a description of our approach). The theorem is, thus, a combination of the following results (we assume everywhere $X(\mathsf{k}) \neq \varnothing$):

- rationality for threefolds of type $\mathsf{X}_{(1,1,1)}$ is proved in Corollary 3.4;
- rationality for threefolds of type $\mathsf{X}_{(2,2)}$ is proved in Proposition 4.1;
- rationality for threefolds of type $\mathsf{X}_{(2,2,2)}$ is proved in Proposition 4.3;
- rationality for threefolds of type $\mathsf{X}_{(4,4)}$ is proved in Proposition 5.5;
- unirationality for threefolds of type $\mathsf{X}_{(1,1,1,1)}$ is proved in Proposition 4.5;
- unirationality for threefolds of type $\mathsf{X}_{(3,3)}$ is proved in Proposition 6.9;
- nonrationality for threefolds of type $\mathsf{X}_{(3,3)}$ is proved in Corollary 6.11.

Theorem 1.2 provides nice criteria for rationality of five out of the six types of Fano threefolds listed in Table 1. For the remaining type $\mathsf{X}_{(1,1,1,1)}$, we have a conjecture and a partial result.

Conjecture 1.3. *If X has type $\mathsf{X}_{(1,1,1,1)}$ and $\rho(X) = 1$, then X is never k-rational.*

To explain the partial result, we need to introduce some notation. Let X be a Fano threefold of type $\mathsf{X}_{(1,1,1,1)}$. As we show in Lemma 2.5, the action of the Galois group $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$ on $\mathrm{Pic}(X_{\bar{\mathsf{k}}})$ factors through the group $\mathfrak{S}_4$ that acts by permutations of the pullbacks of the point classes of the factors of the ambient $(\mathbb{P}^1_{\mathsf{k}})^4$, and the assumption $\rho(X) = 1$ means that the subgroup

$$\mathrm{G}_X := \mathrm{Im}(\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}) \longrightarrow \mathfrak{S}_4) \subset \mathfrak{S}_4$$

is *transitive*, hence, belongs to the following list of (conjugacy classes) of transitive subgroups of $\mathfrak{S}_4$:

$$\mathrm{G}_X \in \{\mathfrak{S}_4, \mathfrak{A}_4, \mathrm{D}_4, \mathrm{V}_4, \mathrm{C}_4\},$$

where $\mathfrak{A}_4$ is the alternating subgroup, D_4 is the dihedral group of order 8 (a Sylow 2-subgroup in $\mathfrak{S}_4$), V_4 is the Klein group of order 4 and C_4 is the cyclic group of order 4. Note that all of these groups contain V_4 except for C_4.

Theorem 1.4. *Let $\mathrm{G} \subset \mathfrak{S}_4$ be a subgroup containing the Klein group $\mathrm{V}_4 \subset \mathfrak{S}_4$. Let k be a field, such that there is an epimorphism $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}) \twoheadrightarrow \mathrm{G}$. Then for the field of rational functions $K = \mathsf{k}(t)$ there exists a variety X over K of type $\mathsf{X}_{(1,1,1,1)}$, such that $\mathrm{G}_X = \mathrm{G}$, $\rho(X) = 1$ and $X(K) \neq \varnothing$, but X is not stably rational over K.*

1.2. The proofs

For (uni)rationality constructions, it is natural to use k-Sarkisov links:

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\sigma"'] \arrow[dr, "\phi"] \arrow[rr, dashed, "\psi"] & & \tilde{X}^+ \arrow[dl, "\phi_+"'] \arrow[dr, "\sigma_+"] & \\ X & & \bar{X} & & X^+, \end{tikzcd} \tag{1.2.1}$$

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where σ is the blowup of a k-irreducible subvariety, ϕ and ϕ_+ are small crepant birational contractions, ψ is a flop and σ_+ is a Mori extremal contraction. Note that such a link is completely determined by the center of the blowup σ — the contractions and the flop are obtained by the k-Minimal Model Program applied to $\tilde{X}$ (note that $\rho(\tilde{X}) = 2$, so the output of the MMP is unambiguous); in particular, the link is defined over k. For our purpose, it is enough to consider two types of Sarkisov links:

- Sarkisov links where σ is the blowup of a k-point;
- Sarkisov links where σ is the blowup of a reduced k-irreducible singular conic.

We construct the corresponding links accurately for threefolds of type $\mathsf{X}_{(4,4)}$ in §5 (see Theorem 5.1) by using standard MMP arguments. Of course, a similar construction could be given for other types of Fano threefolds from Table 1, but to make the argument less tedious, we use the fact that all others among these threefolds are k-forms of complete intersections in products of projective spaces and deduce the required (uni)rationality constructions from an appropriate birational transformation for a product of projective spaces.

With this goal in mind, we construct in §3 a toric birational transformation between the product $(\mathbb{P}^n)^r$ of projective spaces and a $\mathbb{P}^r$-bundle over the product $(\mathbb{P}^{n-1})^r$ of smaller projective spaces, see Theorem 3.1 (in fact, we construct a birational transformation in a slightly more general situation, but the setup described above is the only one that we need for applications in the paper). This theorem has a consequence of independent interest, Corollary 3.3, saying that a k-form of a product of projective spaces is k-rational if and only if it has a k-point. This corollary immediately gives the required rationality construction for Fano threefolds of type $\mathsf{X}_{(1,1,1)}$ (Corollary 3.4) and with a bit of more work provides rationality constructions for threefolds of types $\mathsf{X}_{(2,2)}$ (Proposition 4.1) and $\mathsf{X}_{(2,2,2)}$ (Proposition 4.3), as well as a unirationality construction for threefolds of type $\mathsf{X}_{(1,1,1,1)}$ (Proposition 4.5).

In the case of a variety X of type $\mathsf{X}_{(3,3)}$ with a k-point x, we again use the toric transformation of Theorem 3.1 to construct a birational equivalence of X with a divisor X^+ of bidegree (2,2) in a k-form of $\mathbb{P}^2 \times \mathbb{P}^2$. If x lies on a $\bar{\mathsf{k}}$-line in X, we check that X^+ contains a k-form of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ and use this to deduce unirationality of X (Proposition 6.9). If x does not lie on a line, we check in Proposition 6.6 that X^+ described above is, in fact, the midpoint of a Sarkisov link that ends with a conic bundle over $\mathbb{P}^2$, which has a smooth quartic curve $\Gamma \subset \mathbb{P}^2$ as discriminant. We also check that the discriminant double covering $\tilde{\Gamma} \to \Gamma$ associated to this conic bundle is trivial over a quadratic extension k' of the base field k but nontrivial over k, and that the conic bundle has a rational section over k'. We check in Theorem 6.10 that these geometric properties characterise the nonrational conic bundles constructed by Benoist and Wittenberg in [3] and deduce in Corollary 6.11 nonrationality of X from [3, Proposition 3.4].

In the last part of the paper, §7, we discuss Fano threefolds of type $\mathsf{X}_{(1,1,1,1)}$. To prove Theorem 1.4, we use a degeneration technique. Namely, we construct a family of Fano threefolds of type $\mathsf{X}_{(1,1,1,1)}$ over $\mathbb{P}^1_{\mathsf{k}}$ with the special fibre a singular toric threefold (with ordinary double points), which is well known not to be stably rational. Since stable rationality is specialisation-closed by a result of Nicaise and Shinder [19], we conclude that the general fibre of the constructed family is also not stably rational.

2. Extremal contractions and Hilbert schemes of curves

In this section, we describe the geometry of Fano threefolds of index 1 from Table 1. In particular, we describe their extremal contractions over $\bar{\mathsf{k}}$, as well as their Hilbert schemes of lines and conics and of twisted cubic curves passing through a fixed point.

To start with, recall that for most Fano threefolds, the anticanonical linear system is very ample and the anticanonical image is an intersection of quadrics; in fact, Fano threefolds which do not enjoy these nice properties (hyperelliptic and trigonal ones) have been classified and listed in [6]. It is easy to check that Fano threefolds from Table 1 are not in this list; therefore, we obtain:

Theorem 2.1 ([6, Chapter 2, Theorems 2.2 and 3.4]). *Let X be a Fano threefold from Table 1. The anticanonical class $-K_X$ is very ample and the anticanonical image*

$$X = X_{2g-2} \subset \mathbb{P}^{g+1}$$

is an intersection of quadrics (as a scheme), where $g = \mathrm{g}(X)$.

2.1. Contractions over $\bar{\mathsf{k}}$

Assume X is a Fano threefold of index 1 from Table 1, that is, a threefold of either of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(3,3)}$ or $\mathsf{X}_{(1,1,1,1)}$. Then there is an embedding

$$X_{\bar{\mathsf{k}}} \subset Y \cong (\mathbb{P}^n)^r, \tag{2.1.1}$$

(we will see in Lemma 2.5 that $r = \rho(X_{\bar{\mathsf{k}}})$, hence, the notation), where

$$(n,r) = (2,3),\ (4,2),\ (3,2) \text{ or } (1,4).$$

Indeed, for types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(3,3)}$, $\mathsf{X}_{(1,1,1,1)}$, this holds by definition, and for type $\mathsf{X}_{(4,4)}$, this follows from the following:

Lemma 2.2. *Let $\Gamma_1 \subset Q_1 \subset \mathbb{P}^4$ be a linearly normal smooth rational quartic curve in a smooth quadric threefold. If H_1 is the hyperplane class of Q_1, then the linear system $|2H_1 - \Gamma_1|$ of quadrics through Γ_1 defines a birational morphism $\pi_2\colon \mathrm{Bl}_{\Gamma_1} Q_1 \to Q_2 \subset \mathbb{P}^4$ onto another smooth quadric threefold Q_2, and this morphism is itself the blowup of a linearly normal smooth rational quartic curve $\Gamma_2 \subset Q_2$, so that*

$$\mathrm{Bl}_{\Gamma_1}(Q_1) \cong \mathrm{Bl}_{\Gamma_2}(Q_2).$$

Moreover, if X is a Fano threefold of type $\mathsf{X}_{(4,4)}$, there is a natural embedding

$$X_{\bar{\mathsf{k}}} \longrightarrow Q_1 \times Q_2 \subset \mathbb{P}^4 \times \mathbb{P}^4,$$

such that $-K_{X_{\bar{\mathsf{k}}}}$ is the sum of the pullbacks of the hyperplane classes of the factors.

Proof. The curve Γ_1 is an intersection of six quadrics in $\mathbb{P}^4$; therefore, it is an intersection of five quadrics in Q_1. Hence, if E_1 is the exceptional divisor of the blowup $\pi_1\colon X_{\bar{\mathsf{k}}} \to Q_1$ and H_1 is the pullback of the hyperplane class of Q_1, the linear system $|2H_1 - E_1|$ on $X_{\bar{\mathsf{k}}}$ is four-dimensional and base-point free. Therefore, this linear system defines a morphism $\pi_2\colon \mathrm{Bl}_{\Gamma_1}(Q_1) \to \mathbb{P}^4$; moreover, standard intersection theory gives $(2H_1 - E_1)^3 = 2$. Hence,

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the image of π_2 (which is not contained in a hyperplane by definition) is a quadric $Q_2 \subset \mathbb{P}^4$ and π_2 is birational. Since $-K_{X_{\bar{\mathsf{k}}}}$ is ample on the fibres of π_2 and $\rho(X_{\bar{\mathsf{k}}}) = 2$, we see that π_2 is an extremal Mori contraction. By [16], the quadric Q_2 is smooth and π_2 is the blowup of a curve, which must be a linearly normal smooth rational quartic curve. For the last statement, just note that $H_1 + (2H_1 - E_1) = 3H_1 - E_1$ is the anticanonical class of $X_{\bar{\mathsf{k}}}$. □

We denote by H_i, $1 \le i \le r$, the pullbacks to $Y = (\mathbb{P}^n)^r$ of the hyperplane classes of the factors and, abusing the notation, also their restrictions to $X_{\bar{\mathsf{k}}}$ via the embedding (2.1.1).

Lemma 2.3. *If X is a threefold of either of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(3,3)}$ or $\mathsf{X}_{(1,1,1,1)}$, then the Picard group $\mathrm{Pic}(X_{\bar{\mathsf{k}}})$ is freely generated by the classes H_i*

$$\mathrm{Pic}(X_{\bar{\mathsf{k}}}) = \bigoplus_{i=1}^{r} \mathbb{Z}H_i.$$

Moreover,

$$-K_{X_{\bar{\mathsf{k}}}} = H := H_1 + \cdots + H_r. \tag{2.1.2}$$

Proof. For type $\mathsf{X}_{(4,4)}$, this follows from Lemma 2.2, and for the other types, the first statement follows from the Lefschetz hyperplane theorem and the second from adjunction and the description of Table 1. □

For each subset $I \subset \{1,\dots,r\}$ we consider the projection

$$\pi_I \colon X_{\bar{\mathsf{k}}} \hookrightarrow Y \longrightarrow \prod_{i \in I} \mathbb{P}^n \cong (\mathbb{P}^n)^{|I|}. \tag{2.1.3}$$

Especially useful are the morphisms π_I for I of cardinality $r-1$, so we introduce the notation

$$\widehat{\imath} := \{1,\dots,r\} \setminus \{i\},$$

and write

$$\pi_{\widehat{\imath}} \colon X_{\bar{\mathsf{k}}} \longrightarrow (\mathbb{P}^n)^{r-1} \tag{2.1.4}$$

for the corresponding morphisms. Note that in the case $r = 2$, we have $\widehat{\imath} = \{3 - i\}$, so these morphisms are the same as morphisms π_{3-i}. The next lemma describes $X_{\bar{\mathsf{k}}}$ in terms of the $\pi_{\widehat{\imath}}$.

Lemma 2.4. *The morphism $\pi_{\widehat{\imath}}$ is birational onto its image, and the exceptional divisor $E_{\widehat{\imath}}$ of $\pi_{\widehat{\imath}}$ is irreducible. More precisely, the morphism $\pi_{\widehat{\imath}}$ identifies $X_{\bar{\mathsf{k}}}$ as follows:*

(i) *if X has type $\mathsf{X}_{(2,2,2)}$, the map $\pi_{\widehat{\imath}}$ is the blowup of a smooth divisor $W_{\widehat{\imath}} \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$ along a smooth rational curve $\Gamma_{\widehat{\imath}} \subset W_{\widehat{\imath}}$ of bidegree $(2,2)$, whose projections to the factors $\mathbb{P}^2$ are closed embeddings; the divisor class H_i is equal to $\sum_{j \ne i} H_j - E_{\widehat{\imath}}$;*

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(ii) *if X has type $\mathsf{X}_{(4,4)}$, the map π_i is the blowup of a three-dimensional quadric Q_i along a smooth linearly normal rational curve $\Gamma_i \subset Q_i$ of degree 4; the divisor class $H_{\hat{\imath}}$ is equal to $2H_i - E_i$;*

(iii) *if X has type $\mathsf{X}_{(3,3)}$, the map π_i is the blowup of $\mathbb{P}^3$ along a smooth curve $\Gamma_i \subset \mathbb{P}^3$ of genus 3 and degree 6; the divisor class $H_{\hat{\imath}}$ is equal to $3H_i - E_i$;*

(iv) *if X has type $\mathsf{X}_{(1,1,1,1)}$, the map $\pi_{\hat{\imath}}$ is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a smooth elliptic curve $\Gamma_{\hat{\imath}} \subset (\mathbb{P}^1)^3$ of multidegree (2,2,2); the divisor class H_i is equal to $\sum_{j \neq i} H_j - E_{\hat{\imath}}$.*

Proof. Part (ii) is proved in Lemma 2.2. So, assume X is a variety of either of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(3,3)}$ or $\mathsf{X}_{(1,1,1,1)}$. Birationality of the projection $\pi_{\hat{\imath}}$ is clear from the descriptions of Table 1; and it also follows that all fibres of $\pi_{\hat{\imath}}$ are linear subspaces in $\mathbb{P}^n$ and $-K_{X_{\bar{\mathsf{k}}}}$ restricts to each of them as the hyperplane class by (2.1.2). Also, it is easy to see that the image of $\pi_{\hat{\imath}}$ is smooth in all cases (for type $\mathsf{X}_{(2,2,2)}$, if $W_{\hat{\imath}} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is singular, then its preimage in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is singular along a plane, hence, $X_{\bar{\mathsf{k}}}$, which is the intersection of this preimage with two other divisors, must be singular; and for types $\mathsf{X}_{(3,3)}$ and $\mathsf{X}_{(1,1,1,1)}$, the image is just $\mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, respectively).

By Lemma 2.3, the relative Picard number of $\pi_{\hat{\imath}}$ is 1 and $-K_{X_{\bar{\mathsf{k}}}}$ is ample, hence, $\pi_{\hat{\imath}}$ is an extremal Mori contraction. Since both the source and target of $\pi_{\hat{\imath}}$ are smooth, it follows from [16] that the morphism $\pi_{\hat{\imath}}$ is either the blowup of a smooth curve or the blowup of a smooth point. In the latter case, the restriction of $-K_{X_{\bar{\mathsf{k}}}}$ to the nontrivial fibre $\mathbb{P}^2$ of $\pi_{\hat{\imath}}$ would be isomorphic to $\mathscr{O}_{\mathbb{P}^2}(2)$, contradicting to the above observation, hence, $\pi_{\hat{\imath}}$ is the blowup of a smooth curve.

The remaining assertions are easy and left to the reader (see also [15]). □

Lemma 2.5. *The classes H_i are semiample and generate the nef cone of $X_{\bar{\mathsf{k}}}$. The Galois group $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$ permutes these classes in a transitive way. In other words, the natural group homomorphism $\varpi_X \colon \mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}) \to \mathrm{Aut}(\mathrm{Pic}(X_{\bar{\mathsf{k}}}))$ factors through the permutation subgroup $\mathfrak{S}_r \subset \mathrm{Aut}(\mathrm{Pic}(X_{\bar{\mathsf{k}}}))$, and its image*

$$\mathrm{G}_X := \mathrm{Im}(\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}) \xrightarrow{\varpi_X} \mathfrak{S}_r) \tag{2.1.5}$$

is a transitive subgroup of $\mathfrak{S}_r$.

Proof. The classes H_i are pullbacks of ample classes on $\mathbb{P}^n$, hence, semiample, and they generate $\mathrm{Pic}(X_{\bar{\mathsf{k}}})$ by Lemma 2.3. If Λ_i is the class of a nontrivial fibre of $\pi_{\hat{\imath}}$, we have

$$H_j \cdot \Lambda_i = \delta_{ij},$$

therefore, H_j generate the rays of the nef cone. It follows that the Galois group permutes the H_i, hence, its action on $\mathrm{Pic}(X_{\bar{\mathsf{k}}})$ factors through the permutation group. Transitivity of the subgroup $\mathrm{G}_X \subset \mathfrak{S}_r$ follows from the equality $\rho(X) = 1$. □

We say that a surface $\Pi \subset X_{\bar{\mathsf{k}}}$ is an H-plane if $\Pi \cong \mathbb{P}^2_{\bar{\mathsf{k}}}$ and $H|_\Pi$ is the line class.

Corollary 2.6. *Fano threefolds of index 1 from Table 1 contain no H-planes over $\bar{\mathsf{k}}$.*

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Proof. If $\Pi \subset X_{\bar{\mathsf{k}}}$ is an H-plane, the restriction $(H_1 + \cdots + H_r)|_\Pi$ is the line class. Since all the H_i are nef, it follows that $H_j|_\Pi \sim 0$ for all $j \neq i$ and some i, hence, Π is contracted to a point by the projection $\pi_{\hat{\imath}}$. It remains to note that the fibres of $\pi_{\hat{\imath}}$ are at most one-dimensional by Lemma 2.4. □

2.2. Lines

By a line on X, we understand a curve (defined over $\bar{\mathsf{k}}$) of anticanonical degree 1. We denote by $\mathrm{F}_1(X)$, the Hilbert scheme of lines on X. Note that $\mathrm{F}_1(X)_{\bar{\mathsf{k}}} \cong \mathrm{F}_1(X_{\bar{\mathsf{k}}})$.

