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# Explicit deformation of the horospherical variety of type $\mathrm{G}_2$

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**Abstract.** We give two simple geometric constructions of a smooth family of projective varieties with central fiber isomorphic to the horospherical variety of type $\mathrm{G}_2$ and all other fibers isomorphic to the isotropic orthogonal Grassmannian $\mathrm{OGr}(2,7)$, and we discuss briefly the derived category of this family.

Bibliography: 8 titles.

**Keywords:** horospherical varieties, smooth degeneration, exceptional collection.

## § 1. Introduction

Let $\mathbf{G}$ be a simple algebraic group of Dynkin type $\mathrm{G}_2$. If $V_1$ and $V_2$ are the fundamental representations of $\mathbf{G}$ of dimension 7 and 14, respectively, then the highest weight vector orbits in $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ are as follows:

- $X_1 \subset \mathbb{P}(V_1)$ is a smooth quadric of dimension 5;
- $X_2 \subset \mathbb{P}(V_2)$ is the so-called *adjoint variety* of $\mathbf{G}$.

In fact, $X_2$ can be realized as a subvariety of $\mathrm{Gr}(2, V_1)$ (see Lemma 2.1 below for details), and if $\mathcal{U}_{X_2}$ is the restriction to $X_2$ of the tautological rank 2 subbundle from the Grassmannian, then

$$\widetilde{X} := \mathbb{P}_{X_2}(\mathcal{U}_{X_2}) \xrightarrow{p_2} X_2 \qquad (1.1)$$

is the flag variety of $\mathbf{G}$; in particular, the bundle $\mathcal{U}_{X_2}$ and the $\mathbb{P}^1$-fibration $p_2$ are $\mathbf{G}$-equivariant.

Similarly, there is a $\mathbf{G}$-equivariant vector bundle $\mathcal{C}_{X_1}$ of rank 2 on the quadric $X_1$ such that

$$\widetilde{X} \cong \mathbb{P}_{X_1}(\mathcal{C}_{X_1}) \xrightarrow{p_1} X_1 \qquad (1.2)$$

is a $\mathbf{G}$-equivariant $\mathbb{P}^1$-fibration; $\mathcal{C}_{X_1}$ is known as the *Cayley bundle* (see [2], Lemma 8.3, and [6]).

The *horospherical variety* $X$ *of type* $\mathrm{G}_2$ can be constructed out of these data in several ways. We outline below three related constructions; for details, see [7] and [1]. Let $H_1$ and $H_2$ denote the hyperplane classes of $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$; by abusing

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the notation we denote their pullbacks to $X_1$ and $X_2$, respectively, and to $\widetilde{X}$ in the same way. Consider the following projective bundles:

$$\mathbb{P}_{X_1}(\mathcal{O}_{X_1}(-H_1)\oplus\mathcal{C}_{X_1}), \quad \mathbb{P}_{X_2}(\mathcal{U}_{X_2}\oplus\mathcal{O}_{X_2}(-H_2)) \quad \text{and} \quad \mathbb{P}_{\widetilde{X}}(\mathcal{O}_{\widetilde{X}}(-H_1)\oplus\mathcal{O}_{\widetilde{X}}(-H_2))$$

over $X_1$, $X_2$ and $\widetilde{X}$, respectively. Note that the summands of rank 1 induce sections of the first two projective bundles, and a pair of sections of the last one. It is not hard to prove that the relative hyperplane class of each of these projective bundles is base point free and induces a $\mathbf{G}$-equivariant morphism into $\mathbb{P}(V_1\oplus V_2)$. The image of each morphism is the horospherical variety $X \subset \mathbb{P}(V_1 \oplus V_2)$, and the images of the sections are the disjoint subvarieties

$$X_1 = X \cap \mathbb{P}(V_1) \quad \text{and} \quad X_2 = X \cap \mathbb{P}(V_2),$$

and in this way we obtain isomorphisms

$$\mathrm{Bl}_{X_2}(X) \cong \mathbb{P}_{X_1}(\mathcal{O}_{X_1}(-H_1) \oplus \mathcal{C}_{X_1}), \tag{1.3}$$

$$\mathrm{Bl}_{X_1}(X) \cong \mathbb{P}_{X_2}(\mathcal{U}_{X_2} \oplus \mathcal{O}_{X_2}(-H_2)) \tag{1.4}$$

and

$$\mathrm{Bl}_{X_1\sqcup X_2}(X) \cong \mathbb{P}_{\widetilde{X}}\ (\mathcal{O}_{\widetilde{X}}(-H_1) \oplus \mathcal{O}_{\widetilde{X}}(-H_2)). \tag{1.5}$$

All these isomorphisms are $\mathbf{G}$-equivariant.