Lemma 2.7. *Let X be a Fano threefold of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(3,3)}$, or $\mathsf{X}_{(1,1,1,1)}$. A line on X is a fibre of the exceptional divisor of one of the projections (2.1.4). In particular,*

$$\mathrm{F}_1(X_{\bar{\mathsf{k}}}) \cong \bigsqcup_{i=1}^{r} \Gamma_{\hat{\imath}},$$

where the smooth curves $\Gamma_{\hat{\imath}}$ have been described in Lemma 2.4. The normal bundle of each line is

$$\mathscr{N}_{L/X_{\bar{\mathsf{k}}}} \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1). \tag{2.2.1}$$

Finally, the action of the Galois group $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$ on the set of connected components of the Hilbert scheme of lines factors through the group G_X and is transitive.

Proof. Since the classes H_i are semiample, it follows from (2.1.2) that for each $\bar{\mathsf{k}}$-line L on X, there is a unique i, such that $L \cdot H_i = 1$ and $L \cdot H_j = 0$ for $j \neq i$ (i.e. $[L] = \Lambda_i$ in the notation of Lemma 2.5). Thus, L is contracted by the projection $\pi_{\hat{\imath}}$, hence, it is equal to a fibre of the exceptional divisor of this projection. Taking into account the description of the projections $\pi_{\hat{\imath}}$ from Lemma 2.4, we obtain the description of $\mathrm{F}_1(X_{\bar{\mathsf{k}}})$.

Further, the description of the normal bundle of L follows from the exact sequence

$$0 \longrightarrow \mathscr{N}_{L/E_{\hat{\imath}}} \longrightarrow \mathscr{N}_{L/X_{\bar{\mathsf{k}}}} \longrightarrow \mathscr{N}_{E_{\hat{\imath}}/X_{\bar{\mathsf{k}}}}|_L \longrightarrow 0,$$

because the first term is trivial and the last is $\mathscr{O}_L(-1)$. Finally, factorisation of the Galois action on the set of connected components of $\mathrm{F}_1(X_{\bar{\mathsf{k}}})$ and its transitivity follow from Lemma 2.5. □

For a $\bar{\mathsf{k}}$-point $x \in X$, we denote by $\mathrm{F}_1(X_{\bar{\mathsf{k}}}, x) \subset \mathrm{F}_1(X_{\bar{\mathsf{k}}})$ the subscheme parameterising lines passing through x. We will need the following observation.

Lemma 2.8. *Let X be a Fano threefold of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(3,3)}$ or $\mathsf{X}_{(1,1,1,1)}$. If $x \in X(\bar{\mathsf{k}})$, then the scheme $\mathrm{F}_1(X_{\bar{\mathsf{k}}}, x)$ is a finite reduced scheme of length at most $r = \rho(X_{\bar{\mathsf{k}}})$. If, moreover, $x \in X(\mathsf{k})$, then either $\mathrm{F}_1(X_{\bar{\mathsf{k}}}, x) = \varnothing$ or $\mathrm{F}_1(X_{\bar{\mathsf{k}}}, x)$ is a reduced scheme of length r and the Galois group $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$ action on $\mathrm{F}_1(X_{\bar{\mathsf{k}}}, x)$ factors through the group G_X and is transitive.*

Proof. By Lemma 2.7 for each $\bar{\mathsf{k}}$-point x of X, there is at most one line from each of the connected components of the Hilbert scheme $\mathrm{F}_1(X_{\bar{\mathsf{k}}})$ passing through x. This proves that $\mathrm{F}_1(X_{\bar{\mathsf{k}}}, x)$ is finite and reduced and gives the bound for its length.

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Now, assume x is a point of X defined over k, and let L be a $\bar{\mathsf{k}}$-line through x. Then for any $g \in \mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$, the line $g(L)$ also passes through x. Transitivity of the Galois action on the set of components of $\mathrm{F}_1(X_{\bar{\mathsf{k}}})$ then implies that there is a unique line of each type through x, hence, the length of $\mathrm{F}_1(X_{\bar{\mathsf{k}}},x)$ is r and the $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$-action on $\mathrm{F}_1(X_{\bar{\mathsf{k}}},x)$ factors through G_X and is transitive. □

2.3. Conics

By a conic on X, we understand a connected curve (defined over $\bar{\mathsf{k}}$) of anticanonical degree 2. We denote by $\mathrm{F}_2(X)$, the Hilbert scheme of conics on X. As before note that $\mathrm{F}_2(X)_{\bar{\mathsf{k}}} \cong \mathrm{F}_2(X_{\bar{\mathsf{k}}})$.

Lemma 2.9. *Let X be a Fano threefold of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(3,3)}$ or $\mathsf{X}_{(1,1,1,1)}$. We have the following descriptions of the Hilbert schemes of conics $\mathrm{F}_2(X_{\bar{\mathsf{k}}})$:*

$$\begin{aligned}
\mathrm{F}_2((X_{(2,2,2)})_{\bar{\mathsf{k}}}) &\cong \mathbb{P}^2_{\bar{\mathsf{k}}} \sqcup \mathbb{P}^2_{\bar{\mathsf{k}}} \sqcup \mathbb{P}^2_{\bar{\mathsf{k}}},\\
\mathrm{F}_2((X_{(4,4)})_{\bar{\mathsf{k}}}) &\cong \Gamma_1 \times \Gamma_2,\\
\mathrm{F}_2((X_{(3,3)})_{\bar{\mathsf{k}}}) &\cong \mathrm{Sym}^2\Gamma_1 \cong \mathrm{Sym}^2\Gamma_2,\\
\mathrm{F}_2((X_{(1,1,1,1)})_{\bar{\mathsf{k}}}) &\cong \bigsqcup_6 (\mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}),
\end{aligned}$$

where Γ_i are the curves described in Lemma 2.4.

Moreover, the morphism from each component of the universal conic to $X_{\bar{\mathsf{k}}}$ is dominant.

Proof. First, note that no conic on X is contracted by the projections $\pi_{\hat{\imath}}$, since by Lemma 2.4 any reduced connected curve contracted by $\pi_{\hat{\imath}}$ is a line, and lines do not support nonreduced conics by (2.2.1) and [12, Remark 2.1.7]. Therefore, we deduce from (2.1.2), that for each $\bar{\mathsf{k}}$-conic $C \subset X_{\bar{\mathsf{k}}}$, there is a pair of indices $1 \le i_1 < i_2 \le r$, such that

$$H_{i_1} \cdot C = H_{i_2} \cdot C = 1 \qquad \text{and} \qquad H_j \cdot C = 0 \quad \text{for } j \notin \{i_1, i_2\}. \tag{2.3.1}$$

If $r \ge 3$, that is, if X is of type $\mathsf{X}_{(2,2,2)}$ or $\mathsf{X}_{(1,1,1,1)}$, such C is contracted by one of the projections

$$\pi_i \colon X_{\bar{\mathsf{k}}} \longrightarrow \mathbb{P}^2_{\bar{\mathsf{k}}} \qquad \text{or} \qquad \pi_{i_1,i_2} \colon X_{\bar{\mathsf{k}}} \longrightarrow \mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}, \tag{2.3.2}$$

respectively. It is easy to see that the maps (2.3.2) are flat conic bundles, hence, C is a fibre of one of them, and, therefore, $\mathrm{F}_2(X_{\bar{\mathsf{k}}})$ is the disjoint union of $\mathbb{P}^2_{\bar{\mathsf{k}}}$ or of $\mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}$, respectively.

Assume X is of type $\mathsf{X}_{(4,4)}$. Applying Corollary A.2 twice, we obtain a morphism

$$\varphi = (\varphi_1, \varphi_2) \colon \mathrm{F}_2(X_{\bar{\mathsf{k}}}) \longrightarrow \Gamma_1 \times \Gamma_2$$

that takes a smooth conic $C \subset X_{\bar{\mathsf{k}}}$ to the unique pair of lines (L_2, L_1) of different types, such that $C \cap L_i \ne \varnothing$. We will show that φ is an isomorphism.

First, note that by (2.3.1), if $C \subset X$ is a conic, then $\pi_1(C) \subset Q_1$ and $\pi_2(C) \subset Q_2$ are lines, and by Lemma 2.4(ii), they intersect the curves Γ_1 and Γ_2, respectively. Thus, by Corollary A.2, for $x_1 \in \Gamma_1$ if $[C] \in \varphi_1^{-1}(x_1)$, the line $\pi_1(C) \subset Q_1$ passes through x_1.

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Since any line on Q_1 through x_1 lies in the embedded tangent space to Q_1 at x_1, and the intersection of this tangent space with Q_1 is a two-dimensional quadratic cone with vertex at x_1, it follows that

$$\varphi_1^{-1}(x_1) \cong \mathbb{P}^1$$

for any $x_1 \in \Gamma_1$. Since $\Gamma_2 \cong \mathbb{P}^1$ also, the morphism φ is a morphism of $\mathbb{P}^1$-bundles over Γ_1 and, to show that it is an isomorphism, it is enough to check that it is birational.

So, consider a general pair (L_2, L_1) of lines on X of different types. It follows from (2.3.1) and Lemma 2.4(ii) that $\bar{L}_1 := \pi_1(L_1)$ is a line on Q_1 bisecant to Γ_1, $x_1 := \pi_1(L_2)$ is a point on Γ_1 and by Corollary A.2, the preimage $\varphi^{-1}(L_2, L_1)$ is the Hilbert scheme of lines $L \subset Q_1$ passing through x_1 and intersecting $\bar{L}_1$. By genericity, we may assume $x_1 \notin \bar{L}_1$ (i.e. that the lines L_1 and L_2 do not intersect). Then any line L as above is contained in the intersection of the plane spanned by $\bar{L}_1$ and x_1 with Q_1, which is equal to the union of the line $\bar{L}_1$ with a residual line. Therefore, L must be equal to the residual line, hence, the scheme $\varphi^{-1}(L_2, L_1)$ consists of a single point, so φ is birational, and, hence, it is an isomorphism.

Since the embedded tangent space to Q_1 at a general point $x \in Q_1$ intersects the quartic curve Γ_1 at four points, there are four lines on Q_1 through x intersecting Γ_1, hence, the universal conic is dominant of degree 4 over $X_{\bar{\mathsf{k}}}$.

Finally, assume X is of type $\mathsf{X}_{(3,3)}$. By (2.3.1) and Lemma 2.4(iii), the image of C with respect to the blowup $\pi_i \colon X_{\bar{\mathsf{k}}} \to \mathbb{P}^3$ is a line intersecting the curve $\Gamma_i \subset \mathbb{P}^3$ at two points. This defines a morphism

$$\mathrm{F}_2(X_{\bar{\mathsf{k}}}) \longrightarrow \mathrm{Sym}^2 \Gamma_i,$$

and it is easy to see that it is an isomorphism. It is also easy to see that for a general point $x \in \mathbb{P}^3$, there are seven lines passing through x and bisecant to Γ_1; therefore, the universal conic on $X_{\bar{\mathsf{k}}}$ is dominant of degree 7 over $X_{\bar{\mathsf{k}}}$. □

Remark 2.10. Let X be a threefold of type $\mathsf{X}_{(4,4)}$. Clearly, a general line on the quadric Q_1 passing through a point $x \in \Gamma_1$ is not bisecant to Γ_1, and its strict transform in X intersects the line $L_2 = \pi_1^{-1}(x)$ transversally. This means that a general conic intersecting L_2 is smooth and intersects L_2 transversally.

For a given curve $\Theta \subset X$, we denote by $\mathrm{F}_2(X, \Theta)$ the subscheme of the Hilbert scheme $\mathrm{F}_2(X)$ that parameterises conics intersecting the curve Θ and by $\mathscr{C}_2(X, \Theta) \subset \mathrm{F}_2(X, \Theta) \times X$ the restriction of the universal family of conics.

Lemma 2.11. *If X is of type $\mathsf{X}_{(4,4)}$ and Θ is a singular conic, then $\mathrm{F}_2(X_{\bar{\mathsf{k}}}, \Theta) \cong \Gamma_1 \cup \Gamma_2$ is the union of the two rulings of the surface $\mathrm{F}_2(X_{\bar{\mathsf{k}}}) \cong \Gamma_1 \times \Gamma_2$. Moreover, the natural projection $\mathscr{C}_2(X, \Theta) \to X$ is birational onto an anticanonical divisor $R_\Theta \subset X$ passing through each component of the curve Θ with multiplicity* 3.

Proof. Let L_1 and L_2 be the irreducible components (over $\bar{\mathsf{k}}$) of the conic Θ. The argument of Lemma 2.9 shows that L_i are lines of two different types and

$$\mathrm{F}_2(X_{\bar{\mathsf{k}}}, \Theta) = \mathrm{F}_2(X_{\bar{\mathsf{k}}}, L_1) \cup \mathrm{F}_2(X_{\bar{\mathsf{k}}}, L_2).$$

Recall that by Lemma 2.7, the curves Γ_1 and Γ_2 can be identified with the two connected components of $\mathrm{F}_1(X_{\bar{\mathsf{k}}})$ and the isomorphism $\mathrm{F}_2(X_{\bar{\mathsf{k}}}) \cong \Gamma_1 \times \Gamma_2$ of Lemma 2.9 is defined by taking a conic C to the unique pair of lines of different types intersecting C. This means that

$$\mathrm{F}_2(X_{\bar{\mathsf{k}}},\Theta) \cong (\Gamma_1 \times [L_1]) \cup ([L_2] \times \Gamma_2) \subset \Gamma_1 \times \Gamma_2;$$

thus, $\mathrm{F}_2(X_{\bar{\mathsf{k}}},\Theta)$ is the union of two rulings of the surface $\mathrm{F}_2(X)$ and we have the equality $\mathscr{C}_2(X_{\bar{\mathsf{k}}},\Theta) = \mathscr{C}_2(X_{\bar{\mathsf{k}}},L_1) \cup \mathscr{C}_2(X_{\bar{\mathsf{k}}},L_2)$.

Furthermore, it follows from the description of Lemma 2.9 that the natural morphism $\mathscr{C}_2(X_{\bar{\mathsf{k}}},L_2) \to X_{\bar{\mathsf{k}}}$ is birational onto the hyperplane section tangent to Q_1 at the point $\pi_1(L_2)$; it contains the line $\pi_1(L_1)$ with multiplicity 1 and has multiplicity 2 at the point $\pi_1(L_2)$. Similarly, the morphism $\mathscr{C}_2(X_{\bar{\mathsf{k}}},L_1) \to X_{\bar{\mathsf{k}}}$ is birational onto the hyperplane section containing the line $\pi_2(L_2)$ with multiplicity 1 and having multiplicity 2 at the point $\pi_2(L_1)$. Thus, the morphism $\mathscr{C}_2(X,\Theta) \to X$ is birational onto a divisor of class $(H_1 - L_1 - 2L_2) + (H_2 - L_2 - 2L_1) = H - 3\Theta$. □

2.4. Twisted cubic curves

Finally, we describe the Hilbert scheme $\mathrm{F}_3(X,x)$ of subschemes of X with Hilbert polynomial $3t+1$ with respect to H that pass through a point x; since X is an intersection of quadrics (Theorem 2.1) and contains no planes (Corollary 2.6), every such subscheme is a union of rational curves (see [10, Lemma 2.9]), so we will use the name rational normal cubic curves for subschemes parameterised by $\mathrm{F}_3(X,x)$. We denote by $\mathscr{C}_3(X,x) \subset \mathrm{F}_3(X,x) \times X$, the restriction of the universal family of curves. Recall the curves $\Gamma_{\hat{\imath}}$ described in Lemma 2.4.

Lemma 2.12. *Let X be a Fano threefold of types $\mathsf{X}_{(2,2,2)}$, $\mathsf{X}_{(4,4)}$, $\mathsf{X}_{(3,3)}$ or $\mathsf{X}_{(1,1,1,1)}$. If x is a k-point on X not lying on a $\bar{\mathsf{k}}$-line, one has the following descriptions of the schemes $\mathrm{F}_3(X_{\bar{\mathsf{k}}},x)$*

$$\begin{aligned}
\mathrm{F}_3((X_{(2,2,2)})_{\bar{\mathsf{k}}},x) &\cong \Gamma_{1,2} \cong \Gamma_{1,3} \cong \Gamma_{2,3},\\
\mathrm{F}_3((X_{(4,4)})_{\bar{\mathsf{k}}},x) &\cong \mathbb{P}^1_{\bar{\mathsf{k}}} \sqcup \mathbb{P}^1_{\bar{\mathsf{k}}},\\
\mathrm{F}_3((X_{(3,3)})_{\bar{\mathsf{k}}},x) &\cong \Gamma_1 \sqcup \Gamma_2,\\
\mathrm{F}_3((X_{(1,1,1,1)})_{\bar{\mathsf{k}}},x) &\cong \bigsqcup_8 \mathbb{P}^1_{\bar{\mathsf{k}}}.
\end{aligned}$$

Moreover, for threefolds of type $\mathsf{X}_{(4,4)}$, the natural projection $\mathscr{C}_3(X,x) \to X$ is birational onto an anticanonical divisor $R_x \subset X$ passing through the point x with multiplicity 4.

Proof. First, consider a threefold X of type $\mathsf{X}_{(2,2,2)}$. If C is a rational normal cubic curve and $H_i \cdot C = 0$ for some i, then C is contracted by one of the conic bundles (2.3.2), hence, the curve C is supported on a fibre of (2.3.2). But the conormal bundle of any such fibre is trivial, hence, it cannot support a nonreduced curve of arithmetic genus 0 and degree more than 2. This means that we have $H_i \cdot C = 1$ for each i, and we conclude from this, and Lemma 2.4, that the image of C under the map $\pi_{1,2}\colon X_{\bar{\mathsf{k}}} \to W_{1,2}$ is a rational curve of bidegree $(1,1)$ intersecting the curve $\Gamma_{1,2}$ and passing through x. The argument analogous

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to that of Corollary A.2 shows that there is a morphism

$$\varphi_{1,2}\colon \mathrm{F}_3(X,x) \longrightarrow \Gamma_{1,2}$$

that takes a twisted cubic curve $C \subset X$ to the unique point $x_{1,2} \in \Gamma_{1,2}$, such that $C \cap \pi_{1,2}^{-1}(x_{1,2}) \neq \varnothing$. This morphism is an isomorphism, because on $W_{1,2}$, there is a unique curve of bidegree $(1,1)$ through a given pair of points (unless they lie on a fibre of either of the projections $W_{1,2} \to \mathbb{P}^2_{\bar{\mathsf{k}}}$, in which case, x lies on a line in X). The same argument proves isomorphisms of $\mathrm{F}_3(X,x)$ with the curves $\Gamma_{1,3}$ and $\Gamma_{2,3}$.