The constructions described above are quite general and can be applied to horospherical varieties with Picard number 1 of other types (see [7], Theorem 0.1, for classification). The special property of the horospherical variety of type $\mathrm{G}_2$ is its relationship with another smooth projective $\mathbf{G}$-variety, the *orthogonal isotropic Grassmannian*

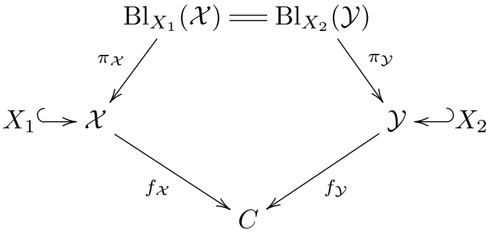
$$Y = \mathrm{OGr}(2, V_1), \tag{1.6}$$

which is the subvariety of $\mathrm{Gr}(2, V_1)$ that parameterizes the two-dimensional subspaces isotropic with respect to the quadratic equation of $X_1 \subset \mathbb{P}(V_1)$.

It can be observed that $X$ and $Y$ share all numerical invariants; in particular, they have the same rank of the Grothendieck groups (equal to 12), the same Fano index (equal to 4), the same dimension of the spaces of global sections of $\mathcal{O}(1)$ (equal to 21), and so on. A nice explanation for these coincidences was given in [8], Proposition 2.3, where a smooth degeneration of $Y$ to $X$, that is, a smooth projective variety over $\mathbb{A}^1$ with central fiber isomorphic to $X$ and all other fibers isomorphic to $Y$, was constructed.

This degeneration was constructed in [8] as a certain orbit closure, which makes it slightly implicit and hard to use. The goal of this paper is to give two geometric constructions of such a family, which are more convenient for applications; these constructions work over an arbitrary smooth pointed curve $(C, 0)$.

**Theorem 1.1.** *Let $(C, 0)$ be a smooth pointed curve, and set $\mathcal{Y} := Y \times C$. Then there is a commutative diagram*



(1.7)

*where*

- *$f_{\mathcal{X}} : \mathcal{X} \to C$ is a smooth projective morphism such that*

$$\mathcal{X}_0 := f_{\mathcal{X}}^{-1}(0) \cong X,$$

- *$f_{\mathcal{Y}} : \mathcal{Y} \to C$ is the projection onto the second factor, so that*

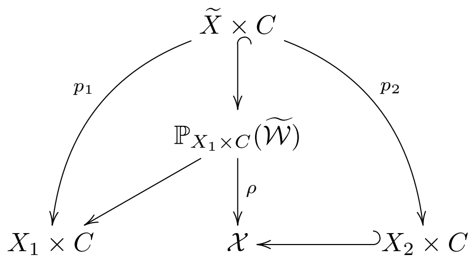
$$\mathcal{Y}_0 := f_{\mathcal{Y}}^{-1}(0) \cong Y,$$

- *$\pi_{\mathcal{X}}$ and $\pi_{\mathcal{Y}}$ are the blowups of the smooth subvarieties*

$$X_1 \subset X = \mathcal{X}_0 \subset \mathcal{X} \quad \text{and} \quad X_2 \subset Y = \mathcal{Y}_0 \subset \mathcal{Y}.$$

*In particular, $f_{\mathcal{X}}^{-1}(C \setminus \{0\}) \cong f_{\mathcal{Y}}^{-1}(C \setminus \{0\}) \cong Y \times (C \setminus \{0\})$, so that $X$ is a smooth degeneration of $Y$.*

**Theorem 1.2.** *There is a vector bundle $\widetilde{\mathcal{W}}$ of rank 3 on $X_1 \times C$ and a commutative diagram*



(1.8)

*over $C$, where $\mathcal{X}$ is the same as in Theorem 1.1 and $\rho$ is the blowup of $X_2 \times C \subset \mathcal{X}$.*

The crucial observation (Proposition 2.3) on which the proof of both theorems relies is that there is a natural embedding $X_2 \hookrightarrow Y$ and that the blowup $\mathrm{Bl}_{X_2}(Y)$ has the structure of a projective bundle over $X_1$ which is analogous to (1.3).