Next, consider a threefold of type $\mathsf{X}_{(4,4)}$. If $H_i \cdot C = 0$ for some i, then C is contracted by π_i, hence, is supported on a line. But the conormal bundle of a line is globally generated by (2.2.1), hence, a line cannot support a nonreduced curve of arithmetic genus 0 and degree more than 1. This means that C has bidegree $(1,2)$ or $(2,1)$. In the first case, the image of C under π_1 is a line on the quadric Q_1 passing through x; hence, the corresponding component of $\mathrm{F}_3(X,x)_{\bar{\mathsf{k}}}$ is isomorphic to $\mathbb{P}^1_{\bar{\mathsf{k}}}$. It also follows that the corresponding component of $\mathscr{C}_3(X,x)$ is a Hirzebruch surface that maps birationally onto the hyperplane section of Q_1 tangent at x, that is, a divisor of class H_1 passing through x with multiplicity 2. The second component is described analogously. The total divisor class of the image R_x of $\mathscr{C}_3(X,x) \to X$ is $H_1 + H_2 - 4x$, that is, it is the anticanonical class passing through x with multiplicity 4.

Next, consider a threefold of type $\mathsf{X}_{(3,3)}$. The same argument as above shows that C has bidegree $(1,2)$ or $(2,1)$. In the first case, the image of C under π_1 is a line on $\mathbb{P}^3_{\bar{\mathsf{k}}}$ passing through x and intersecting the curve Γ_1. Since for any point of Γ_1 there is a unique line through it and x, the corresponding component of $\mathrm{F}_3(X_{\bar{\mathsf{k}}},x)$ is isomorphic to Γ_1. Analogously, the second component is isomorphic to Γ_2.

Finally, consider a threefold of type $\mathsf{X}_{(1,1,1,1)}$. Then, of course, $H_i \cdot C = 0$ for some i. The argument used for threefolds of type $\mathsf{X}_{(2,2,2)}$ shows this cannot hold for two distinct i. So, assume this holds for $i = 1$. By Lemma 2.4(iv), the image of C under the map $\pi_{1,2,3}$ is a curve of multidegree $(0,1,1)$ on $\mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}$ intersecting the curve $\Gamma_{1,2,3}$ and passing through x. In other words, it is a curve of bidegree $(1,1)$ on the surface $\mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}$ passing through x and either of the two points of intersection of $\Gamma_{1,2,3}$ with this surface (note that these points cannot collide because otherwise x would lie on a line in X). Therefore, there are two pencils of such curves. Using the same argument for other i, we see that altogether there are eight pencils of twisted cubic curves on X passing through x. □

Remark 2.13. Let X be a threefold of type $\mathsf{X}_{(4,4)}$. Since a general line on Q_1 passing through a point $x \notin \Gamma_1$ does not intersect Γ_1, it follows that a general twisted cubic curve on X passing through x is smooth.

3. A birational transformation for a product of projective spaces

In this section, we construct a birational transformation for a product of projective spaces and deduce a consequence for the rationality of its k-forms; in particular, we prove the rationality criterion for threefolds $\mathsf{X}_{(1,1,1)}$.

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3.1. Product of projective spaces

Consider the product

$$Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r} = \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_r)$$

of projective spaces. Assume that $r = p+q$ and

$$n_1 \geq n_2 \geq \cdots \geq n_p \geq 2, \qquad n_{p+1} = \cdots = n_{p+q} = 1. \tag{3.1.1}$$

Let $y \in Y$ be a point, and let $(v_1, v_2, \dots, v_r)$, $0 \neq v_i \in V_i$, be the corresponding collection of vectors. Consider the blowup

$$\tilde{Y} = \mathrm{Bl}_y(Y),$$

and let $E \subset \tilde{Y}$ be its exceptional divisor. Let $\mathrm{PGL}(V_i)_{v_i} \subset \mathrm{PGL}(V_i)$ be the stabiliser of the point $[v_i] \in \mathbb{P}(V_i)$ in the projective linear group $\mathrm{PGL}(V_i)$. The group

$$G = \prod_{i=1}^{r} \mathrm{PGL}(V_i)_{v_i}$$

acts naturally on $\tilde{Y}$ and has finitely many orbits, which can be described as follows. First, for each $1 \leq i \leq r$, let

$$\tilde{Y}_i := \mathrm{Bl}_y\Big(\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_{i-1}) \times [v_i] \times \mathbb{P}(V_{i+1}) \times \cdots \times \mathbb{P}(V_r)\Big) \subset \tilde{Y}. \tag{3.1.2}$$

Furthermore, for any subset $I \subsetneq \{1, \dots, r\}$ denote

$$\tilde{Y}_I := \bigcap_{i \in I} \tilde{Y}_i \qquad \text{and} \qquad E_I := E \cap \tilde{Y}_I. \tag{3.1.3}$$

Finally, set

$$\tilde{Y}_I^\circ := \tilde{Y}_I \setminus \left(E_I \cup \bigcup_{I \subsetneq J} \tilde{Y}_J \right) \qquad \text{and} \qquad E_I^\circ := E_I \setminus \left(\bigcup_{I \subsetneq J} E_J \right). \tag{3.1.4}$$

Then $\tilde{Y}_\varnothing^\circ$ is the open orbit, $E_\varnothing^\circ$ and $\tilde{Y}_i^\circ$, $p+1 \leq i \leq q$, are the orbits of codimension 1 and all other orbits have higher codimension.

To describe the other side of the transformation, denote

$$\bar{V}_i := V_i / \mathsf{k} v_i$$

and choose splittings $V_i = \mathsf{k} v_i \oplus \bar{V}_i$. They induce a direct sum decomposition:

$$V_1 \otimes \cdots \otimes V_r = \bigoplus_{I \subset \{1, \dots, r\}} \bar{V}_I, \qquad \text{where} \quad \bar{V}_I := \bigotimes_{i \in I} \bar{V}_i.$$

Note that the point y corresponds to the summand $\bar{V}_\varnothing = \mathsf{k}$ and the tangent space to Y at y corresponds to the sum of the summands $\bar{V}_I$ with $|I| = 1$.

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Note also that for $i \ge p+1$, one has $\mathbb{P}(\bar{V}_i) \cong \operatorname{Spec}(\mathsf{k})$. Let

$$Y^+ := \prod_{i=1}^{r} \mathbb{P}(\bar{V}_i) = \mathbb{P}(\bar{V}_1) \times \mathbb{P}(\bar{V}_2) \times \cdots \times \mathbb{P}(\bar{V}_p) \cong \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} \times \cdots \times \mathbb{P}^{n_p-1}.$$

Denote by h_i the pullback to Y^+ of the hyperplane class of the i-th factor (note that $h_i = 0$ for $i \ge p+1$ because, as we noticed above, $\mathbb{P}(\bar{V}_i)$ is just a point), and for $I \subset \{1,\dots,r\}$, set

$$h_I := \sum_{i \in I} h_i.$$

Consider the vector bundle $\mathscr{E}$ of rank $r+1$ on Y^+ defined by

$$\mathscr{E} := \bigoplus_{|I| \ge r-1} \mathscr{O}(-h_I). \tag{3.1.5}$$

Denote by

$$s_i \colon Y^+ \longrightarrow \mathbb{P}_{Y^+}(\mathscr{E})$$

the section of (3.1.5) corresponding to the summand with $I = \{1,\dots,i-1,i+1,\dots,r\}$. Set

$$\hat{Y}^+ := \mathbb{P}_{Y^+}(\mathscr{E}), \qquad \tilde{Y}^+ := \operatorname{Bl}_{s_{p+1}(Y^+) \sqcup \cdots \sqcup s_{p+q}(Y^+)}(\hat{Y}^+), \tag{3.1.6}$$

and let $E_i \subset \tilde{Y}^+$, $p+1 \le i \le p+q$ be the exceptional divisors. The group G acts transitively on Y^+, the vector bundle $\mathscr{E}$ is G-equivariant and its summands $\mathscr{O}(-h_I)$ with $|I| = r-1$ are G-invariant. Therefore, the action of G lifts naturally to $\hat{Y}^+$ and $\tilde{Y}^+$. Moreover, the action of G on $\tilde{Y}^+$ still has a finite number of orbits, which can be described as follows.

For a subset $J \subsetneq \{1,\dots,r\}$ denote

$$\bar{\mathscr{E}}_J = \bigoplus_{J \subset I,\ |I| = r-1} \mathscr{O}(-h_I); \tag{3.1.7}$$

this is a subbundle in $\mathscr{E}$ of corank $1+|J|$. Let $\tilde{Y}^+_J \subset \tilde{Y}^+$ denote the strict transform of $\mathbb{P}_{Y^+}(\bar{\mathscr{E}}_J)$. Then the G-orbits are

$$(\tilde{Y}^+)^\circ = \tilde{Y}^+ \setminus \left(\tilde{Y}^+_\varnothing \cup \bigcup_{i=p+1}^{q} E_i \right), \qquad (\tilde{Y}^+_J)^\circ = \tilde{Y}^+_J \setminus \left(\bigcup_{i=p+1}^{q} E_i \right),$$

$$E_i^\circ = E_i \setminus \tilde{Y}^+_\varnothing, \qquad E_{i,J}^\circ = (E_i \cap \tilde{Y}^+_J) \setminus \left(\bigcup_{J \subsetneq K} E_i \cap \tilde{Y}^+_K \right),$$

where in the last formula, we assume $i \notin J$. Note that $(\tilde{Y}^+)^\circ$ is the open orbit, $(\tilde{Y}^+_\varnothing)^\circ$ and E_i° are the orbits of codimension 1 and all other orbits have higher codimension.

The linear projection out of the point $[v_i]$ defines a $\mathrm{PGL}(V_i)_{v_i}$-equivariant rational map $\mathbb{P}(V_i) \dashrightarrow \mathbb{P}(\bar{V}_i)$, which is regular if $i \ge p+1$. The product of these maps is a G-equivariant

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rational map, which we denote by $\psi_0\colon Y \dashrightarrow Y^+$. It gives rise to the following birational transformation.

Theorem 3.1. *There is a small birational G-equivariant isomorphism $\psi\colon \tilde{Y} \dashrightarrow \tilde{Y}^+$ that fits into the commutative diagram*

$$\begin{tikzcd} \tilde{Y} \arrow[r, dashed, "\psi"] \arrow[dd, "\sigma"'] \arrow[dr, dotted, "\hat{\psi}"'] & \tilde{Y}^+ \arrow[d, "\tilde{\sigma}_+"] \arrow[dd, bend left=60, "\sigma_+"] \\ & \hat{Y}^+ \arrow[d, "\hat{\sigma}_+"] \\ Y \arrow[r, dashed, "\psi_0"'] \arrow[ur, dotted, "\hat{\psi}_0"] & Y^+ \end{tikzcd}, \tag{3.1.8}$$

where $\hat{\sigma}_+\colon \hat{Y}_+ = \mathbb{P}_{Y^+}(\mathscr{E}) \to Y^+$ and $\tilde{\sigma}_+\colon \tilde{Y}_+ \to \mathrm{Bl}_{s_{p+1}(Y^+)\sqcup\cdots\sqcup s_{p+q}(Y^+)}(\hat{Y}^+) \to \hat{Y}^+$ is the projection and the blowup, respectively, $\sigma_+ := \hat{\sigma}_+ \circ \tilde{\sigma}_+$ and, such that ψ induces isomorphisms of G-orbits

$$\tilde{Y}^\circ_\varnothing \cong (\tilde{Y}^+)^\circ, \qquad E^\circ_\varnothing \cong (\tilde{Y}^+_\varnothing)^\circ \qquad \text{and} \qquad \tilde{Y}^\circ_i \cong E^\circ_i$$

of codimension 0 and 1. Moreover, if

- *H_i, $1 \le i \le r$, are the hyperplane classes of $\mathbb{P}(V_i)$ and $H = H_1 + \cdots + H_r$,*
- *E is the exceptional divisor of σ,*
- *h is the relative hyperplane class of the projective bundle $\hat{\sigma}_+$,*
- *h_i, $1 \le i \le p$, are the hyperplane classes of $\mathbb{P}(\bar{V}_i)$ and*
- *e_i, $p+1 \le i \le p+q$, are the exceptional divisor classes of the blowup $\tilde{\sigma}_+$,*

then in the Picard group $\mathrm{Pic}(\tilde{Y}) = \mathrm{Pic}(\tilde{Y}^+)$, there are the following equalities

$$\begin{array}{rcll} h_i & = & H_i - E, & 1 \le i \le p, \\ e_i & = & H_i - E, & p+1 \le i \le p+q, \\ h & = & H - (r-1)E. & \end{array} \tag{3.1.9}$$

Conversely, one has

$$E = h - \sum_{i=1}^{p} h_i - \sum_{j=p+1}^{p+q} e_j, \qquad H_i = \begin{cases} h_i + E, & \text{for } 1 \le i \le p, \\ e_i + E, & \text{for } p+1 \le i \le p+q. \end{cases} \tag{3.1.10}$$

The maps $\hat{\psi}_0$ and $\hat{\psi}$ in (3.1.8) will be defined in the course of proof.

Proof. For each $u_i \in V_i$, denote by $\bar{u}_i \in \bar{V}_i$ the image of u_i under the linear projection from the fixed vector $v_i \in V_i$. Then the rational map $\psi_0\colon Y \dashrightarrow Y^+$ is given by the formula

$$(u_1, \ldots, u_r) \longmapsto (\bar{u}_1, \ldots, \bar{u}_r).$$

This map is regular on the open orbit $Y^\circ \subset Y$ (given by the conditions $\bar{u}_i \ne 0$ for all indices $1 \le i \le r$), and it extends regularly to the orbits $Y^\circ_i \subset Y$ of codimension 1 (given by the condition $\bar{u}_i = 0$ for some $p+1 \le i \le p+q$ and $\bar{u}_j \ne 0$ for all $j \ne i$).

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Now, consider the rational G-equivariant map

$$\hat{\psi}_0\colon Y \dashrightarrow \hat{Y}^+, \qquad (u_1,\dots,u_r) \longmapsto \left((\bar{u}_1,\dots,\bar{u}_r),\ \sum_{|I|\ge r-1} \bigotimes_{i\in I} \bar{u}_i \right). \tag{3.1.11}$$

Here, we consider the summand $\otimes_{i\in I}\bar{u}_i$ as a point in the fibre of the line bundle $\mathscr{O}(-h_I)$ and their sum for $|I|\ge r-1$ as a point in the fibre (of the projectivisation) of the vector bundle $\mathscr{E}$. Obviously, the map $\hat{\psi}_0$ induces an isomorphism of the open orbit $Y^\circ \subset Y$ onto the open orbit $\mathbb{P}_{Y^+}(\mathscr{E})\setminus\mathbb{P}_{Y^+}(\bar{\mathscr{E}}_\varnothing)$ in $\hat{Y}^+$ and contracts each orbit Y_i° of codimension 1 to the section $s_i(Y^+)\subset\hat{Y}^+$, $p+1\le i\le p+q$.

Now, consider the composition $\hat{\psi} = \hat{\psi}_0\circ\sigma\colon \tilde{Y} \dashrightarrow \hat{Y}^+$. The restriction of $\hat{\psi}$ to the exceptional divisor E is given by

$$\begin{aligned} E = \mathbb{P}(\bar{V}_1\oplus\cdots\oplus\bar{V}_r) &\dashrightarrow \hat{Y}^+,\\ (\bar{u}_1+\cdots+\bar{u}_r) &\longmapsto \left((\bar{u}_1,\dots,\bar{u}_r),\ \sum_{|I|=r-1} \bigotimes_{i\in I} \bar{u}_i \right). \end{aligned} \tag{3.1.12}$$

It maps $E^\circ_\varnothing \subset \tilde{Y}$ isomorphically onto the G-orbit $(\tilde{Y}^+_\varnothing)^\circ = \mathbb{P}_{Y^+}(\bar{\mathscr{E}})\setminus\left(\bigcup_{i=1}^r \mathbb{P}_{Y^+}(\bar{\mathscr{E}}_i)\right)$ of codimension 1. By the above arguments, it also gives an isomorphism of open G-orbits and contracts the orbits $\tilde{Y}_i^\circ \cong Y_i^\circ$, $p+1\le i\le p+q$, to the sections $s_i(Y^+)\subset\hat{Y}^+$. Therefore, $\hat{\psi}$ induces a birational isomorphism

$$\psi\colon \tilde{Y} \dashrightarrow \mathrm{Bl}_{s_{p+1}(Y^+)\sqcup\cdots\sqcup s_{p+q}(Y^+)}(\mathbb{P}_{Y^+}(\mathscr{E})) = \tilde{Y}^+.$$

Finally, it is easy to see that the induced map $\tilde{Y}_i^\circ \to E_i^\circ$ is an isomorphism for all indices $p+1\le i\le p+q$. This gives the commutative diagram (3.1.8) and proves that ψ is small.

The first two lines in (3.1.9) follow easily from the formulas (3.1.11), (3.1.12) and (3.1.2). The last line follows from the equality of the canonical classes of $\tilde{Y}$ and $\tilde{Y}^+$ expressed in terms of H_i and E on the one hand, and h_i, h and e_i on the other hand.

Finally, (3.1.10) follows from (3.1.9). □

Remark 3.2. Alternatively, one can use the fact that the varieties $\tilde{Y}$ and $\tilde{Y}^+$, as well as the birational isomorphism ψ, are toric. Thus, to check that ψ is small, it is enough to identify the generators of rays of the corresponding fans. Moreover, comparing the other cones in the fans, one can check that the map ψ factors as the composition

$$\tilde{Y} \overset{\psi_1}{\dashrightarrow} \tilde{Y}' \overset{\psi_2}{\dashrightarrow} \dots \overset{\psi_{r-2}}{\dashrightarrow} \tilde{Y}^{(r-2)} \overset{\psi_{r-1}}{\dashrightarrow} \tilde{Y}^+$$

of standard (anti)flips ψ_l in the strict transforms of $\tilde{Y}_I$ for $|I| = l$, $1\le l\le r-2$ and for $|I| = r-1$ with $\{p+1,\dots,p+q\}\subset I$, respectively.

3.2. Rationality of forms of products of projective spaces

Here, we apply the birational transformation of the previous subsection to deduce the following corollary (see [23] for a different proof).

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Corollary 3.3. *Let Y be a k-form of $\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\times\cdots\times\mathbb{P}^{n_r}$. For any $y\in Y(\mathsf{k})$, the diagram (3.1.8) is defined over k, and if $Y(\mathsf{k})\neq\varnothing$, then Y is k-rational.*

Proof. First, we prove that for any $y\in Y(\mathsf{k})$, the diagram (3.1.8) is defined over k. The divisor classes $H=\sum_{i=1}^{r}H_i$ and $H':=\sum_{i=1}^{p}H_i$ on $Y_{\bar{\mathsf{k}}}$ are Galois-invariant, and since we have $Y(\mathsf{k})\neq\varnothing$ by assumption, we conclude that they are defined over k. Also $\tilde{Y}$ and E are defined over k as y is a k-point. Therefore, the divisor classes

$$\sum_{i=1}^{p}h_i=H'-pE,\qquad h=H-(r-1)E\qquad\text{and}\qquad -\sum_{i=p+1}^{p+q}e_i=H-H'-qE$$

(which, by Theorem 3.1, are equal to the strict transforms of the classes that are ample on Y^+, relatively ample for $\hat{Y}^+\to Y^+$ and for $\tilde{Y}^+\to\hat{Y}^+$, respectively) are defined over k, hence, the varieties Y^+, $\hat{Y}^+$ and $\tilde{Y}^+$, equal to the images of $\tilde{Y}$ under the maps given by their appropriate linear combinations, are defined over k, as well as the remaining arrows in the diagram.

Now to prove k-rationality of Y, we argue by induction in $\dim(Y)=\sum n_i$. If the dimension is zero, there is nothing to prove. So, assume the dimension is positive and consider the diagram (3.1.8). By Theorem 3.1, the variety Y^+ is a k-form of the product $Y^+_{\bar{\mathsf{k}}}=\mathbb{P}^{n_1-1}_{\bar{\mathsf{k}}}\times\mathbb{P}^{n_2-1}_{\bar{\mathsf{k}}}\times\cdots\times\mathbb{P}^{n_r-1}_{\bar{\mathsf{k}}}$. By the Nishimura lemma (see [18]), we have $Y^+(\mathsf{k})\neq\varnothing$, hence, Y^+ is k-rational by the induction assumption. Furthermore, the morphism $\hat{Y}^+\to Y^+$ is a k-form of a projective bundle and, by (3.1.10), the strict transform of the exceptional divisor E of $\tilde{Y}$ provides for it a relative hyperplane section. But E is defined over k, therefore, $\tilde{Y}^+$ is rational over Y^+, hence, it is k-rational. It remains to note that the morphisms σ, ψ and $\tilde{\sigma}_+$ in (3.1.8) are birational, hence, Y is k-rational as well. □

Applying this to the case of a k-form of $(\mathbb{P}^1)^3$, we obtain

Corollary 3.4. *If X is a Fano threefold of type $\mathsf{X}_{(1,1,1)}$ with $X(\mathsf{k})\neq\varnothing$, then X is k-rational.*

For other applications of the theorem, we will often use the following observation. Recall the definitions (3.1.2) and (3.1.7) of the subvarieties $Y_i\subset Y$ of codimension n_i and subbundles $\bar{\mathscr{E}}_i\subset\mathscr{E}$ of corank 2.

Proposition 3.5. *Let Y be a k-form of $(\mathbb{P}^n)^r$ where $n\geq 2$, and assume Y has a k-point $y\in Y(\mathsf{k})$. Let $X\subset Y$ be a closed k-subvariety containing the point y, such that*

$$X_{\bar{\mathsf{k}}}=\bigcap_{\alpha=1}^{c}D_\alpha\subset(\mathbb{P}^n_{\bar{\mathsf{k}}})^r$$

is a complete intersection of divisors $D_\alpha\subset(\mathbb{P}^n_{\bar{\mathsf{k}}})^r$, where $1\leq\alpha\leq c$. Let $\tilde{D}_\alpha\subset\tilde{Y}_{\bar{\mathsf{k}}}$ and $\tilde{D}^+_\alpha\subset\tilde{Y}^+_{\bar{\mathsf{k}}}=\hat{Y}^+_{\bar{\mathsf{k}}}$ be the strict transforms of D_α, and set

$$\tilde{X}^+_{\bar{\mathsf{k}}}:=\bigcap_{\alpha=1}^{c}\tilde{D}^+_\alpha\subset\tilde{Y}^+_{\bar{\mathsf{k}}}=\hat{Y}^+_{\bar{\mathsf{k}}}=\mathbb{P}_{(\mathbb{P}^{n-1}_{\bar{\mathsf{k}}})^r}(\mathscr{E}).$$

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If X is smooth at y and for each $1 \le i \le r$ one has

$$\dim(X_{\bar{\mathsf{k}}} \cap (Y_i)_{\bar{\mathsf{k}}}) < \dim(X_{\bar{\mathsf{k}}}) \qquad \textit{and} \qquad \dim(\tilde{X}^+_{\bar{\mathsf{k}}} \cap \mathbb{P}_{Y^+_{\bar{\mathsf{k}}}}(\bar{\mathscr{E}}_i)) < \dim(X_{\bar{\mathsf{k}}}), \tag{3.2.1}$$

then the strict transform $\tilde{X}^+ = \psi_*(\mathrm{Bl}_y(X))$ of X in $\tilde{Y}^+$ is a k-form of the complete intersection $\tilde{X}^+_{\bar{\mathsf{k}}}$ and X is birational to $\tilde{X}^+$ over k.

Proof. By Corollary 3.3 the diagram (3.1.8) is defined over k, so it is enough to check that the complete intersection $\tilde{X}^+_{\bar{\mathsf{k}}}$ is equal to the strict transform of $\mathrm{Bl}_y(X_{\bar{\mathsf{k}}})$. First, note that the assumption that y is a smooth point of X implies that the strict transform $\mathrm{Bl}_y(X_{\bar{\mathsf{k}}})$ of $X_{\bar{\mathsf{k}}}$ in $\tilde{Y}$ is the complete intersection of the divisors $\tilde{D}_\alpha$. Furthermore, the first part of the assumptions (3.2.1) implies that the intersection of $\mathrm{Bl}_y(X_{\bar{\mathsf{k}}})$ with the open G-orbit in $\tilde{Y}_{\bar{\mathsf{k}}}$ is dense in $\mathrm{Bl}_y(X_{\bar{\mathsf{k}}})$. Therefore, the strict transform of $\mathrm{Bl}_y(X_{\bar{\mathsf{k}}})$ in $\tilde{Y}^+_{\bar{\mathsf{k}}}$ is contained in $\tilde{X}^+_{\bar{\mathsf{k}}}$. So, it remains to check that $\tilde{X}^+_{\bar{\mathsf{k}}}$ is irreducible of dimension $\dim(X_{\bar{\mathsf{k}}})$. This is definitely true for the intersection of $\tilde{X}^+_{\bar{\mathsf{k}}}$ with the complement of the union of projective subbundles $\mathbb{P}_{Y^+_{\bar{\mathsf{k}}}}(\bar{\mathscr{E}}_i)$, because the map ψ defines an isomorphism of this complement with an open subset of $\tilde{Y}_{\bar{\mathsf{k}}}$. On the other hand, the second part of the assumptions (3.2.1) gives a bound for the dimension of the intersections with these projective subbundles, which implies the irreducibility. □

4. Rationality and unirationality of types $\mathsf{X}_{(2,2)}$, $\mathsf{X}_{(2,2,2)}$ and $\mathsf{X}_{(1,1,1,1)}$

In this section, we prove rationality of Fano threefolds of types $\mathsf{X}_{(2,2)}$ and $\mathsf{X}_{(2,2,2)}$ as well as unirationality of threefolds of type $\mathsf{X}_{(1,1,1,1)}$ under the assumption $X(\mathsf{k}) \ne \varnothing$.

4.1. Rationality of $\mathsf{X}_{(2,2)}$

To start with, we deal with threefolds of type $\mathsf{X}_{(2,2)}$.

Proposition 4.1. *Let X be a Fano threefold of type $\mathsf{X}_{(2,2)}$. If $X(\mathsf{k}) \ne \varnothing$, then X is k-rational.*

Proof. Let x be a k-point of X. By definition, X is a smooth divisor of bidegree (1,1) in a k-form Y of $\mathbb{P}^2 \times \mathbb{P}^2$. Since the birational isomorphism $\psi\colon \mathrm{Bl}_x(Y) = \tilde{Y} \dashrightarrow \tilde{Y}^+ = \mathbb{P}_{Y^+}(\mathscr{E})$ is small by Theorem 3.1, it follows that X is birational to a k-form of a divisor

$$\tilde{X}^+_{\bar{\mathsf{k}}} \subset \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathscr{E}) = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathscr{O}(-1,-1) \oplus \mathscr{O}(-1,0) \oplus \mathscr{O}(0,-1)),$$

which by (3.1.10) has type $H_1 + H_2 - E = h$. Any such divisor corresponds to a morphism

$$\xi\colon \mathscr{O}(-1,-1) \oplus \mathscr{O}(-1,0) \oplus \mathscr{O}(0,-1) \longrightarrow \mathscr{O}.$$

Furthermore, the divisor $\tilde{X}^+ \subset \tilde{Y}^+$ comes with a morphism $\sigma_+\colon \tilde{X}^+ \to Y^+$ defined over k. By the Nishimura lemma, we have $\tilde{X}^+(\mathsf{k}) \ne \varnothing$, hence, $Y^+(\mathsf{k}) \ne \varnothing$, and since Y^+ is a k-form of $\mathbb{P}^1 \times \mathbb{P}^1$, it is k-rational by Corollary 3.3. Finally, the general fibre of the morphism $\sigma_+\colon \tilde{X}^+ \to Y^+$ is a 1-dimensional linear section of a form of a projective plane, hence, it is isomorphic to $\mathbb{P}^1$, hence, $\tilde{X}^+$ is rational over Y^+, hence, is k-rational, hence, so is X. □

Remark 4.2. One can check that the birational isomorphism $\psi\colon \tilde{X} \dashrightarrow \tilde{X}^+$is a flop in the union of the strict transforms of two $\bar{\mathsf{k}}$-lines passing through the point $x \in X$, that $\sigma_+\colon \tilde{X}^+ \to Y^+$ is the projectivisation of the vector bundle $\mathrm{Ker}(\xi)$ of rank 2 over Y^+ and that these maps provide a Sarkisov link (1.2.1). This is an example of a pseudoisomorphism between almost del Pezzo varieties of degree 5 (see [11, Lemma 5.4 and Proof of Theorem 1.2]).

4.2. Rationality of $\mathsf{X}_{(2,2,2)}$

A similar argument works for threefolds of type $\mathsf{X}_{(2,2,2)}$.

Proposition 4.3. *Let X be a Fano threefold of type $\mathsf{X}_{(2,2,2)}$. If $X(\mathsf{k}) \neq \varnothing$, then X is k-rational.*

Proof. Let x be a k-point of X. By definition, $X_{\bar{\mathsf{k}}}$ is a complete intersection of three divisors in $Y_{\bar{\mathsf{k}}} = \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \mathbb{P}(V_3) \cong \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ of multidegree (1,1,0), (1,0,1) and (0,1,1), respectively. Denote by

$$F_{12} \in V_1^\vee \otimes V_2^\vee, \qquad F_{13} \in V_1^\vee \otimes V_3^\vee, \qquad F_{23} \in V_2^\vee \otimes V_3^\vee,$$

their equations. We apply Proposition 3.5; for this, we consider the intersection

$$\begin{aligned}\tilde{X}^+_{\bar{\mathsf{k}}} &\subset \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(\mathscr{E}) \\ &= \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(\mathscr{O}(-1,-1,-1) \oplus \mathscr{O}(-1,-1,0) \oplus \mathscr{O}(-1,0,-1) \oplus \mathscr{O}(0,-1,-1))\end{aligned}$$

of the three strict transforms of the above divisors, which, by (3.1.10), have types

$$H_1 + H_2 - E = h - h_3, \qquad H_1 + H_3 - E = h - h_2, \qquad H_2 + H_3 - E = h - h_1,$$

hence, correspond to a morphism of vector bundles

$$\begin{aligned}\xi\colon \mathscr{O}(-1,-1,-1) \oplus \mathscr{O}(-1,-1,0) \oplus \mathscr{O}(-1,0,-1) \oplus \mathscr{O}(0,-1,-1) \longrightarrow \\ \longrightarrow \mathscr{O}(0,0,-1) \oplus \mathscr{O}(0,-1,0) \oplus \mathscr{O}(-1,0,0).\end{aligned}$$

It is easy to see that ξ is given by the matrix

$$\xi = \begin{pmatrix} \bar{F}_{12} & 0 & F_{12}(-,v_2) & F_{12}(v_1,-) \\ \bar{F}_{13} & F_{13}(-,v_3) & 0 & F_{13}(v_1,-) \\ \bar{F}_{23} & F_{23}(-,v_3) & F_{23}(v_2,-) & 0 \end{pmatrix}, \tag{4.2.1}$$

where we write $x = (v_1, v_2, v_3)$, choose splittings $V_i = \bar{\mathsf{k}}v_i \oplus \bar{V}_i$, write $\bar{F}_{ij}$ for the restriction of F_{ij} to $\bar{V}_i \otimes \bar{V}_j$ and consider $F_{ij}(v_i,-)$ and $F_{ij}(-,v_j)$ as linear functions on $\bar{V}_j$ and $\bar{V}_i$ by restriction.

Let us check the dimension conditions (3.2.1). Since $(Y_i)_{\bar{\mathsf{k}}}$ is a fibre of the projection

$$\pi_i\colon Y_{\bar{\mathsf{k}}} = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2,$$

it follows from Lemma 2.9 that $X_{\bar{\mathsf{k}}} \cap (Y_i)_{\bar{\mathsf{k}}}$ is a conic, hence, the first part of the dimension conditions is satisfied. To check the second part, we need to show that the restriction

$$\xi_{2,3}\colon \mathscr{O}(-1,0,-1) \oplus \mathscr{O}(0,-1,-1) \longrightarrow \mathscr{O}(0,0,-1) \oplus \mathscr{O}(0,-1,0) \oplus \mathscr{O}(-1,0,0)$$

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of ξ to the last two summands of $\mathscr{E}$ (given by the last two columns of (4.2.1)) cannot be everywhere degenerate and similarly for the restrictions $\xi_{1,3}$ and $\xi_{1,2}$. Assuming that $\xi_{2,3}$ is everywhere degenerate, we conclude from (4.2.1) that

$$\begin{aligned} F_{12}(v_1,-) = F_{13}(v_1,-) = 0 \quad &\text{or} \\ F_{12}(-,v_2) = F_{23}(v_2,-) = 0 \quad &\text{or} \\ F_{13}(v_1,-) = F_{23}(v_2,-) = 0. & \end{aligned}$$

In any case, it would follow that at least two of the bilinear forms $F_{i,j}$ are degenerate, hence, at least two of the divisors $W_{i,j} \subset \mathbb{P}(V_i) \times \mathbb{P}(V_j)$ (defined by the equation $F_{i,j}$) are singular, which contradicts Lemma 2.4(i).

Thus, the conditions (3.2.1) are satisfied and we conclude from Proposition 3.5 that X is k-birational to a k-form of the complete intersection $\tilde{X}^+_{\bar{\mathsf{k}}}$.

Finally, the subvariety $\tilde{X}^+ \subset \tilde{Y}^+$ comes with a morphism $\sigma_+ \colon \tilde{X}^+ \to Y^+$ defined over k. By the Nishimura lemma, we have $\tilde{X}^+(\mathsf{k}) \neq \varnothing$, hence, $Y^+(\mathsf{k}) \neq \varnothing$, and since Y^+ is a k-form of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, it is k-rational by Corollary 3.3. Moreover, the general fibre of the morphism $\sigma_+ \colon \tilde{X}^+ \to Y^+$ is a zero-dimensional linear section of a form of a projective space, hence, this morphism is birational, hence, $\tilde{X}^+$ is k-rational, hence, so is X. $\square$

Remark 4.4. If the point x does not lie on a $\bar{\mathsf{k}}$-line, that is, $\mathrm{F}_1(X,x) = \varnothing$, one can check that the birational isomorphism $\psi \colon \tilde{X} \dashrightarrow \tilde{X}^+$ is a flop in the union of the strict transforms of three smooth $\bar{\mathsf{k}}$-conics passing through the point $x \in X$, that $\sigma_+ \colon \tilde{X}^+ \to Y^+$ is the blowup of a smooth geometrically rational curve of multidegree $(2,2,2)$ and that these maps provide a Sarkisov link (1.2.1).

4.3. Unirationality of $\mathsf{X}_{(1,1,1,1)}$

Finally, we deal with threefolds of type $\mathsf{X}_{(1,1,1,1)}$.

Proposition 4.5. *Let X be a Fano threefold of type $\mathsf{X}_{(1,1,1,1)}$. If $X(\mathsf{k}) \neq \varnothing$, then X is k-unirational.*

Proof. Let x be a k-point of X. By definition, the variety X is a smooth divisor of multidegree $(1,1,1,1)$ in a k-form Y of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The birational isomorphism $\psi \colon \mathrm{Bl}_x(Y) = \tilde{Y} \dashrightarrow \tilde{Y}^+ = \mathrm{Bl}_{s_1,s_2,s_3,s_4}(\mathbb{P}^4)$ is small by Theorem 3.1, so it follows that X is birational to a k-form of a divisor

$$\tilde{X}^+_{\bar{\mathsf{k}}} \subset \mathrm{Bl}_{s_1,s_2,s_3,s_4}(\mathbb{P}^4),$$

which, by (3.1.10), has type $H_1 + H_2 + H_3 + H_4 - E = 3h - 2\sum_{i=1}^4 e_i$, in particular, $\tilde{X}^+$ is a cubic hypersurface. Moreover, the exceptional divisor $E \cap \mathrm{Bl}_x(X) \subset \tilde{Y} = \mathrm{Bl}_x(Y)$ of the blowup $\mathrm{Bl}_x(X) \to X$ is an irreducible k-rational surface birational to a k-form of the complete intersection of $\tilde{X}^+_{\bar{\mathsf{k}}}$ with the linear span $\mathbb{P}^3 \subset \mathbb{P}^4$ of the points s_i.

Let us prove that $\tilde{X}^+_{\bar{\mathsf{k}}}$ is not a cone. Indeed, $\tilde{X}^+_{\bar{\mathsf{k}}}$ is smooth away from the linear span $\mathbb{P}^3$ of the s_i, because the map ψ from Theorem 3.1 is an isomorphism over its complement and $X_{\bar{\mathsf{k}}}$ is smooth, so if $\tilde{X}^+_{\bar{\mathsf{k}}}$ is a cone, its vertex belongs to the $\mathbb{P}^3$. But then its intersection with the $\mathbb{P}^3$ (which has been shown to be an irreducible k-rational surface) is itself a cone and has a singular point at each of the s_i. But it is easy to see that any such cone is reducible; this contradiction proves the claim. Now, we conclude that $\tilde{X}^+$ is k-unirational by [9, Theorem 1.2]. □

5. Rationality of type $\mathsf{X}_{(4,4)}$

In this section, we prove rationality of Fano threefolds of type $\mathsf{X}_{(4,4)}$.

5.1. Sarkisov links

We start with a construction of two Sarkisov links. Recall that $\mathrm{F}_1(X,x)$ denotes the Hilbert scheme of lines on X passing through x, see §2.2; and that by Lemma 2.8, if $x \in X(\mathsf{k})$ and $\mathrm{F}_1(X,x)$ is not empty, then $\mathrm{F}_1(X,x)$ is the union of two reduced $\bar{\mathsf{k}}$-points swapped by the Galois action. If L_1 and L_2 are the corresponding $\bar{\mathsf{k}}$-lines on X (passing through x), then

$$\Theta(x) := L_1 \cup L_2$$

is a singular k-conic on X irreducible over k and with $\mathrm{Sing}(\Theta(x)) = \{x\}$.

Recall that a quintic del Pezzo threefold is a Fano threefold of index 2 and half-anticanonical degree 5. Over an algebraically closed field, it can be realised as a complete intersection of the Grassmannian $\mathrm{Gr}(2,5)$ with a linear subspace of codimension 3 (see [6, Chapter 2, Theorem 1.1]).

Theorem 5.1. *Let X be a Fano threefold of type $\mathsf{X}_{(4,4)}$, and let $x \in X(\mathsf{k})$ be a k-point.*

(i) *If $\mathrm{F}_1(X,x) = \varnothing$, there exists a Sarkisov link (1.2.1) defined over k, where:*
- *σ is the blowup of the point x,*
- *X^+ is a smooth quintic del Pezzo threefold and*
- *σ_+ is the blowup of a smooth k-irreducible curve $B^+ \subset X^+$ of degree 4 with two geometrically rational $\bar{\mathsf{k}}$-components.*

(ii) *If $\mathrm{F}_1(X,x) \neq \varnothing$, there exists a Sarkisov link (1.2.1) defined over k, where:*
- *σ is the blowup of the singular k-irreducible conic $\Theta(x)$,*
- *X^+ is a smooth Fano threefold of type $\mathsf{X}_{(2,2)}$ and*
- *σ_+ is the blowup of a singular k-irreducible curve $B^+ \subset X^+$ of degree 6 with two geometrically rational $\bar{\mathsf{k}}$-components.*

The proof of the theorem takes §5.1 and §5.2: in the rest of §5.1, we prove the existence of the links, and in §5.2, we describe them in detail. The proofs of cases (i) and (ii) are completely analogous, so to carry them on simultaneously, we introduce the following convenient notation:

$$m = m(x) := \begin{cases} 2, & \text{if } \mathrm{F}_1(X,x) = \varnothing, \\ 1, & \text{if } \mathrm{F}_1(X,x) \neq \varnothing. \end{cases} \tag{5.1.1}$$

The proof of the existence of the links is analogous to the first parts of [10, Theorems 5.9 and 5.17], so we use some results from [10, §5.1] below.