## § 2. The key observation

Recall that $V_1$ denotes the fundamental 7-dimensional representation of the group $\mathbf{G}$ and $Y$ was defined in (1.6). We denote by $\mathcal{U} \subset V_1 \otimes \mathcal{O}$ and $\mathcal{U}^\perp \subset V_1^\vee \otimes \mathcal{O}$ the tautological subbundles of rank 2 and 5 on $\mathrm{Gr}(2, V_1)$, respectively, and write $\mathcal{O}(1)$ for the Plücker line bundle. We also denote by $\mathcal{S}$ the spinor bundle on $Y$ with the convention opposite to that in § 6 of [3], so that $\wedge^2 \mathcal{S} \cong \mathcal{O}(-1)$.

**Lemma 2.1.** *There exists a chain of embeddings*

$$X_2 \hookrightarrow Y \hookrightarrow \mathrm{Gr}(2, V_1) \subset \mathbb{P}(\wedge^2 V_1)$$

*such that*

- $Y \subset \mathrm{Gr}(2, V_1)$ *is the zero locus of a regular section of* $\mathrm{Sym}^2 \mathcal{U}^\vee$,
- $X_2 \subset \mathrm{Gr}(2, V_1)$ *is the zero locus of a regular section of* $\mathcal{U}^\perp(1)$ *and*
- $X_2 \subset Y$ *is the zero locus of a regular section of* $\mathcal{S}^\vee$.

*Moreover, the restrictions of* $\mathcal{U}$ *and* $\mathcal{S}$ *to* $X_2$ *are isomorphic.*

*Proof.* The description of $X_2$ as a zero locus in $\mathrm{Gr}(2, V_1)$ was discovered by Mukai (a global section of $\mathcal{U}^\perp(1)$ is given by a 3-form $\lambda \in \wedge^3 V_1^\vee$ whose stabilizer is the group $\mathbf{G}$). The restriction of $\mathcal{O}(1)$ to $X_2$ generates $\mathrm{Pic}(X_2)$, the linear span of $X_2$ in $\mathbb{P}(\wedge^2 V_1)$ is $\mathbb{P}(V_2)$, and the corresponding embedding $V_2 \hookrightarrow \wedge^2 V_1$ extends to an exact sequence

$$0 \longrightarrow V_2 \longrightarrow \wedge^2 V_1 \xrightarrow{\lambda} V_1^\vee \longrightarrow 0 \tag{2.1}$$

of representations of $\mathbf{G}$. Note that the $\mathbf{G}$-action on $V_1$ preserves a nondegenerate quadratic form (the equation of the quadric $X_1 \subset \mathbb{P}(V_1)$), hence we have the chain of group embeddings

$$\mathbf{G} \subset \mathrm{SO}(V_1) \subset \mathrm{GL}(V_1).$$

Moreover, a highest weight vector of $V_2$ with respect to $\mathbf{G}$ is also a highest weight vector of $\wedge^2 V_1$ with respect to $\mathrm{SO}(V_1)$ and $\mathrm{GL}(V_1)$, and therefore we have the chain of highest-weight vector orbits

$$X_2 \subset Y \subset \mathrm{Gr}(2, V_1)$$

of the respective groups.

The description of $Y \subset \mathrm{Gr}(2, V_1)$ as the zero locus is tautological (the section corresponds to a quadratic form on $V_1$ preserved by $\mathrm{SO}(V_1)$), and the description of $X_2 \subset Y$ as the zero locus of a section of $\mathcal{S}^\vee$ and an isomorphism $\mathcal{S}|_{X_2} \cong \mathcal{U}|_{X_2}$ were established in [2], Lemma 8.3. □

**Corollary 2.2.** *There is an equalty of schemes* $X_2 = Y \cap \mathbb{P}(V_2)$.

The intersection on the right-hand side is, of course, highly non-transverse.