Let

$$\sigma\colon \tilde{X} \longrightarrow X$$

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be the blowup of X at x or at $\Theta(x)$, respectively. We denote by H (the pullback to $\tilde{X}$ of) the anticanonical class of X and by E the exceptional divisor of σ.

First, note that for $m = m(x)$, the anticanonical linear system

$$|-K_{\tilde{X}}| = |H - mE|$$

is base-point free by Theorem 2.1 and [10, Lemmas 5.5 and 5.7]. Moreover, combining [10, (5.1.7) and (5.1.9)], we can uniformly write

$$H^3 = 2g-2, \qquad H^2 \cdot E = 0, \qquad H \cdot E^2 = 2(m-2), \qquad E^3 = m-1, \tag{5.1.2}$$

where we recall from Table 1 that $g = \mathrm{g}(X) = 15$. We will also need the following observation.

Lemma 5.2. *The linear system $\mathscr{M} := |H-(m+1)E|$ on the blowup $\tilde{X}$ of X has positive dimension:*

$$\dim \mathscr{M} \geq g - m - 7 \geq 6 \tag{5.1.3}$$

and has no fixed components.

Proof. The dimension is estimated in [10, Lemma 5.4(i) and (iii)]. To prove that $\mathscr{M}$ has no fixed components, note that the linear system $|kE|$ is zero-dimensional for any $k \geq 0$ (since E is the exceptional divisor of a blowup), hence, the only possibility for a fixed component of $\mathscr{M}$ is provided by the divisor E with some multiplicity. So, assume

$$|H-(m+1)E| = (a-m-1)E + |H-aE|,$$

where $a \geq m+2$ and E is not a fixed component of the linear system $|H-aE|$. Since the linear system $|H-mE|$ is base-point free and $|H-aE|$ has no fixed components, using (5.1.2) we obtain

$$0 \leq (H-aE)^2 \cdot (H-mE) = 2g-2-a^2(m^2-3m+4)+4am(m-2).$$

When $m=2$, this gives $a^2 \leq 14$, hence, $a \leq 3$, and when $m=1$, this gives $(a+1)^2 \leq 15$, hence, $a \leq 2$. In both cases, this contradicts the assumption $a \geq m+2$. □

Now, we can deduce the existence of the Sarkisov links.

Proposition 5.3. *Let X be a Fano threefold of type $\mathsf{X}_{(4,4)}$ with a k-point x.*

(i) *If $\mathrm{F}_1(X,x) = \varnothing$, there exists a Sarkisov link (1.2.1), where σ is the blowup of x.*

(ii) *If $\mathrm{F}_1(X,x) \neq \varnothing$, there exists a Sarkisov link (1.2.1), where σ is the blowup of $\Theta(x)$.*

In both cases, the link is defined over k.

Proof. We use notation (5.1.1). Recall that the anticanonical class $H = -K_X$ is very ample and the image of the anticanonical embedding $X \subset \mathbb{P}^{g+1}$ is an intersection of quadrics (see Theorem 2.1). The anticanonical morphism $\phi\colon \tilde{X} \to \mathbb{P}^{g-m-1}$ cannot contract a divisor D, because by [10, Lemmas 5.5 and 5.7], this divisor is then a fixed component of $\mathscr{M}$, but by Lemma 5.2, this linear system has no fixed components. Therefore, the required link exists and is defined over k by [10, Lemmas 5.5 and 5.7]. □

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5.2. The second contraction

By Proposition 5.3 we have the diagram (1.2.1), so to finish the proof of Theorem 5.1, it remains to describe the extremal contraction σ_+. During this step, we systematically use the classification of extremal contractions from [5] and [16].

We denote by $H^+, E^+ \in \mathrm{Pic}(\tilde{X}^+)$, the strict transforms of the classes $H, E \in \mathrm{Pic}(\tilde{X})$. Note that

$$-K_{\tilde{X}^+} = H^+ - mE^+$$

because $-K_{\tilde{X}} = H - mE$ by definition of $\tilde{X}$ and the map ψ is an isomorphism in codimension one. Consider also the strict transform

$$\mathscr{M}^+ := |H^+ - (m+1)E^+|$$

of the linear system $\mathscr{M}$.

We denote by $\Upsilon_i \subset \tilde{X}$ the flopping curves and by $\Upsilon_i^+ \subset \tilde{X}^+$ the corresponding flopped curves. Finally, when $\mathrm{F}_1(X,x) = \varnothing$, we denote by C a general twisted cubic curve on X passing through x, and otherwise, we denote by C a general conic meeting $\Theta(x)$ (recall Lemmas 2.11 and 2.12 for the description of the corresponding Hilbert schemes). Note that

$$H \cdot C = m+1. \tag{5.2.1}$$

We denote by $\tilde{C}$ the strict transform of C in $\tilde{X}$ and by $\tilde{C}^+$ the strict transform of $\tilde{C}$ in $\tilde{X}^+$. Note that by Remarks 2.10 and 2.13, the curve $\tilde{C}$ is smooth and

$$E \cdot \tilde{C} = 1; \tag{5.2.2}$$

in particular, $(H - mE) \cdot \tilde{C} = 1$ and $\tilde{C}$ does not contain the curves Υ_i.

Lemma 5.4. *The nef cone of $\tilde{X}^+$ is generated by the anticanonical class $-K_{\tilde{X}^+}$ and $M^+ \in \mathscr{M}^+$, and the Mori cone of $\tilde{X}^+$ is generated by the class of the curves Υ_i^+ and the class of $\tilde{C}^+$. In particular, the extremal contraction σ_+ is given by a multiple of the linear system $\mathscr{M}^+$ and contracts the extremal ray generated by $\tilde{C}^+$.*

Proof. Since ϕ is crepant and ψ is a flop, the morphism ϕ_+ is crepant as well. Moreover, the anticanonical linear system $|-K_{\tilde{X}}|$ is base-point free by [10, Lemma 5.7], hence, $|-K_{\tilde{X}^+}|$ is base-point free as well.

On the other hand, we have $(H - mE) \cdot \Upsilon_i = -K_{\tilde{X}} \cdot \Upsilon_i = 0$, and since $H \cdot \Upsilon_i > 0$, we conclude that $E \cdot \Upsilon_i > 0$. Therefore, for $M \in \mathscr{M}$ we have

$$M \cdot \Upsilon_i = (H - (m+1)E) \cdot \Upsilon_i = -E \cdot \Upsilon_i < 0.$$

If $M^+ \in \mathscr{M}^+$ is the strict transform of M, this implies that $M^+ \cdot \Upsilon_i^+ > 0$ by one of the definitions of a flop (see, for example, [7, Definition 6.10]). Now if M^+ is not nef, it is negative on the extremal ray R corresponding to the contraction $\sigma_+ \colon \tilde{X}^+ \to X^+$. Since the canonical class is also negative on R, the contraction σ_+ cannot be small (see [2, Theorem 0] or [17, Corollary 6.3.4]), hence, curves in R sweep a subvariety of $\tilde{X}^+$ of dimension ≥ 2, hence, the base locus of $\mathscr{M}^+$ is at least two-dimensional, which contradicts Lemma 5.2. This proves that M^+ is nef.

Now, we have $-K_{\tilde{X}} \cdot \tilde{C} = (H - mE) \cdot \tilde{C} = 1$ by (5.2.1) and (5.2.2), hence, $-K_{\tilde{X}^+} \cdot \tilde{C}^+ = 1$. On the other hand, since a general divisor H meets C away from the indeterminacy locus of the map $X \dashrightarrow \tilde{X}^+$, we have $H^+ \cdot \tilde{C}^+ \geq H \cdot C = m+1$ and so $E^+ \cdot \tilde{C}^+ \geq 1$. Thus,

$$M^+ \cdot \tilde{C}^+ = (-K_{\tilde{X}^+} - E^+) \cdot \tilde{C}^+ = 1 - E^+ \cdot \tilde{C}^+ \leq 0.$$

Since M^+ is nef, $M^+ \cdot \tilde{C}^+ = 0$.

Combining the above computations, we conclude that the nef cone of $\tilde{X}^+$ is generated by $-K_{\tilde{X}^+}$ and M^+ and the Mori cone is generated by Υ_i^+ and $\tilde{C}^+$. The rest of the lemma follows from the Mori contraction theorem [16, Theorems 3.1 and 3.2]. $\square$

Now, we can finally prove Theorem 5.1.

Proof of Theorem 5.1. Let X be a Fano threefold of type $\mathsf{X}_{(4,4)}$. By Proposition 5.3, there exists a Sarkisov link (1.2.1), and it remains to describe the contraction σ_+. Since σ_+ is an extremal contraction, we have $\uprho(X^+) = \uprho(\tilde{X}^+) - 1 = \uprho(\tilde{X}) - 1 = \uprho(X)$, hence, $\uprho(X^+) = 1$. Similarly, we have $\uprho(X^+_{\bar{\mathsf{k}}}) \leq \uprho(\tilde{X}^+_{\bar{\mathsf{k}}}) - 1 = \uprho(\tilde{X}_{\bar{\mathsf{k}}}) - 1 = \uprho(X_{\bar{\mathsf{k}}}) = 2$, hence

$$\uprho(X^+_{\bar{\mathsf{k}}}) \leq 2. \tag{5.2.3}$$

On the other hand, in the case $\mathrm{F}_1(X,x) \neq \varnothing$, the varieties $\tilde{X}$ and $\tilde{X}^+$ are not smooth and, arguing as in the proof of [10, Theorem 5.9], we obtain

$$\operatorname{rk}\mathrm{Cl}(\tilde{X}^+) = 2, \qquad \operatorname{rk}\mathrm{Cl}(\tilde{X}^+_{\bar{\mathsf{k}}}) = 5 - m. \tag{5.2.4}$$

Since ϕ and ϕ_+ are crepant morphisms, the projection formula implies that any triple intersection product of divisor classes on $\tilde{X}^+$ which includes $K_{\tilde{X}^+}$ is equal to the analogous triple product on $\tilde{X}$, so using (5.1.2), we compute (recall that $g = \mathrm{g}(X) = 15$)

$$\begin{aligned} (-K_{\tilde{X}^+})^3 &= 2(g - m - 3) = 24 - 2m, \\ (-K_{\tilde{X}^+})^2 \cdot E^+ &= 4, \\ (-K_{\tilde{X}^+}) \cdot (E^+)^2 &= -2. \end{aligned} \tag{5.2.5}$$

On the other hand, by Lemma 5.4 and primitivity of $H^+ - (m+1)E^+$, we have

$$H^+ - (m+1)E^+ = \sigma_+^* A^+, \tag{5.2.6}$$

where A^+ is the ample generator of the Picard group of X^+. We have

$$\begin{aligned} (\sigma_+^* A^+)^2 \cdot (-K_{\tilde{X}^+}) &= (H^+ - (m+1)E^+)^2 \cdot (-K_{X^+}) \\ &= 2(g - m - 8) = 14 - 2m > 0, \end{aligned} \tag{5.2.7}$$

therefore, σ_+ is not a del Pezzo fibration. Similarly, if σ_+ is a conic bundle, it follows that

$$(A^+)^2 = 7 - m,$$

hence, X^+ is a smooth quintic or sextic del Pezzo surface, which of course contradicts the inequality (5.2.3). Therefore, the morphism σ_+ is birational.

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By Lemma 5.4, the morphism σ_+ contracts the strict transform R^+ of the divisor swept by curves C, that is, the strict transform of the divisor $R_x \subset X$ if $\mathrm{F}_1(X,x) = \varnothing$, or of the divisor $R_{\Theta(x)}$ otherwise. In both cases, Lemmas 2.11 and 2.12 show that R^+ has over $\bar{\mathsf{k}}$ two irreducible components swapped by the Galois group. Therefore, it follows from (5.2.4) that

$$\operatorname{rk}\operatorname{Cl}(X^+) = 1, \qquad \operatorname{rk}\operatorname{Cl}(X^+_{\bar{\mathsf{k}}}) = 3 - m,$$

and σ_+ is the blowup of two $\bar{\mathsf{k}}$-curves or two $\bar{\mathsf{k}}$-points. Furthermore, by Lemmas 2.11 and 2.12, we have

$$R^+ \sim H^+ - (m+2)E^+.$$

Denoting by i_+ the index of X^+ and by a_+ the discrepancy of the exceptional divisor R^+ of σ_+, and computing the anticanonical class of $\tilde{X}^+$ in two ways, we obtain the equality

$$H^+ - mE^+ = i_+(H^+ - (m+1)E^+) - a_+(H^+ - (m+2)E^+).$$

Solving this equation, we obtain $i_+ = 2$ and $a_+ = 1$. Thus, X^+ is a Fano threefold of index 2 and σ_+ is either the blowup of a k-irreducible curve B^+ with two $\bar{\mathsf{k}}$-components or of two rational double $\bar{\mathsf{k}}$-points on X^+ swapped by the Galois action. Moreover, using the equality from (5.2.7), we obtain

$$14 - 2m = (\sigma_+^* A^+)^2 \cdot (-K_{\tilde{X}^+}) = (\sigma_+^* A^+)^2 \cdot (2\sigma_+^* A^+ - R^+) = 2(\sigma_+^* A^+)^3,$$

hence, X^+ is a quintic or sextic del Pezzo threefold, respectively. Finally, if X^+ is singular, its class group $\operatorname{Cl}(X^+)$ has rank greater than 1 (see [20, Theorem 1.7]), which contradicts to the equality $\operatorname{rk}\operatorname{Cl}(X^+) = 1$ obtained above. Thus, X^+ is smooth and σ_+ is the blowup of a curve B^+, such that $B^+_{\bar{\mathsf{k}}}$ has two irreducible $\bar{\mathsf{k}}$-components swapped by the Galois group.

To finally compute the degree of B^+, recall that R^+ is the exceptional divisor of σ_+. Note that on the one hand, equalities (5.2.5) and (5.2.6) imply that

$$(\sigma_+^* A^+) \cdot (-K_{\tilde{X}^+})^2 = 2g - 2m - 10 = 20 - 2m,$$

and on the other hand, this expression is equal to

$$(\sigma_+^* A^+) \cdot (2\sigma_+^* A^+ - R^+)^2 = 4(\sigma_+^* A^+)^3 + (\sigma_+^* A^+) \cdot (R^+)^2 = 4(7-m) - \deg(B^+)$$

(where the degree is computed with respect to A^+). Thus,

$$\deg(B^+) = 8 - 2m,$$

hence, B^+ is a quartic or sextic curve with two connected $\bar{\mathsf{k}}$-components (swapped by the Galois action), that is, a union of two conics or two cubic curves. □

5.3. Rationality

Now, we use the constructed links to prove rationality of threefolds of type $\mathsf{X}_{(4,4)}$.

Proposition 5.5. *Let X be a Fano threefold of type $\mathsf{X}_{(4,4)}$. If $X(\mathsf{k}) \neq \varnothing$, then X is k-rational.*

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Proof. Let $x \in X(\mathsf{k})$ be a k-point. First, assume that $\mathrm{F}_1(X,x) = \varnothing$. Then, by Theorem 5.1(i), the variety X is birational to a smooth quintic del Pezzo threefold X^+. But X^+ is k-rational by [10, Theorem 3.3], hence, so is X.

Now, assume that $\mathrm{F}_1(X,x) \neq \varnothing$. Then, by Theorem 5.1(ii), the variety X is birational to a smooth Fano threefold X^+ of type $\mathsf{X}_{(2,2)}$. Moreover, by the Nishimura lemma, we have $X^+(\mathsf{k}) \neq \varnothing$. Therefore, X^+ is k-rational by Proposition 4.1, hence, so is X. □

6. Fano threefolds of type $\mathsf{X}_{(3,3)}$

In this section, we prove that a Fano threefold X of type $\mathsf{X}_{(3,3)}$ is k-unirational if $X(\mathsf{k}) \neq \varnothing$ but not k-rational if $\rho(X) = 1$.

6.1. The discriminant curve

Let X be a Fano threefold of type $\mathsf{X}_{(3,3)}$ with $X(\mathsf{k}) \neq \varnothing$. Recall from Lemma 2.5 that the image G_X of the Galois group $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$ in $\mathrm{Aut}(\mathrm{Pic}(X_{\bar{\mathsf{k}}}))$ is a group of order 2 swapping the generators H_1 and H_2 of $\mathrm{Pic}(X_{\bar{\mathsf{k}}})$. The homomorphism $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}) \to \mathrm{G}_X$, therefore, defines a quadratic extension k'/k, such that H_1 and H_2 are defined on $X_{\mathsf{k}'}$, hence,

$$X_{\mathsf{k}'} \cong \Big(\mathbb{P}(V_1) \times \mathbb{P}(V_2)\Big) \cap \mathbb{P}(A^\perp),$$

where V_i are k'-vector spaces of dimension 4 and $A \subset V_1^\vee \otimes V_2^\vee$ is the three-dimensional subspace of linear equations of $X_{\mathsf{k}'}$. Note that the k'-spaces $V_1 \otimes V_2$ and A are defined over k, as well as the inclusion $A \subset V_1^\vee \otimes V_2^\vee$. We think of vectors $a \in A$ as bilinear forms on $V_1 \otimes V_2$ and denote by

$$\Gamma \hookrightarrow \mathbb{P}(A)$$

the discriminant curve parameterising degenerate bilinear forms; it is also defined over k.

Lemma 6.1. *The curve Γ is a smooth plane quartic curve; in particular, it is a nonhyperelliptic curve of genus* 3.

Proof. The discriminant divisor in $\mathbb{P}(V_1^\vee \otimes V_2^\vee)$, that is, the divisor parameterising degenerate bilinear forms, is a quartic hypersurface, hence, Γ is a quartic curve or the entire plane. To prove that Γ is a smooth curve, we can work over k', and it is enough to show that the tangent space to Γ at any point is one-dimensional. Assume to the contrary, that the tangent space at a point $[a] \in \mathbb{P}(A)$ is two-dimensional; then

(i) either the bilinear form $a(-,-) \in V_1^\vee \otimes V_2^\vee$ has corank at least 2,

(ii) or a has corank 1, and if the vectors $v_1 \in V_1$ and $v_2 \in V_2$ generate its left and right kernels, respectively, the point $([v_1],[v_2]) \in \mathbb{P}(V_1) \times \mathbb{P}(V_2)$ belongs to $X_{\mathsf{k}'}$.