*Proof of Corollary* 2.2. Let $\mathcal{I}_{X_2}$ be the ideal of $X_2 \subset Y = \mathrm{OGr}(2, V_1)$. By Lemma 2.1 we have an exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{S}^\vee \longrightarrow \mathcal{I}_{X_2}(1) \longrightarrow 0.$$

The space of global sections of $\mathcal{S}^\vee$ on $Y$ is the spinor 8-dimensional representation $\mathbb{S}$ of $\mathrm{Spin}(V_1)$; when restricted to $\mathbf{G}$, it is isomorphic to the direct sum $V_1 \oplus \Bbbk$, where the second summand corresponds to the section of $\mathcal{S}^\vee$ defining $X_2 \subset Y$. Since, moreover, $\mathcal{S}^\vee$ is globally generated, the above exact sequence gives an epimorphism $V_1 \otimes \mathcal{O}_Y \twoheadrightarrow \mathcal{I}_{X_2}(1)$. This means that $X_2 \subset \mathrm{Gr}(2, V_1)$ is scheme-theoretically cut out by the hyperplanes corresponding to the subspace

$$V_1 \subset \mathrm{H}^0(Y, \mathcal{O}_Y(1)) = \wedge^2 V_1^\vee.$$

This embedding is obviously $\mathbf{G}$-equivariant, hence it is given by the dual of the second map in (2.1), and therefore $X_2 = Y \cap \mathbb{P}(V_2)$. □

The following proposition is an analogue of (1.3); it is the key to the proof of Theorems 1.1 and 1.2.

**Proposition 2.3.** *There is a* **G***-equivariant isomorphism*

$$\mathrm{Bl}_{X_2}(Y) \cong \mathbb{P}_{X_1}(\mathcal{W}_{X_1}), \tag{2.2}$$

*where* $\mathcal{W}_{X_1}$ *is a* **G***-equivariant vector bundle on* $X_1$ *of rank* 3; *it fits into an exact sequence*

$$0 \longrightarrow \mathcal{C}_{X_1} \longrightarrow \mathcal{W}_{X_1} \longrightarrow \mathcal{O}_{X_1}(-H_1) \longrightarrow 0. \tag{2.3}$$

*Proof.* Consider the orthogonal isotropic partial flag variety and its two projections:

$$\begin{array}{ccc} & \mathrm{OFl}(2,3;V_1) & \\ \swarrow & & \searrow \\ \mathrm{OGr}(2,V_1) & & \mathrm{OGr}(3,V_1). \end{array}$$

The fibers of the first are nondegenerate conics; in fact, it is a $\mathbb{P}^1$-bundle, the corresponding vector bundle on $\mathrm{OGr}(2, V_1)$ is precisely the spinor bundle $\mathcal{S}$, hence we have an isomorphism

$$\mathrm{OFl}(2,3;V_1) \cong \mathbb{P}_{\mathrm{OGr}(2,V_1)}(\mathcal{S}) = \mathbb{P}_Y(\mathcal{S}).$$

Similarly, the second map is a $\mathbb{P}^2$-bundle. We denote the corresponding $\mathrm{SO}(V_1)$-equivariant vector bundle of rank 3 on $\mathrm{OGr}(3, V_1)$ by $\mathcal{W}$, so that we have an isomorphism

$$\mathrm{OFl}(2,3;V_1) \cong \mathbb{P}_{\mathrm{OGr}(3,V_1)}(\mathcal{W}). \tag{2.4}$$

Note that $\mathrm{OGr}(3, V_1)$ is a smooth 6-dimensional quadric in the projectivization of the 8-dimensional spinor representation $\mathbb{S}$ of $\mathrm{Spin}(V_1)$ (the universal covering of $\mathrm{SO}(V_1)$). The restriction of this representation to the group **G** splits as $\mathbb{S} = V_1 \oplus \Bbbk$, and the hyperplane section of $\mathrm{OGr}(3, V_1) \subset \mathbb{P}(\mathbb{S})$ by $\mathbb{P}(V_1)$ is the quadric $X_1$. Its preimage

$$\mathrm{OFl}(2,3;V_1) \times_{\mathrm{OGr}(3,V_1)} X_1 \cong \mathbb{P}_{X_1}(\mathcal{W}|_{X_1})$$

is a relative (over $Y = \mathrm{OGr}(2, V_1)$) hyperplane section of $\mathbb{P}_Y(\mathcal{S})$; therefore, it is isomorphic to the blowup of $Y$ along the zero locus of the corresponding section of $\mathcal{S}^\vee$. By Lemma 2.1 this zero locus is $X_2 \subset Y$, hence we obtain the required isomorphism (2.2), where $\mathcal{W}_{X_1} := \mathcal{W}|_{X_1}$.

To construct the exact sequence (2.3) note that by Lemma 2.1 the normal bundle of $X_2 \subset Y$ is

$$\mathcal{N}_{X_2/Y} \cong \mathcal{S}|_{X_2}^\vee \cong \mathcal{U}_{X_2}^\vee;$$

hence, using (1.1) and (1.2), we deduce that the exceptional divisor of the blowup is

$$\mathbb{P}_{X_2}(\mathcal{N}_{X_2/Y}) \cong \mathbb{P}_{X_2}(\mathcal{U}_{X_2}^\vee) \cong \mathbb{P}_{X_2}(\mathcal{U}_{X_2}) = \widetilde{X} = \mathbb{P}_{X_1}(\mathcal{C}_{X_1}).$$

Moreover, the induced embedding $\mathbb{P}_{X_1}(\mathcal{C}_{X_1}) \hookrightarrow \mathbb{P}_{X_1}(\mathcal{W}_{X_1})$ is compatible with the projection onto $X_1$ and also with the relative hyperplane classes, hence it induces an embedding of vector bundles $\mathcal{C}_{X_1} \hookrightarrow \mathcal{W}_{X_1}$. The quotient is a line bundle, so it can be identified with $\mathcal{O}_{X_1}(-H_1)$ by the determinant computation, taking into account the isomorphisms

$$\det(\mathcal{C}_{X_1}) \cong \mathcal{O}_{X_1}(-3H_1) \quad \text{and} \quad \det(\mathcal{W}_{X_1}) \cong \mathcal{O}_{X_1}(-4H_1),$$

which follow from (1.2) and (2.4) by a canonical class computation. □

*Remark* 2.4. The crucial difference between (2.2) and (1.3) is that the extension (2.3) defining the vector bundle $\mathcal{W}_{X_1}$ is *nontrivial*. One can see this as follows: if the extension (2.3) were split, then the embedding $\mathcal{O}_{X_1}(-H_1) \hookrightarrow \mathcal{W}_{X_1}$ would give an embedding $X_1 \hookrightarrow Y$ of the 5-dimensional quadric $X_1$, but it is well known that $Y = \mathrm{OGr}(2, V_1)$ (and even the ambient Grassmannian $\mathrm{Gr}(2, V_1)$) does not contain quadrics of dimension greater than 4.

## § 3. Proof of Theorem 1.1

Recall that $\mathcal{Y} = Y \times C$ and the map $f_{\mathcal{Y}} \colon \mathcal{Y} \to C$ is the projection. Consider the subvariety

$$X_2 \hookrightarrow Y = \mathcal{Y}_0 \hookrightarrow \mathcal{Y}$$

(where we recall that $\mathcal{Y}_0 \subset \mathcal{Y}$ denotes the central fiber of $f_{\mathcal{Y}}$) and the blowup $\pi_{\mathcal{Y}} \colon \mathrm{Bl}_{X_2}(\mathcal{Y}) \to \mathcal{Y}$. This gives us the right half of the diagram (1.7). To construct the left half we need two lemmas.

**Lemma 3.1.** *The scheme central fiber of* $\mathrm{Bl}_{X_2}(\mathcal{Y}) \xrightarrow{\pi_{\mathcal{Y}}} \mathcal{Y} \xrightarrow{f_{\mathcal{Y}}} C$ *is the normal crossing divisor*

$$\mathbb{P}_{X_2}(\mathcal{U}_{X_2} \oplus \mathcal{O}_{X_2}(-H_2)) \bigcup_{\widetilde{X}} \mathbb{P}_{X_1}(\mathcal{W}_{X_1}),$$

*where the first component is the exceptional divisor of* $\pi_{\mathcal{Y}}$ *and the second is the strict transform of* $\mathcal{Y}_0 \cong Y$.

*Proof*. Since $\mathcal{Y}$ is the product $Y \times C$, using Lemma 2.1 we compute the normal bundle

$$\mathcal{N}_{X_2/\mathcal{Y}} \cong \mathcal{N}_{X_2/Y} \oplus \mathcal{O}_{X_2} \cong \mathcal{U}_{X_2}^{\vee} \oplus \mathcal{O}_{X_2}.$$

This is a twist of $\mathcal{U}_{X_2} \oplus \mathcal{O}_{X_2}(-H_2)$, hence we obtain a description of the first component of the central fiber of $\mathrm{Bl}_{X_2}(\mathcal{Y})$. The second component is isomorphic to the blowup $\mathrm{Bl}_{X_2}(Y)$, so Proposition 2.3 applies. Finally, the intersection of the components is the projectivization of $\mathcal{N}_{X_2/Y} \cong \mathcal{U}_{X_2}^{\vee}$, hence (1.1) shows that it is isomorphic to $\widetilde{X}$. □