In case (i), if $K_1 \subset V_1$ and $K_2 \subset V_2$ are the left and right kernels of a (they have dimension ≥ 2 by assumption), then the form a vanishes on $\mathbb{P}(K_1) \times \mathbb{P}(K_2)$, hence, the intersection $\big(\mathbb{P}(K_1) \times \mathbb{P}(K_2)\big) \cap X_{\mathsf{k}'}$ is a codimension-2 linear section of $\mathbb{P}(K_1) \times \mathbb{P}(K_2)$, hence, it is nonempty. Therefore, in case (i), similarly to the case (ii), there is a point $([v_1],[v_2]) \in X_{\mathsf{k}'}$, such that v_1 and v_2 belong to the left and right kernels of some a. Then, the hyperplane

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section of $\mathbb{P}(V_1)\times\mathbb{P}(V_2)$ by the hyperplane corresponding to a is singular at $([v_1],[v_2])$, hence, $X_{\mathsf{k}'}$ is also singular at this point. □

As explained in Lemma 2.4(iii), the projections $\pi_1\colon X_{\mathsf{k}'}\to\mathbb{P}(V_1)$ and $\pi_2\colon X_{\mathsf{k}'}\to\mathbb{P}(V_2)$ defined over k', but not over k, are the blowups of smooth curves $\Gamma_i\subset\mathbb{P}(V_i)$ of genus 3 and degree 6 also defined over k'. The next lemma relates the k'-curves Γ_i to the discriminant curve Γ defined over k.

Lemma 6.2. *There is a natural isomorphism $\Gamma_i\cong\Gamma_{\mathsf{k}'}$ of curves over k'.*

Proof. The fibre of the projection π_1 over a point $[v_1]\in\mathbb{P}(V_1)$ is the intersection of the projectivisations of the orthogonals of v_1 with respect to all bilinear forms $a\in A$. Therefore, it has positive dimension if and only if v_1 belongs to the left kernel of one of the forms. Furthermore, if v_1 belongs to the left kernel of two distinct forms in A, the fibre of π_1 over $[v_1]$ contains a plane, which contradicts Corollary 2.6. This means that the morphism

$$\gamma_i\colon\Gamma_{\mathsf{k}'}\longrightarrow\mathbb{P}(V_i),\qquad a\longmapsto\operatorname{Ker}_i(a),$$

where Ker_1 and Ker_2 denote the left and right kernels of the bilinear form a, respectively, is an isomorphism $\Gamma_{\mathsf{k}'}\to\Gamma_i$. □

Remark 6.3. It is also easy to check that if $H_i|_\Gamma$ are the pullbacks of the hyperplane classes of $\Gamma_i\subset\mathbb{P}(V_i)$ to $\Gamma_{\mathsf{k}'}$ under the isomorphism of Lemma 6.2, then $H_1|_\Gamma+H_2|_\Gamma=3K_\Gamma$, and that the divisor classes $H_i|_\Gamma-K_\Gamma$ are noneffective and swapped by the $\mathrm{G}(\mathsf{k}'/\mathsf{k})$-action. Conversely, given two such classes on a curve $\Gamma_{\mathsf{k}'}$ one can reconstruct the variety X.

6.2. The double projection from a point

Recall the quadratic extension k'/k defined in §6.1. Recall also the canonical embedding $X\subset Y$, where Y is a k-form of $\mathbb{P}^3\times\mathbb{P}^3$. We consider the birational transformation of Theorem 3.1 for the variety $Y_{\mathsf{k}'}=\mathbb{P}(V_1)\times\mathbb{P}(V_2)$ associated with a k-point

$$x_0=([v_1],[v_2])\in X\subset Y.$$

As in §3.1, we denote $\bar V_i:=V_i/\mathsf{k}'v_i$ and choose a splitting $V_i=\mathsf{k}'v_i\oplus\bar V_i$. The transformation of Theorem 3.1 in this case looks as follows:

$$\begin{tikzcd}\tilde Y \arrow[r, dashed, "\psi"] \arrow[d, "\sigma"'] & \tilde Y^+ \arrow[d, "\sigma_+"] \\ Y & Y^+\end{tikzcd}, \tag{6.2.1}$$

where σ is the blowup of x_0, $Y^+_{\mathsf{k}'}\cong\mathbb{P}(\bar V_1)\times\mathbb{P}(\bar V_2)$, σ_+ is the projectivisation of the vector bundle

$$\mathscr{E}=\mathscr{O}(-h_1-h_2)\oplus\mathscr{O}(-h_1)\oplus\mathscr{O}(-h_2) \tag{6.2.2}$$

(here, h_i stand for the hyperplane classes of $\mathbb{P}(\bar V_i)$) over $Y^+_{\mathsf{k}'}$ and the map ψ is a small birational isomorphism. Note that all varieties and maps in (6.2.1) are defined over k.

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Recall also the relations (3.1.9) in $\operatorname{Pic}(\tilde{Y}_{\mathsf{k}'}) = \operatorname{Pic}(\tilde{Y}^+_{\mathsf{k}'})$ between the hyperplane classes H_i of the factors $\mathbb{P}(V_i)$ of $Y_{\mathsf{k}'}$, the class E of the exceptional divisor of σ, the hyperplane classes h_i and the relative hyperplane class h of $\tilde{Y}^+_{\mathsf{k}'} = \mathbb{P}_{Y^+_{\mathsf{k}'}}(\mathscr{E})$

$$\begin{cases} h_1 = H_1 - E, \\ h_2 = H_2 - E, \\ h \ = H_1 + H_2 - E \end{cases} \qquad \begin{cases} H_1 = h - h_2, \\ H_2 = h - h_1, \\ E = h - h_1 - h_2. \end{cases} \tag{6.2.3}$$

Since X is a smooth linear section of Y, containing the point x_0, it is a complete intersection of three divisors D_α, $1 \le \alpha \le 3$, in the linear system $|H_1 + H_2|$, whose strict transforms on $\tilde{Y}$ belong to the linear system $|H_1 + H_2 - E|$. Now, it follows from (6.2.3) that their strict transforms $\tilde{D}^+_\alpha$ on $\tilde{Y}^+$ belong to the linear system $|h|$. As in Proposition 3.5, we consider the complete intersection

$$\tilde{X}^+_{\mathsf{k}'} := \tilde{D}^+_1 \cap \tilde{D}^+_2 \cap \tilde{D}^+_3 \subset \mathbb{P}_{Y^+_{\mathsf{k}'}}(\mathscr{E}).$$

It follows that $\tilde{X}^+_{\mathsf{k}'}$ is determined by a morphism of vector bundles

$$\xi \colon \mathscr{E} \longrightarrow A^\vee \otimes \mathscr{O},$$

and if we choose a basis a_1, a_2, a_3 in A, it is easy to see that ξ is given by the matrix

$$\xi = \begin{pmatrix} \bar{a}_1(-,-) & a_1(-,v_2) & a_1(v_1,-) \\ \bar{a}_2(-,-) & a_2(-,v_2) & a_2(v_1,-) \\ \bar{a}_3(-,-) & a_3(-,v_2) & a_3(v_1,-) \end{pmatrix}, \tag{6.2.4}$$

where $\bar{a}_i \in \bar{V}_1^\vee \otimes \bar{V}_2^\vee$ denotes the restriction of the bilinear form a_i to $\bar{V}_1 \otimes \bar{V}_2$, while $a_i(-,v_2) \in V_1^\vee$ and $a_i(v_1,-) \in V_2^\vee$ are considered as linear functions on $\bar{V}_1$ and $\bar{V}_2$, respectively.

Proposition 6.4. *The threefold X is k-birational to the k-form $\tilde{X}^+$ of the threefold $\tilde{X}^+_{\mathsf{k}'}$ defined by (6.2.4) and to a k-form X^+ of its image in Y^+*

$$X^+_{\mathsf{k}'} = \sigma_+(\tilde{X}^+_{\mathsf{k}'}) = \{\det(\xi) = 0\} \subset \mathbb{P}(\bar{V}_1) \times \mathbb{P}(\bar{V}_2),$$

which is a geometrically irreducible and normal divisor of bidegree $(2,2)$. Moreover,

- *if $\mathrm{F}_1(X,x_0) = \varnothing$, then the variety $\tilde{X}^+ \cong \tilde{X} = \mathrm{Bl}_x(X)$ is smooth, the morphism $\sigma_+ \colon \tilde{X}^+ \to X^+$ is induced by the double projection from x_0 and it is a small resolution of singularities;*
- *if $\mathrm{F}_1(X,x_0) \ne \varnothing$, then the variety X^+ contains a k-form of a quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}(\bar{V}_1) \times \mathbb{P}(\bar{V}_2)$ rational over k.*

Proof. To prove birationality of $\tilde{X}$ and $\tilde{X}^+ = \psi_*(\tilde{X})$, we apply Proposition 3.5, so we need to verify the dimension conditions (3.2.1). We have $(Y_1)_{\mathsf{k}'} = [v_1] \times \mathbb{P}(V_2)$, hence

$$X_{\mathsf{k}'} \cap (Y_1)_{\mathsf{k}'} = ([v_1] \times \mathbb{P}(V_2)) \cap \mathbb{P}(A^\perp)$$

is a fibre of the projection $\pi_1 \colon X_{\mathsf{k}'} \to \mathbb{P}(V_1)$. By Lemma 2.4, it is a point or a line. A similar argument for $X_{\mathsf{k}'} \cap (Y_2)_{\mathsf{k}'}$ shows that the first part of (3.2.1) holds. Moreover, this

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argument also shows that in the case $\mathrm{F}_1(X,x_0) = \varnothing$, the blowup $\tilde{X}_{\mathsf{k}'}$ of $X_{\mathsf{k}'}$ has an empty intersection with the indeterminacy locus $(\tilde{Y}_1)_{\mathsf{k}'} \sqcup (\tilde{Y}_2)_{\mathsf{k}'}$ of the map ψ.

On the other hand, the subbundle $\bar{\mathscr{E}}_1 \subset \mathscr{E}$ is just the summand $\mathscr{O}(-h_2)$ in (6.2.2), hence, the corresponding intersection $\tilde{X}^+_{\mathsf{k}'} \cap \mathbb{P}_{Y^+_{\mathsf{k}'}}(\bar{\mathscr{E}}_1)$ is the zero locus of the morphism

$$\xi_2\colon \mathscr{O}(-h_2) \longmapsto A^\vee \otimes \mathscr{O}$$

given by the last column of (6.2.4). It is easy to see that this is empty, if $\mathrm{F}_1(X,x_0) = \varnothing$, or isomorphic to a line otherwise. A similar argument works for $\tilde{X}^+_{\mathsf{k}'} \cap \mathbb{P}_{Y^+_{\mathsf{k}'}}(\bar{\mathscr{E}}_2)$; therefore, the second part of (3.2.1) also holds. This proves that $\tilde{X}^+ = \psi_*(\tilde{X})$ is a k-form of $\tilde{X}^+_{\mathsf{k}'}$, it is k-birational to X, and if $\mathrm{F}_1(X,x_0) = \varnothing$, it is isomorphic to $\tilde{X}$, and, in particular, in this case, it is smooth.

Now, we describe the image of $\tilde{X}^+$ in Y^+. By definition, $\tilde{X}^+_{\mathsf{k}'}$ parameterises points in the projectivisations of kernel spaces of ξ; therefore, its image in $Y^+_{\mathsf{k}'} = \mathbb{P}(\bar{V}_1) \times \mathbb{P}(\bar{V}_2)$ is the degeneracy locus $X^+_{\mathsf{k}'}$ of ξ, which is, of course, given by the equation $\det(\xi) = 0$. Since $\det(\mathscr{E}) \cong \mathscr{O}(-2h_1 - 2h_2)$ by (6.2.2), this is a divisor of bidegree (2,2), which is geometrically irreducible because $\tilde{X}^+_{\mathsf{k}'}$ is. Moreover, fibres of the morphism $\sigma_+\colon \tilde{X}^+_{\mathsf{k}'} \to X^+_{\mathsf{k}'}$ are linear spaces, so, since both the source and the target are three-dimensional, the morphism is birational. To prove that $X^+_{\bar{\mathsf{k}}}$ is normal, we consider the Koszul resolution

$$\begin{aligned}0 &\longrightarrow \mathscr{O}_{\mathbb{P}_{Y^+_{\bar{\mathsf{k}}}}(-\mathscr{E})}(-3h) \longrightarrow \mathscr{O}_{\mathbb{P}_{Y^+_{\bar{\mathsf{k}}}}(-\mathscr{E})}(-2h)^{\oplus 3}\\ &\longrightarrow \mathscr{O}_{\mathbb{P}_{Y^+_{\bar{\mathsf{k}}}}(-\mathscr{E})}(-h)^{\oplus 3} \longrightarrow \mathscr{O}_{\mathbb{P}_{Y^+_{\bar{\mathsf{k}}}}(-\mathscr{E})} \longrightarrow \mathscr{O}_{\tilde{X}^+_{\bar{\mathsf{k}}}} \longrightarrow 0.\end{aligned}$$

Pushing it forward to $Y^+_{\bar{\mathsf{k}}}$, we obtain the following exact sequence

$$0 \longrightarrow \mathscr{O}_{Y^+_{\bar{\mathsf{k}}}}(-2h_1 - 2h_2) \longrightarrow \mathscr{O}_{Y^+_{\bar{\mathsf{k}}}} \longrightarrow \sigma_{+*}\mathscr{O}_{\tilde{X}^+_{\bar{\mathsf{k}}}} \longrightarrow 0.$$

It follows that $\sigma_{+*}\mathscr{O}_{\tilde{X}^+_{\bar{\mathsf{k}}}} \cong \mathscr{O}_{X^+_{\bar{\mathsf{k}}}}$, and since $\tilde{X}^+_{\bar{\mathsf{k}}}$ is normal, so is $X^+_{\bar{\mathsf{k}}}$.

Now, assume $\mathrm{F}_1(X,x_0) = \varnothing$. In this case, the pullback along the morphism σ_+ of the ample divisor class $h_1 + h_2$ on $\mathbb{P}(\bar{V}_1) \times \mathbb{P}(\bar{V}_2)$ by (6.2.3) equals $H_1 + H_2 - 2E$, the anticanonical class of $\tilde{X}^+ \cong \tilde{X}$, hence, the morphism σ_+ is the double projection from the point x_0. Consequently, it is small by the argument of [10, Theorem 5.17]. Indeed, by [10, Lemma 5.4(iii)], we have $\dim|H_1 + H_2 - 3E| \geq g - 9 = 2$ (recall that $g = \mathrm{g}(X) = 11$, see Table 1), hence, by [10, Lemma 5.7(ii)], any divisor D contracted by σ_+ must be a fixed component of $|H_1 + H_2 - 3E|$, and at the same time by [10, (5.1.8)], its class should be a multiple of $H_1 + H_2 - 5E$, and these two conclusions are incompatible.

Finally, assume that $\mathrm{F}_1(X,x_0) \neq \varnothing$. As it was explained in Lemma 6.2, this means that (for appropriate k'-basis in A) we have

$$a_1(v_1,-) = 0 \qquad \text{and} \qquad a_2(-,v_2) = 0$$

as linear functions on $\bar{V}_2$ and $\bar{V}_1$, respectively; moreover, $[a_1],[a_2] \in \mathbb{P}(A)$ as above are unique and swapped by the Galois action. Consider the surface

$$\{a_1(-,v_2) = 0,\ a_2(v_1,-) = 0\} \subset \mathbb{P}(\bar{V}_1) \times \mathbb{P}(\bar{V}_2) = Y^+_{\mathsf{k}'}$$

(isomorphic to $\mathbb{P}^1_{\mathsf{k}'} \times \mathbb{P}^1_{\mathsf{k}'}$). The equations defining it are Galois-conjugate, hence, it comes from a k-surface in Y^+. This surface is the image of the exceptional divisor E of σ, hence, it is k-rational. It is clear from (6.2.4) that this surface is contained in the degeneracy locus X^+ of ξ. □

6.3. A conic bundle structure

In this section, we work under the assumption $\mathrm{F}_1(X, x_0) = \varnothing$ and show that, in this case, X admits a nice conic bundle structure.

We will need a general result about what we call Springer resolutions. Let M be a variety, let $\xi\colon \mathscr{E}_1 \to \mathscr{E}_2^\vee$ be a morphism of vector bundles on M of the same rank and let $\xi^\vee\colon \mathscr{E}_2 \to \mathscr{E}_1^\vee$ be its dual morphism. Assume the degeneracy locus $Z \subset M$ of ξ is a geometrically integral divisor. Let $Z_1 \subset \mathbb{P}_M(\mathscr{E}_1)$ and $Z_2 \subset \mathbb{P}_M(\mathscr{E}_2)$ be the zero loci of the morphisms

$$\mathscr{O}(-h_{\mathscr{E}_1}) \hookrightarrow p_1^*\mathscr{E}_1 \xrightarrow{p_1^*\xi} p_1^*\mathscr{E}_2^\vee \qquad \text{and} \qquad \mathscr{O}(-h_{\mathscr{E}_2}) \hookrightarrow p_2^*\mathscr{E}_2 \xrightarrow{p_2^*\xi^\vee} p_2^*\mathscr{E}_1^\vee,$$

where $p_i\colon \mathbb{P}_M(\mathscr{E}_i) \to M$ are the projections, $h_{\mathscr{E}_i}$ are their relative hyperplane classes and the first arrows are the tautological embeddings.

Lemma 6.5. *If one of the morphisms*

$$p_1|_{Z_1}\colon Z_1 \longrightarrow Z \qquad \text{or} \qquad p_2|_{Z_2}\colon Z_2 \longrightarrow Z$$

is birational, then so is the other. Moreover, if one of them is small, then so is the other, and there is an equality

$$(h_{\mathscr{E}_1} + \mathrm{c}_1(p_1^*\mathscr{E}_1)) + (h_{\mathscr{E}_2} + \mathrm{c}_1(p_2^*\mathscr{E}_2)) = 0 \tag{6.3.1}$$

in the group $\mathrm{Cl}(Z_1) \cong \mathrm{Cl}(Z) \cong \mathrm{Cl}(Z_2)$, *where the isomorphisms of the class groups are induced by the small birational morphisms* p_i.

Proof. Let $Z^{\ge c} \subset Z$ be the locus of points where the corank of ξ is at least c (so that $Z = Z^{\ge 1}$). Then, both morphisms $p_i|_{Z_i}$ are $\mathbb{P}^{c-1}$-fibrations over $Z^{\ge c} \setminus Z^{\ge c+1}$. In particular, if one of the morphisms is birational, then $\dim Z^{\ge c} \le \dim Z - c$ for $c \ge 2$, and then the other morphism is also birational. Similarly, if one of the morphisms is small, then $\dim Z^{\ge c} \le \dim Z - c - 1$ for $c \ge 2$, and then the other morphism is also small. Finally, assuming that the morphisms are small, we have

$$\mathrm{Cl}(Z_1) = \mathrm{Cl}(Z_1 \setminus p_1^{-1}(Z^{\ge 2})) = \mathrm{Cl}(Z \setminus Z^{\ge 2}) = \mathrm{Cl}(Z_2 \setminus p_2^{-1}(Z^{\ge 2})) = \mathrm{Cl}(Z_2),$$

and when restricted to $Z \setminus Z^{\ge 2}$, the morphism ξ has constant corank 1, the summands in (6.3.1) are equal to $\mathrm{c}_1(\mathrm{Im}(\xi))$ and $\mathrm{c}_1(\mathrm{Im}(\xi^\vee))$ (respectively) and (6.3.1) follows from the natural duality isomorphism $\mathrm{Im}(\xi^\vee) \cong \mathrm{Im}(\xi)^\vee$. □

Now, coming back to the threefold X of type $\mathsf{X}_{(3,3)}$ and assuming $\mathrm{F}_1(X, x_0) = \varnothing$, we recall that the subvariety $X^+ \subset Y^+$ is the degeneracy locus of $\xi\colon \mathscr{E} \to A^\vee \otimes \mathscr{O}$ and note that $\tilde{X}^+ \subset \mathbb{P}_{Y^+}(\mathscr{E})$ is one of its Springer resolutions. Consider the other Springer resolution

$$\tilde{X}^{++} \subset \mathbb{P}_{Y^+}(A \otimes \mathscr{O}) \cong Y^+ \times \mathbb{P}(A),$$

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which, by definition, is the zero locus of the morphism

$$\mathscr{O}(-h_A) \hookrightarrow A \otimes \mathscr{O} \xrightarrow{\xi^\vee} \mathscr{E}^\vee,$$

where h_A is the hyperplane class of $\mathbb{P}(A)$ and we suppress the pullbacks in the notation. In view of (6.2.2), the scheme $X^{++}_{\mathsf{k}'}$ is just a complete intersection of divisors of types $h_1+h_2+h_A$, h_1+h_A and h_2+h_A in $\mathbb{P}(\bar{V}_1)\times\mathbb{P}(\bar{V}_2)\times\mathbb{P}(A)$ that correspond to the columns of (6.2.4).