Now consider the trivial vector bundle $\wedge^2 V_1 \otimes \mathcal{O}_C$ and the filtration (2.1) on its central fiber. It induces a vector bundle $\mathcal{V}$ on $C$ and a morphism $\alpha \colon \wedge^2 V_1 \otimes \mathcal{O}_C \to \mathcal{V}$, that fit into an exact sequence

$$0 \longrightarrow \wedge^2 V_1 \otimes \mathcal{O}_C \xrightarrow{\alpha} \mathcal{V} \longrightarrow V_2 \otimes \mathcal{O}_{\{0\}} \longrightarrow 0,$$

where $\mathcal{O}_{\{0\}}$ is the structure sheaf of the point $\{0\} \in C$; the central fiber of $\mathcal{V}$ is canonically an extension

$$0 \longrightarrow V_1^{\vee} \longrightarrow \mathcal{V}_{\{0\}} \longrightarrow V_2 \longrightarrow 0 \tag{3.1}$$

(opposite to (2.1)), and the morphism $\alpha_{\{0\}}$ factors as $\wedge^2 V_1 \xrightarrow{\lambda} V_1^\vee \longrightarrow \mathcal{V}_{\{0\}}$. Note that $\alpha$ is an isomorphism over the punctured curve $C \setminus \{0\}$, hence it induces a birational map $\mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C) \dashrightarrow \mathbb{P}_C(\mathcal{V})$ of projective bundles over $C$.

**Lemma 3.2.** *Consider the embeddings*

$$\mathbb{P}(V_2) \subset \mathbb{P}(\wedge^2 V_1) \hookrightarrow \mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C) \quad \textit{and} \quad \mathbb{P}(V_1^\vee) \subset \mathbb{P}(\mathcal{V}_{\{0\}}) \hookrightarrow \mathbb{P}_C(\mathcal{V})$$

*into the central fibers of projective bundles. The birational map $\alpha$ induces an isomorphism of blowups*

$$\mathrm{Bl}_{\mathbb{P}(V_2)}(\mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C)) \cong \mathrm{Bl}_{\mathbb{P}(V_1^\vee)}(\mathbb{P}_C(\mathcal{V}))$$

*over $C$ such that the exceptional divisor of each side coincides with the strict transform of the central fiber of the projective bundle of the other side.*

This is an elementary transformation of projective bundles, so the argument is standard.

Now we construct the left half of (1.7). Consider the natural embedding

$$\mathcal{Y} = Y \times C = \mathrm{OGr}(2, V_1) \times C \hookrightarrow \mathbb{P}(\wedge^2 V_1) \times C = \mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C).$$

By Corollary 2.2 the strict transform of $\mathcal{Y}$ under the blowup $\mathrm{Bl}_{\mathbb{P}(V_2)}(\mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C))$ from Lemma 3.2 is isomorphic to $\mathrm{Bl}_{X_2}(\mathcal{Y})$. Consider the composition

$$\mathrm{Bl}_{X_2}(\mathcal{Y}) \hookrightarrow \mathrm{Bl}_{\mathbb{P}(V_2)}(\mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C)) \cong \mathrm{Bl}_{\mathbb{P}(V_1^\vee)}(\mathbb{P}_C(\mathcal{V})) \to \mathbb{P}_C(\mathcal{V})$$

of the induced embedding with the isomorphism from Lemma 3.2 and the obvious contraction. We denote its image by $\mathcal{X} \subset \mathbb{P}_C(\mathcal{V})$ and consider the resulting morphisms

$$\mathrm{Bl}_{X_2}(\mathcal{Y}) \xrightarrow{\pi_{\mathcal{X}}} \mathcal{X} \xrightarrow{f_{\mathcal{X}}} C.$$

It remains to prove that $f_{\mathcal{X}}$ is smooth, its central fiber is isomorphic to $X$ and $\pi_{\mathcal{X}}$ is the blowup of $X_1 \subset \mathcal{X}$.

By Lemma 3.2 the morphism $\mathrm{Bl}_{\mathbb{P}(V_2)}(\mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C)) \to \mathbb{P}_C(\mathcal{V})$ contracts the strict transform of the central fiber of $\mathbb{P}_C(\wedge^2 V_1 \otimes \mathcal{O}_C)$ and is an isomorphism on its complement. It follows that $\pi_{\mathcal{X}}$ contracts the strict transform of the central fiber of $\mathcal{Y}$ and is an isomorphism on its complement.