Proposition 6.6. *If $\mathrm{F}_1(X,x_0)=\varnothing$, there is a commutative diagram defined over* k

$$\begin{tikzcd}
\tilde{X} \arrow[r, equal, "\psi"] \arrow[d, "\sigma"'] & \tilde{X}^+ \arrow[rr, dashed, "\psi_+"] \arrow[dr, "\sigma_+"] & & \tilde{X}^{++} \arrow[dl, "\sigma_{++}"'] \arrow[d, "f"] \\
X & & X^+ & \mathbb{P}(A),
\end{tikzcd} \tag{6.3.2}$$

where $\tilde{X}^+$ and $\tilde{X}^{++}$ are the Springer resolutions of the degeneracy locus $X^+\subset Y^+$ of ξ, the morphisms σ_+ and σ_{++} are small birational contractions and $\psi_+=\sigma_{++}^{-1}\circ\sigma_+$ is a flop.

Moreover, $\tilde{X}^{++}$ is smooth, f is a flat conic bundle whose discriminant curve is the curve Γ defined in §6.1, the map $f\circ\psi_+\circ\psi\colon \tilde{X} \dashrightarrow \mathbb{P}(A)$ is given by the linear system $|H_1+H_2-3E|$ and the exceptional divisor $E\subset\tilde{X}$ of σ dominates $\mathbb{P}(A)$.

Proof. The morphism σ_+ is small by Proposition 6.4, hence, σ_{++} is small by Lemma 6.5; moreover, it follows that both morphisms are crepant. Now, the relation (6.3.1) implies that the σ_+-antiample class $-h$ is σ_{++}-ample, hence, $\psi_+ := \sigma_{++}^{-1}\circ\sigma_+$ is a flop. Since $\tilde{X}^+\cong\tilde{X}$ is smooth and ψ_+ is a flop, $\tilde{X}^{++}$ is smooth as well (see [8, Theorem 2.4]).

Next, we show that f is a conic bundle and identify its discriminant. For this, note that by definition, the fibre of f over a point $[a]\in\mathbb{P}(A)$ is given in $\mathbb{P}(\bar{V}_1)\times\mathbb{P}(\bar{V}_2)$ by the equations

$$a(-,v_2)=a(v_1,-)=\bar{a}(-,-)=0.$$

The first is a linear function on $\bar{V}_1$, the second is a linear function on $\bar{V}_2$ and both are nonzero because $\mathrm{F}_1(X,x_0)=\varnothing$, so their common zero locus is $\mathbb{P}^1_{\mathsf{k}'}\times\mathbb{P}^1_{\mathsf{k}'}\subset\mathbb{P}(\bar{V}_1)\times\mathbb{P}(\bar{V}_2)$. The last equation $\bar{a}(-,-)=0$ cuts a divisor of bidegree (1,1) on this $\mathbb{P}^1_{\mathsf{k}'}\times\mathbb{P}^1_{\mathsf{k}'}$, that is, a conic; and if $\bar{a}(-,-)$ vanishes identically, then the corresponding bilinear form a vanishes on $\mathbb{P}^2_{\mathsf{k}'}\times\mathbb{P}^2_{\mathsf{k}'}\subset\mathbb{P}(V_1)\times\mathbb{P}(V_2)$, hence, has corank 2, which is impossible by the argument of Lemma 6.1 as X is smooth. This shows that f is a flat conic bundle. Finally, note that if $\bar{v}_i'$, $\bar{v}_i''$, $\bar{v}_i'''$ are bases of vector spaces $\bar{V}_i$, such that,

$$a(v_1,\bar{v}_2')=a(v_1,\bar{v}_2'')=a(\bar{v}_1',v_2)=a(\bar{v}_1'',v_2)=0$$

then the matrix of a has the form

$$\begin{pmatrix} 0 & 0 & 0 & a(v_1,\bar{v}_2''') \\ 0 & \bar{a}(\bar{v}_1',\bar{v}_2') & \bar{a}(\bar{v}_1',\bar{v}_2'') & * \\ 0 & \bar{a}(\bar{v}_1'',\bar{v}_2') & \bar{a}(\bar{v}_1'',\bar{v}_2'') & * \\ a(\bar{v}_1''',v_2) & * & * & * \end{pmatrix}$$

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with nonzero entries $a(v_1, \bar{v}_2''')$ and $a(\bar{v}_1''', v_2)$ and with the 2-by-2 matrix in the middle giving the equation of the conic $f^{-1}(a)$ in $\mathbb{P}^1_{\bar{\mathsf{k}}} \times \mathbb{P}^1_{\bar{\mathsf{k}}}$. Therefore, the conic is singular if and only if $\det(a) = 0$, that is, if and only if $[a] \in \Gamma$. Thus, the discriminant curve of f equals Γ.

Finally, using (6.3.1), (6.2.2) and (6.2.3), we deduce that the map $f \circ \psi_+ \circ \psi$ is given by the linear system

$$h_A = 2(h_1 + h_2) - h = H_1 + H_2 - 3E.$$

Since the canonical class of $\tilde{X}$ is equal to $H_1 + H_2 - 2E$ and ψ_+ is a flop, it follows that $\psi_*(E)$ is a relative anticanonical divisor for f, hence, it dominates $\mathbb{P}(A)$. □

If $f\colon \mathscr{X} \to S$ is a flat conic bundle over a surface S (not necessarily proper) with a smooth discriminant curve $\Delta \subset S$, consider the preimage $\mathscr{X}_\Delta := f^{-1}(\Delta)$, its normalisation $\mathscr{X}_\Delta^\nu \to \mathscr{X}_\Delta$ and the Stein factorisation

$$\mathscr{X}_\Delta^\nu \longrightarrow \tilde{\Delta} \longrightarrow \Delta.$$

Then, the first arrow is a $\mathbb{P}^1$-bundle and the second arrow is an étale double covering (because Δ was assumed to be smooth). We will say that the étale covering $\tilde{\Delta} \to \Delta$ is the discriminant double covering of the conic bundle f.

Lemma 6.7. *The discriminant double covering of the conic bundle $f\colon \tilde{X}^{++} \to \mathbb{P}(A)$ has the form*

$$\tilde{\Gamma} \cong \Gamma \times_{\mathsf{k}} \mathsf{k}' \longrightarrow \Gamma.$$

Proof. If the conic $f^{-1}([a])$ is singular, it is a union of two irreducible components that correspond to the two factors $\mathbb{P}(\bar{V}_i)$ in $Y^+_{\mathsf{k}'}$ and each of them is contracted by appropriate projection $Y^+_{\mathsf{k}'} \to \mathbb{P}(\bar{V}_i)$. Therefore, the discriminant double covering $\tilde{\Gamma} \to \Gamma$ becomes trivial after the extension of scalars to k', while it is nontrivial over k, hence, the claim. □

6.4. Unirationality

In this section, we prove unirationality of X assuming that $X(\mathsf{k}) \neq \varnothing$. We start with the following observation, which might be useful in other situations.

Lemma 6.8. *Let Y be a k-form of $\mathbb{P}^2 \times \mathbb{P}^2$, and let $W \subset Y$ be a k-rational k-form of a quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^2$. Any geometrically irreducible normal divisor $Z \subset Y$ of bidegree $(2,2)$, such that $W \subset Z$ is k-unirational.*

Proof. Consider the toric birational isomorphism

$$\begin{tikzcd}
 & \mathrm{Bl}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathbb{P}^2 \times \mathbb{P}^2) \arrow[r, equal] \arrow[dl] & \mathrm{Bl}_{\mathbb{P}^1 \sqcup \mathbb{P}^1}(\mathbb{P}^4) \arrow[dr] & \\
\mathbb{P}^2 \times \mathbb{P}^2 \arrow[rrr, dashed, "\chi"] & & & \mathbb{P}^4,
\end{tikzcd}$$

analogous to the birational transformation of Theorem 3.1 (see also [23, Proposition 3]). Here, the map χ is the projection from the linear span of W under the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and the right arrow is the blowup of two skew lines in $\mathbb{P}^4$. Denoting by $\hat{e}$ the

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class of the exceptional divisor of the left blowup, by $\hat{h}$ the hyperplane class of $\mathbb{P}^4$ and by e_1 and e_2 the classes of the exceptional divisors of the right blowup, it is easy to check that we have the relations

$$\begin{cases} \hat{h} = h_1 + h_2 - \hat{e}, \\ e_1 = h_1 - \hat{e}, \\ e_2 = h_2 - \hat{e}, \end{cases} \quad \text{and} \quad \begin{cases} h_1 = \hat{h} - e_2, \\ h_2 = \hat{h} - e_1, \\ \hat{e} = \hat{h} - e_1 - e_2. \end{cases}$$

In particular, the map χ is given by the linear system $|h_1 + h_2 - \hat{e}|$, hence, it is defined over k. Furthermore, we have $2h_1 + 2h_2 - \hat{e} = 3\hat{h} - e_1 - e_2$, and since Z is normal, the strict transform of Z under the map χ is a cubic threefold $\hat{Z} \subset \mathbb{P}^4$ passing through the pair of skew lines $\mathbb{P}^1 \sqcup \mathbb{P}^1$. Moreover, we have $\hat{e} = \hat{h} - e_1 - e_2$, hence, the image of $\hat{E}$ is the hyperplane section of this cubic threefold (by the linear span of these lines). On the other hand, $\hat{E}$ is birational to the k-rational surface W, hence, it is k-rational. In particular, $\hat{Z}(\mathsf{k}) \neq \varnothing$.

Now, if $\hat{Z}$ is not a cone, it is k-unirational by Kollár's theorem [9, Theorem 1.2]. Otherwise, if $\hat{Z}$ is a cone, and its vertex lies away from the hyperplane spanned by the two skew lines, then the base of the cone is the k-rational surface $\hat{E}$, hence, the cone $\hat{Z}$ is also k-rational. Finally, if the vertex of the cone lies on $\hat{E}$, then $\hat{E}$ itself must be a cubic cone in $\mathbb{P}^3$, and since it also contains two skew lines, it is not geometrically irreducible, which is absurd. □

Now, we can deduce unirationality of X.

Proposition 6.9. *If X is a Fano threefold of type $\mathsf{X}_{(3,3)}$ with $X(\mathsf{k}) \neq \varnothing$, then X is k-unirational.*

Proof. Let x_0 be a k-point on X.

First, assume $\mathrm{F}_1(X,x_0) = \varnothing$. By Proposition 6.6, we have a k-birational map $X \dashrightarrow \tilde{X}^{++}$, where $f\colon \tilde{X}^{++} \to \mathbb{P}(A)$ is a conic bundle, and the k-rational surface $E \cong \mathbb{P}(T_{x_0}X) \subset \tilde{X}$ dominates the base of this conic bundle. Therefore, $\tilde{X}$ is k-unirational (see, e.g. [10, Lemma 4.14(i)]); and, hence, so is X.

Now, assume $\mathrm{F}_1(X,x_0) \neq \varnothing$. By Proposition 6.4, we have a birational map $X \dashrightarrow X^+$, where X^+ is a geometrically irreducible normal divisor of bidegree (2,2) in a k-form of $\mathbb{P}^2 \times \mathbb{P}^2$ that contains a k-form of a k-rational quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, X^+ is k-unirational by Lemma 6.8, hence, so is X. □

6.5. Nonrationality

In this section, we prove nonrationality of Fano threefolds of type $\mathsf{X}_{(3,3)}$. We will use the following reformulation of a result of Benoist–Wittenberg from [3].

Theorem 6.10. *Let $\mathscr{X} \to S$ be a flat conic bundle over a smooth k-rational surface S with smooth connected discriminant curve $\Delta \subset S$. Assume the discriminant double covering takes the form*

$$\tilde{\Delta} \cong \Delta \times_{\mathsf{k}} \mathsf{k}' \longrightarrow \Delta,$$

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where k'/k is a quadratic extension of the base field. If the conic bundle $\mathscr{X}_{\mathsf{k}'} \to S_{\mathsf{k}'}$ admits a rational section, and the curve Δ is not hyperelliptic, then $\mathscr{X}$ is not k-rational.

Note that we require neither the surface S nor the curve Δ to be proper; moreover, during the proof, we will further shrink S but keep (the generic point of) the curve Δ in S.

Proof. Since S is normal and f is proper, any rational section of f extends to codimension 1 point, hence, defines a regular section over the complement of a finite subscheme of S. Moreover, over the complement of this finite subscheme, the section does not pass through singular points of fibres of f, hence, it defines a section of the morphism $\mathscr{X}_\Delta^\nu \to \Delta$, where recall that $\mathscr{X}_\Delta^\nu$ is the normalisation of $\mathscr{X}_\Delta = f^{-1}(\Delta)$. Therefore, it also gives a section of the discriminant double covering $\tilde{\Delta} \to \Delta$. If the original section is defined over k, we obtain a contradiction with the isomorphism $\tilde{\Delta} \cong \Delta \times_{\mathsf{k}} \mathsf{k}'$; this means that the morphism f has no rational sections defined over k.

Now, consider a rational section of $f\colon \mathscr{X}_{\mathsf{k}'} \to S_{\mathsf{k}'}$. Removing, if necessary, a finite subscheme from S, we may assume that this section is regular. Its intersection with the conjugate section (with respect to the $\mathrm{G}(\mathsf{k}'/\mathsf{k})$-action) projects to a curve in S which is disjoint from Δ (because a regular section does not pass through singular points of fibres). So, shrinking S further, we may assume that the section and its conjugate do not intersect. Then the union

$$Z \subset \mathscr{X}$$

of the section and its conjugate is a 2-section of f defined over k; moreover, $Z \cong S \times_{\mathsf{k}} \mathsf{k}'$, and, in particular, Z is étale over S.

Consider the bundles $\mathscr{V} := (f_*\omega_{\mathscr{X}}^{-1})^\vee$ of rank 3 and $\mathscr{V}_Z := (f_*\omega_{\mathscr{X}}^{-1}|_Z)^\vee$ of rank 2 on S. The restriction morphism $\omega_{\mathscr{X}}^{-1} \to \omega_{\mathscr{X}}^{-1}|_Z$ induces an embedding of vector bundles $\mathscr{V}_Z \hookrightarrow \mathscr{V}$ and a Cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathbb{P}_S(\mathscr{V}_Z) \arrow[d] \\ \mathscr{X} \arrow[r] & \mathbb{P}_S(\mathscr{V}), \end{tikzcd}$$

where all arrows are the natural embeddings.

Shrinking the surface S again but keeping an open part of the curve Δ in it, we may assume that the bundles $\mathscr{V}$ and $\mathscr{V}_Z$ are trivial, that the subvarieties $\mathscr{X} \subset \mathbb{P}_S(\mathscr{V})$ and $Z \subset \mathbb{P}_S(\mathscr{V}_Z)$ are given by a quadratic form $q \in \operatorname{Sym}^2 \mathscr{V}^\vee$ and its restriction $q_Z \in \operatorname{Sym}^2 \mathscr{V}_Z^\vee$ to $\mathscr{V}_Z$, respectively. Since Z is étale over S, the form q_Z is everywhere nondegenerate and can be written as follows:

$$q_Z = x^2 - \alpha y^2,$$

where (x,y) are homogeneous coordinates in the fibre of $\mathbb{P}_S(\mathscr{V}_Z) \cong S \times \mathbb{P}^1$ and $\alpha \in \mathsf{k}^\times$ is, such that $\mathsf{k}' = \mathsf{k}(\sqrt{\alpha})$. Now, considering the orthogonal complement to $\mathscr{V}_Z$ in $\mathscr{V}$, we see

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that q takes the form

$$q = x^2 - \alpha y^2 - Fz^2,$$

where F is an equation of Δ on S. Thus, the conic bundle $\mathscr{X} \to S$ is birational to conic bundles considered in [3, §3.3.1], hence, $\mathscr{X}$ is not k-rational by [3, Proposition 3.4]. □

Now, we apply this to prove nonrationality of threefolds of type $\mathsf{X}_{(3,3)}$.

Corollary 6.11. *If X is a Fano threefold of type $\mathsf{X}_{(3,3)}$, then X is not k-rational.*

Proof. If X is not k-unirational, there is nothing to prove. So, assume X is k-unirational. Then, there exists a k-point $x_0 \in X$, such that $\mathrm{F}_1(X,x_0) = \varnothing$. Consider the conic bundle $\tilde{X}^{++} \to \mathbb{P}^2$ constructed in Proposition 6.6. By Lemma 6.1, the discriminant curve of f is the smooth nonhyperelliptic curve Γ defined in §6.1, and by Lemma 6.7, the discriminant double covering has the form $\tilde{\Gamma} \cong \Gamma \times_{\mathsf{k}} \mathsf{k}'$. Finally, the description of Proposition 6.6 shows that f admits a rational section after base change to k'. Therefore, Theorem 6.10 applies and proves that $\tilde{X}^{++}$ is not k-rational, hence, X is not k-rational as well. □

7. Fano threefolds of type $\mathsf{X}_{(1,1,1,1)}$

In this section, we apply the degeneration technique of [19] to prove Theorem 1.4.

7.1. Toric degeneration

To start with, we consider $\mathrm{Y}_0 = (\mathbb{P}^1)^4$, denote by $(u_i : v_i)$ the homogeneous coordinates on the i-th factor and consider the point

$$\mathrm{y}_0 := (1,1,1,1) \in \mathrm{Y}_0.$$

Clearly, Y_0 is a toric variety with respect to the action of the split torus $\mathbb{G}_\mathrm{m}^4$ that rescales the v_i. We also consider the action of $\mathfrak{S}_4$ on Y_0 that permutes the factors. It normalises the torus action, and together they generate an action of the group $\mathbb{G}_\mathrm{m}^4 \rtimes \mathfrak{S}_4$. Finally, consider the subtorus

$$\mathrm{T}_0 := \left\{(t_1,t_2,t_3,t_4) \in \mathbb{G}_\mathrm{m}^4 \mid t_1t_2t_3t_4 = 1\right\} \tag{7.1.1}$$

and the collection of three 1-parametric subgroups

$$\mathrm{T}_0^{i_1,i_2;i_3,i_4} := \left\{(t_1,t_2,t_2,t_4) \mid t_{i_1} = t_{i_2} = t_{i_3}^{-1} = t_{i_4}^{-1}\right\} \subset \mathrm{T}_0, \tag{7.1.2}$$

where $(i_1,i_2)(i_3,i_4) \in \mathrm{V}_4 \setminus \{1\} \subset \mathfrak{S}_4$ is a nontrivial element of the Klein subgroup.