The restriction of $\pi_{\mathcal{X}}$ to the exceptional divisor $\mathbb{P}_{X_2}(\mathcal{U}_{X_2} \oplus \mathcal{O}_{X_2}(-H_2))$ of $\pi_{\mathcal{Y}}$ (see Lemma 3.1) is the morphism given by the relative hyperplane class, hence by (1.4) its image is the horospherical variety $X$. This is the scheme central fiber of $f_{\mathcal{X}}$, so the smoothness of $X$ implies that $f_{\mathcal{X}}$ is smooth.

The restriction of $\pi_{\mathcal{X}}$ to the strict transform $\mathrm{Bl}_{X_2}(Y) \cong \mathbb{P}_{X_1}(\mathcal{W}_{X_1})$ of the central fiber of $\mathcal{Y}$ over $C$ coincides by construction with the morphism in Proposition 2.3, and therefore $\pi_{\mathcal{X}}(\mathrm{Bl}_{X_2}(Y)) = X_1 \subset X$.

Finally, the fact that $\pi_{\mathcal{X}}$ is the blowup of $X_1 \subset X = \mathcal{X}_0 \subset \mathcal{X}$ follows from Lemma 2.5 in [5], which completes the proof.

## § 4. Proof of Theorem 1.2

Recall the exact sequence (2.3), and let

$$\epsilon \in \operatorname{Ext}^1(\mathcal{O}_{X_1}(-H_1), \mathcal{C}_{X_1})$$

denote its extension class; note that $\epsilon \neq 0$ by Remark 2.4.

Let $\mathcal{L}$ be the line bundle of degree 1 on $C$ associated with the point $\{0\} \in C$ and let $s_0 \in \mathrm{H}^0(C, \mathcal{L})$ be the corresponding global section. We define a vector bundle $\widetilde{\mathcal{W}}$ on $X_1 \times C$ as an extension

$$0 \longrightarrow \mathcal{C}_{X_1} \boxtimes \mathcal{L} \longrightarrow \widetilde{\mathcal{W}} \longrightarrow \mathcal{O}_{X_1}(-H_1) \boxtimes \mathcal{O}_C \longrightarrow 0 \tag{4.1}$$

with extension class

$$\epsilon \otimes s_0 \in \operatorname{Ext}^1(\mathcal{O}_{X_1}(-H_1), \mathcal{C}_{X_1}) \otimes \mathrm{H}^0(C, \mathcal{L}) \cong \operatorname{Ext}^1(\mathcal{O}_{X_1}(-H_1) \boxtimes \mathcal{O}_C, \mathcal{C}_{X_1} \boxtimes \mathcal{L}).$$

Thus, the extension splits over $\{0\}$, so that

$$\widetilde{\mathcal{W}}|_{X_1 \times \{0\}} \cong \mathcal{O}_{X_1}(-H_1) \oplus \mathcal{C}_{X_1}, \tag{4.2}$$

while for each $0 \neq t \in C$ the extension is isomorphic to (2.3), so that

$$\widetilde{\mathcal{W}}|_{X_1 \times (C \setminus \{0\})} \cong \mathcal{W}_{X_1} \boxtimes \mathcal{O}_{C \setminus \{0\}}. \tag{4.3}$$

Now consider the projective bundle $\mathbb{P}_{X_1 \times C}(\widetilde{\mathcal{W}})$ and its relative hyperplane class $H$. Since both the vector bundles $\mathcal{C}_{X_1}^\vee$ and $\mathcal{O}_{X_1}(H_1)$ are globally generated, the linear system $|H|$ is base point free on each fiber over $C$, and therefore it defines a morphism

$$\mathbb{P}_{X_1 \times C}(\widetilde{\mathcal{W}}) \to \mathbb{P}_C(\mathcal{V}')$$

to an appropriate projective bundle over $C$ (in fact, this bundle can be identified with the bundle $\mathbb{P}_C(\mathcal{V})$ constructed in the proof of Theorem 1.1). We denote the image by $\mathcal{X}$ and claim that it is smooth over $C$ with fibers $X$ and $Y$ over $\{0\} \in C$ and $C \setminus \{0\}$, respectively, and that

$$\mathbb{P}_{X_1 \times C}(\widetilde{\mathcal{W}}) \cong \mathrm{Bl}_{X_2 \times C}(\mathcal{X}).$$