Lemma 7.1. *The subvariety*

$$\mathrm{X}_0^{\mathrm{toric}} := \{u_1u_2u_3u_4 - v_1v_2v_3v_4 = 0\} \subset \mathrm{Y}_0$$

is the unique T_0-invariant divisor of multidegree $(1,1,1,1)$ in Y_0, which contains the point y_0. It is a toric variety with six ordinary double points

$$\mathrm{x}_{p,q} = \left\{((u_1:v_1),(u_2:v_2),(u_3:v_3),(u_4:v_4)) \mid u_i = 0 \textit{ if } \; i \in \{p,q\} \textit{ and } v_i = 0 \textit{ if } i \notin \{p,q\}\right\}.$$

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For each permutation $(i_1,i_2)(i_3,i_4) \in \mathrm{V}_4 \setminus \{1\} \subset \mathfrak{S}_4$ *the curve*

$$\mathrm{C}_{i_1,i_2;i_3,i_4} := \overline{\mathrm{T}_0^{i_1,i_2;i_3,i_4} \cdot \mathrm{y}_0} \subset \mathrm{X}_0^{\mathrm{toric}} \tag{7.1.3}$$

is a smooth rational curve and $\mathrm{C}_{i_1,i_2;i_3,i_4} \cap \mathrm{Sing}(\mathrm{X}_0^{\mathrm{toric}}) = \{\mathrm{x}_{i_1,i_2}, \mathrm{x}_{i_3,i_4}\}$.

Proof. The monomial basis in the space of homogeneous polynomials of multidegree (1,1,1,1) is a weight basis for the action of T_0, and all weights are different except for the weight 0, which has multiplicity 2 and the corresponding weight space is spanned by the monomials $u_1u_2u_3u_4$ and $v_1v_2v_3v_4$. Therefore, every T_0-invariant divisor is either given by a monomial equation (but then it does not contain the point y_0) or by a linear combination of $u_1u_2u_3u_4$ and $v_1v_2v_3v_4$, and if it contains the point y_0, it is equal to $\mathrm{X}_0^{\mathrm{toric}}$. The latter is obviously a toric variety with respect to the natural action of T_0.

In the affine chart $v_1 \neq 0$, $v_2 \neq 0$, $u_3 \neq 0$, $u_4 \neq 0$, we can set $v_1 = v_2 = u_3 = u_4 = 1$ and use u_1, u_2, v_3, v_4 as coordinates. Then, the equation of $\mathrm{X}_0^{\mathrm{toric}}$ takes the form

$$u_1u_2 - v_3v_4 = 0,$$

which means that the origin of the chart, that is, the point $\mathrm{x}_{3,4} = (0,0,\infty,\infty) \in \mathrm{Y}_0$ is an ordinary double point of $\mathrm{X}_0^{\mathrm{toric}}$. Considering similarly the other charts, we see that the singular locus of $\mathrm{X}_0^{\mathrm{toric}}$ is the $\mathfrak{S}_4$-orbit of the point $\mathrm{x}_{3,4}$; in particular, each singular point of $\mathrm{X}_0^{\mathrm{toric}}$ is an ordinary double point. Moreover, we see that the hypersurface $\mathrm{X}_0^{\mathrm{toric}}$ is normal.

The orbits of the point y_0 under the 1-parametric subgroups (7.1.2) are

$$\{(t,t,t^{-1},t^{-1}) \mid t \in \mathbb{G}_{\mathrm{m}}\}, \qquad \{(t,t^{-1},t,t^{-1}) \mid t \in \mathbb{G}_{\mathrm{m}}\}, \qquad \{(t,t^{-1},t^{-1},t) \mid t \in \mathbb{G}_{\mathrm{m}}\}.$$

It is easy to see that the closure of the first orbit is smooth and contains the point $\mathrm{x}_{1,2}$ and $\mathrm{x}_{3,4}$ and similarly for the other two orbits. □

For a field extension k'/k, we denote by $\mathrm{Res}_{\mathsf{k}'/\mathsf{k}} \colon \mathrm{Sch}_{\mathsf{k}'} \to \mathrm{Sch}_{\mathsf{k}}$ the Weil restriction of scalars functor from the category of k'-schemes to the category of k-schemes, the right adjoint to the extension of scalars $- \otimes_{\mathsf{k}} \mathsf{k}' \colon \mathrm{Sch}_{\mathsf{k}} \to \mathrm{Sch}_{\mathsf{k}'}$. Consider the projective line $\mathbb{P}^1_{\mathsf{k}'}$, the torus $\mathbb{G}_{\mathrm{m}}$ acting faithfully on $\mathbb{P}^1_{\mathsf{k}'}$ and denote by $0, \infty \in \mathbb{P}^1_{\mathsf{k}'}$ its fixed points.

Proposition 7.2. *For a field extension* k'/k *of degree* 4 *consider the* k*-forms*

$$Y := \mathrm{Res}_{\mathsf{k}'/\mathsf{k}}(\mathbb{P}^1_{\mathsf{k}'}), \qquad T := \mathrm{Ker}\left(\mathrm{Res}_{\mathsf{k}'/\mathsf{k}} \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}}\right) \tag{7.1.4}$$

of $\mathrm{Y}_0 = (\mathbb{P}^1)^4$ *and of the torus* T_0 *and the natural faithful* T*-action on* Y*. Let*

$$y \in Y$$

be the k*-point that corresponds to the point* $1 \in \mathbb{P}^1_{\mathsf{k}'}$*. Then*

(i) *the half-anticanonical linear system of* Y *is defined over* k*, and it contains a unique* T*-invariant divisor* $X^{\mathrm{toric}} \subset Y$ *passing through* y*.*

(ii) *the divisor* X^{toric} *is integral and has ordinary double points in the sense of [19, Definition 4.2.1] with the singular locus of length* 6*.*

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Proof. The fact that Y and T are k-forms of Y_0 and T_0 is obvious from the definition of Weil restriction of scalars, and the k-point y is obtained from the extension–restriction adjunction. Note that upon extension of scalars to $\bar{\mathsf{k}}$, the triple (Y,T,y) becomes isomorphic to the triple $(\mathrm{Y}_0,\mathrm{T}_0,\mathrm{y}_0)$.

(i) Let H denote the Segre class of Y (the half of the anticanonical class); it is obviously Galois-invariant, and since Y has a k-point, H is defined over k. Therefore, the linear system

$$\mathfrak{P} = |H - y| \cong \mathbb{P}^{14} \tag{7.1.5}$$

of divisors in $|H|$ containing y is defined over k. We define $X^{\mathrm{toric}} \subset Y$ as the closure of the T-orbit of the point y; the uniqueness of X^{toric} follows from Lemma 7.1.

(ii) The extension of scalars of $X^{\mathrm{toric}} \subset Y$ to $\bar{\mathsf{k}}$ coincides with $\mathrm{X}_0^{\mathrm{toric}} \subset \mathrm{Y}_0$, hence, its singular locus $Z := \mathrm{Sing}(X^{\mathrm{toric}})$ has length 6, and if E is the exceptional divisor of the blowup

$$\tilde{X} := \mathrm{Bl}_Z(X^{\mathrm{toric}}),$$

then $E \to Z$ is a smooth quadric bundle. According to [19, Definition 4.2.1], it remains to check that this bundle has a section. For this, note that the union of the 1-parametric subgroups $\mathrm{T}_0^{i_1,i_2;i_3,i_4} \subset \mathrm{T}_0$ defined in (7.1.2) is Galois-invariant, hence, it comes from a k-subset in the torus T, and, hence, the closure of the image of the point y under the action of this subset is a curve $C \subset X$ defined over k. Furthermore, the extension of scalars of C to $\bar{\mathsf{k}}$ is the union of the curves (7.1.3). In particular, the curve C contains the singular locus Z and the intersection of its strict transforms to the blowup $\tilde{X}$, and the exceptional divisor E provides a section for $E \to Z$. □

Now, let X be a smooth Fano threefold of type $\mathsf{X}_{(1,1,1,1)}$. Recall the definition (2.1.5) of the Galois group $\mathrm{G}_X \subset \mathfrak{S}_4$ of X. In the next lemma, we use notation introduced in Proposition 7.2.

Lemma 7.3. *If k'/k is the field extension of degree 4 associated with an epimorphism $\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}) \twoheadrightarrow \mathrm{G}$ onto a transitive subgroup $\mathrm{G} \subset \mathfrak{S}_4$, then a general divisor $X \subset Y$ from the linear system (7.1.5) is a smooth Fano threefold of type $\mathsf{X}_{(1,1,1,1)}$ with $\mathrm{G}_X = \mathrm{G}$, $\rho(X) = 1$ and $X(\mathsf{k}) \neq \varnothing$.*

Proof. The smoothness of a general divisor X in the linear system $\mathfrak{P}$ follows from the Bertini theorem, the property $X(\mathsf{k}) \neq \varnothing$ is obvious because X contains the k-point y, the equality $\mathrm{G}_X = \mathrm{G}$ follows from the construction and $\rho(X) = 1$ follows from transitivity of $\mathrm{G} \subset \mathfrak{S}_4$. □

Remark 7.4. One can also prove the converse statement: any Fano threefold X of type $\mathsf{X}_{(1,1,1,1)}$ with $X(\mathsf{k}) \neq \varnothing$ and $\mathrm{G}_X = \mathrm{G}$ is isomorphic to a divisor in the linear system $\mathfrak{P}$ (see [14, Proposition 7.16]).

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We consider the field extension k'/k as in Lemma 7.3 and use the construction and notation of Proposition 7.2; in particular, the linear system $\mathfrak{P} \cong \mathbb{P}^{14}_{\mathsf{k}}$ of half-anticanonical divisors in Y. Let $\mathfrak{p}_0 \in \mathfrak{P}$ be the point that corresponds to the toric

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divisor $X^{\mathrm{toric}} \subset Y$. Note that the space of lines in $\mathfrak{P}$ through the point $\mathfrak{p}_0$ is the projective space $\mathbb{P}^{13}_{\mathsf{k}}$, in particular, k-points are Zariski dense in it. Therefore, there is a line $L \subset \mathfrak{P}$ through $\mathfrak{p}_0$ defined over k. We denote by

$$\mathscr{X} \to L$$

the corresponding family of half-anticanonical divisors in Y. Then, the general point of L corresponds to a smooth variety $\mathscr{X}_L$ of type $\mathsf{X}_{(1,1,1,1)}$ over the field $\mathsf{k}(L) \cong \mathsf{k}(t)$.

Since for a general k-point $\mathfrak{p} \in L$ the fibre of $\mathscr{X}_{\mathfrak{p}}$ has $\mathrm{G}_{X_{\mathfrak{p}}} = \mathrm{G}$ by Lemma 7.3, and since the natural restriction morphism of Galois groups $\mathrm{G}(\overline{\mathsf{k}(L)}/\mathsf{k}(L)) \to \mathrm{G}(\bar{\mathsf{k}}/\mathsf{k})$ is surjective, we have $\mathrm{G}_{\mathscr{X}_L} = \mathrm{G}$. Since $\mathrm{G} \subset \mathfrak{S}_4$ is transitive, this implies $\rho(\mathscr{X}_L) = 1$. Finally, $\mathscr{X}_L$ by construction contains the $\mathsf{k}(L)$-point $y \times_{\mathsf{k}} \mathsf{k}(L)$, hence, $\mathscr{X}_L(\mathsf{k}(L)) \neq \varnothing$.

Assume $\mathscr{X}_L$ is stably rational. Consider the point $\mathfrak{p}_0 \in L$ as a special point of the family $\mathscr{X}/L$. By Proposition 7.2, the corresponding variety $\mathscr{X}_{\mathfrak{p}_0} = X^{\mathrm{toric}}$ is integral with ordinary double points, hence, by [19, Proposition 4.2.9], the family $\mathscr{X}/L$ is $\mathbb{L}$-faithful in the sense of [19, Definition 4.2.7]. Therefore, by [19, Proposition 4.2.10], the special fibre X^{toric} is stably rational.

On the other hand, by definition, the Galois group of the extension k'/k coincides with the group G_X and contains the Klein group V_4. By [22, §2.4.8], for any smooth compactification $V \supset T$ one has

$$H^1(\mathrm{G}(\bar{\mathsf{k}}/\mathsf{k}), \mathrm{Pic}(V_{\bar{\mathsf{k}}})) = H^1(\mathrm{G}_X, \mathrm{Pic}(V_{\bar{\mathsf{k}}})) \neq 0.$$

Since this group is a stable birational invariant (see, e.g. [4, §2.A] or [22, §4.4]), the torus T and the corresponding toric variety X^{toric} are not stably rational. This contradiction shows that $\mathscr{X}_L$ is not stably rational and completes the proof of the theorem. □

Appendix A. Constructing morphisms of Hilbert schemes

In this section, we show how one can use the technique of derived categories to construct morphisms of Hilbert schemes. For smooth projective varieties X and Y, we denote by π_X and π_Y the projections from $X \times Y$ to the factors, and for an object $\mathscr{K} \in \mathbf{D}(X \times Y)$, we denote by

$$\Phi_{\mathscr{K}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y), \qquad \mathscr{F} \longmapsto \mathbf{R}\pi_{Y*}(\mathbf{L}\pi_X^*(\mathscr{F}) \otimes^{\mathbf{L}} \mathscr{K})$$

the corresponding Fourier–Mukai functor from the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X to that of Y. For an integral valued polynomial $p \in \mathbb{Q}[t]$, we denote by $\mathrm{Hilb}_p(X)$ the Hilbert scheme of subschemes in X with Hilbert polynomials p with respect to a given polarisation.

Proposition A.1. *Let X and Y be smooth projective varieties. If $\mathscr{K} \in \mathbf{D}(X \times Y)$ is an object of the derived category, such that for any subscheme $Z \subset X$ with Hilbert polynomial p the object $\Phi_{\mathscr{K}}(\mathscr{O}_Z) \in \mathbf{D}(Y)$ is isomorphic to the structure sheaf of a point $y(Z) \in Y$, then there is a morphism of schemes*

$$\varphi \colon \mathrm{Hilb}_p(X) \longrightarrow Y,$$

such that $\varphi([Z]) = y(Z)$.

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Proof. Let $Z \subset X \times S$ be an family of subschemes in X flat over S with Hilbert polynomial p. Let

$$\mathscr{F} := \Phi_{\pi_{XY}^*\mathscr{K}}(\mathscr{O}_Z) = \mathbf{R}\pi_{SY*}(\mathbf{L}\pi_{XS}^*(\mathscr{O}_Z) \otimes^{\mathbf{L}} \pi_{XY}^*\mathscr{K}) \in \mathbf{D}(S \times Y)$$

be the image of the structure sheaf of Z under the induced Fourier–Mukai functor from $\mathbf{D}(X \times S)$ to $\mathbf{D}(S \times Y)$, where π_{XS}, π_{SY} and π_{XY} are the projections of $X \times S \times Y$ to the pairwise products of factors. By base change and the projection formula, for each point $s \in S$, we have

$$i_s^*\mathscr{F} \cong \Phi_{\mathscr{K}}(\mathscr{O}_{Z_s}),$$

where $i_s \colon \{s\} \times Y \hookrightarrow S \times Y$ is the natural embedding and $Z_s \subset X$ is the fibre of Z over $s \in S$. Thus, we have $i_s^*\mathscr{F} \cong \mathscr{O}_{y(Z_s)}$ by assumption, therefore, by [13, Lemma 4.4(iii)], there is a unique morphism $\varphi_S \colon S \to Y$, such that $\mathscr{F}$ is isomorphic up to twist to the structure sheaf of the graph of φ_S; in particular, $\varphi_S(s) = y(Z_s)$ for each $s \in S$. Now applying this argument to $S = \mathrm{Hilb}_p(X)$ and Z the universal subscheme, we obtain the required morphism φ. $\square$

In §2.3, we apply Proposition A.1 to the Hilbert scheme of conics on the threefold $X \cong \mathrm{Bl}_{\Gamma_1}(Q_1)$, where $Q_1 \subset \mathbb{P}^4$ is a smooth quadric and $\Gamma_1 \subset Q_1$ is a linearly normal smooth rational quartic curve. Recall that $\mathrm{F}_1(X)$ and $\mathrm{F}_2(X)$ denote the Hilbert schemes of lines and conics on X, and that there is a natural embedding $\Gamma_1 \subset \mathrm{F}_1(X)$ of a connected component (see Lemma 2.7). Recall also that the second connected component $\Gamma_2 \subset \mathrm{F}_1(X)$ corresponds to lines on Q_1 bisecant to Γ_1.

Corollary A.2. *There is a morphism $\varphi_1 \colon \mathrm{F}_2(X) \to \Gamma_1$, such that for a smooth conic $C \subset X$, one has $\varphi_1([C]) = [L]$, where $L \subset X$ is the unique line corresponding to a point of Γ_1, such that $C \cap L \neq \varnothing$. Moreover, if $C = L_1 \cup L_2$ is a reducible conic, so that $L_1 \cap L_2 \neq \varnothing$, and if L_2' is the other line corresponding to a point of Γ_1, such that $L_1 \cap L_2' \neq \varnothing$, then $\varphi_1([C]) = [L_2']$.*

Proof. Let $\pi_1 \colon X \to Q_1$ be the blowup morphism, and let $E_1 \subset X$ be its exceptional divisor; note that E_1 is the universal family of lines on X over the connected component $\Gamma_1 \subset \mathrm{F}_1(X)$ of the Hilbert scheme of lines. Let $\varepsilon \colon E_1 \to X \times \Gamma_1$ be the corresponding embedding and consider

$$\mathscr{K} := \varepsilon_*\mathscr{O}_{E_1}(E_1) \in \mathbf{D}(X \times \Gamma_1).$$

Let us check that the assumption of Proposition A.1 is satisfied for $Y = \Gamma_1$.

If $C \subset X$ is a smooth conic, then $C \cdot E_1 = 1$, and $C \not\subset E_1$, therefore, $C \cap E_1 = \{x\}$ is a single point and the intersection is transverse. Therefore, $\mathscr{O}_C \otimes^{\mathbf{L}} \mathscr{O}_{E_1}(E_1) \cong \mathscr{O}_x$, hence, $\Phi_{\mathscr{K}}(\mathscr{O}_C) \cong \mathscr{O}_{\pi_1(x)}$.

If $C = L_1 \cup L_2$ is a reducible conic, so that we have $L_2 \subset E_1$ and $L_1 \cap E_1 = \{x, x'\}$ with $L_1 \cap L_2 = \{x\}$ (if L_1 is tangent to E_1, we take $x' = x$), then using the exact sequences

$$0 \longrightarrow \mathscr{O}_{L_1}(-1) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_{L_2} \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \mathscr{O}_{L_2}(-1) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_{L_1} \longrightarrow 0,$$

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it is easy to check that $\mathscr{O}_C \otimes^{\mathbf{L}} \mathscr{O}_{E_1}(E_1)$ fits into a distinguished triangle

$$\mathscr{O}_C \otimes^{\mathbf{L}} \mathscr{O}_{E_1}(E_1) \longrightarrow \mathscr{O}_{L_2}(-1) \oplus \mathscr{O}_{x'} \longrightarrow \mathscr{O}_{L_2}(-1)[2]$$

(if L_1 is tangent to E_1, the middle term should be replaced by an extension of $\mathscr{O}_{L_2}(-1)$ by $\mathscr{O}_{x'}$). Since the pushforward functor $\mathbf{R}\pi_{1*}$ kills the sheaf $\mathscr{O}_{L_2}(-1)$, it follows that $\Phi_{\mathscr{K}}(\mathscr{O}_C) \cong \mathscr{O}_{\pi_1(x')}$.

Now, applying Proposition A.1, we conclude that there is a morphism $\varphi_1 \colon \mathrm{F}_2(X) \to \Gamma_1$, such that $\varphi_1([C]) = \pi_1(x)$ if C is smooth and $\varphi_1([L_1 \cup L_2]) = \pi_1(x')$, with the notation for points x and x' introduced above. □

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