Indeed, the fiber $\mathcal{X}_t$ of $\mathcal{X}$ over a point $t \in C$ is the image of $\mathbb{P}_{X_1}(\widetilde{\mathcal{W}}_t)$ under the morphism given by the relative hyperplane class. When $t = 0$, by (4.2) this agrees with the definition (1.3) of the horospherical variety, so that

$$\mathcal{X}_0 \cong X.$$

On the other hand, for $t \neq 0$ we use (4.3) and Proposition 2.3 and deduce that

$$\mathcal{X}_t \cong Y.$$

Finally, note that the exceptional locus of the morphism $\rho\colon \mathbb{P}_{X_1 \times C}(\widetilde{\mathcal{W}}) \to \mathcal{X}$ is the projective subbundle

$$\mathbb{P}_{X_1 \times C}(\mathcal{C}_{X_1} \boxtimes \mathcal{L}) \cong \mathbb{P}_{X_1}(\mathcal{C}_{X_1}) \times C \cong \widetilde{X} \times C \cong \mathbb{P}_{X_2}(\mathcal{U}_{X_2}) \times C,$$

and it is contracted by $\rho$ onto the subvariety $X_2 \times C \subset \mathcal{X}$. $\square$

*Remark* 4.1. One can obtain the vector bundle $\widetilde{\mathcal{W}}$ on $X_1 \times C$ from the (trivial over $C$) vector bundle $\mathcal{W}_{X_1} \boxtimes \mathcal{O}_C$ and the filtration (2.3) of its central fiber by using an elementary transformation similar to the one in Lemma 3.2. Using this one can merge the constructions in Theorems 1.1 and 1.2.

## § 5. Derived categories

There are several ways in which the constructions in Theorems 1.1 and 1.2 can be applied. For instance, one can relate the derived categories of $X$ and $Y$. Recall that both have a full exceptional collection: in the case of $X$ this was proved in [1], Theorem 8.20, and in the case of $Y$ in [3], Theorem 7.1; moreover, Remark 8.22 in [1] points out that these collections have the same structure.

It turns out that these two collections can be glued. In fact, one can define a *relative exceptional collection* on $\mathcal{X}$ over $C$ that coincides with the collection from Theorem 7.1 in [3] over $C \setminus \{0\}$ and with the collection from Theorem 8.20 in [1] on the central fiber.

Explicitly, recall the notation of diagram (1.7) and denote additionally by

- $i_1 \colon E_1 \hookrightarrow \mathrm{Bl}_{X_1}(\mathcal{X})$ the embedding of exceptional divisor of $\pi_{\mathcal{X}}$, and by
- $i_2 \colon E_2 \hookrightarrow \mathrm{Bl}_{X_2}(\mathcal{Y})$ the embedding of exceptional divisor of $\pi_{\mathcal{Y}}$.

Recall from Lemma 3.1 that $E_1 \to X_1$ and $E_2 \to X_2$ are $\mathbb{P}^2$-bundles and the intersection

$$E := E_1 \cap E_2 \cong \widetilde{X}$$

is transverse. Set $\mathcal{U}_{\mathcal{Y}} := \mathcal{U} \boxtimes \mathcal{O}_C$ and $\mathcal{S}_{\mathcal{Y}} := \mathcal{S} \boxtimes \mathcal{O}_C$. Then one can check that on $\mathrm{Bl}_{X_1}(\mathcal{X}) \cong \mathrm{Bl}_{X_2}(\mathcal{Y})$ there are distinguished triangles

$$\begin{aligned} &\pi_{\mathcal{X}}^*\mathcal{S}_{\mathcal{X}} \to \pi_{\mathcal{Y}}^*\mathcal{S}_{\mathcal{Y}} \to i_{1*}\mathcal{O}_{E_1}(-E), \\ &\pi_{\mathcal{Y}}^*\mathcal{U}_{\mathcal{Y}} \to \pi_{\mathcal{X}}^*\mathcal{U}_{\mathcal{X}} \to i_{2*}\mathcal{O}_{E_2}(-H_2 - 2E), \end{aligned} \tag{5.1}$$

which define objects $\mathcal{S}_{\mathcal{X}}$ and $\mathcal{U}_{\mathcal{X}}$ in $\mathbf{D}^{\mathrm{b}}(\mathcal{X})$. Note that both $E_1$ and $E_2$ are supported over $\{0\} \in C$, hence over $C \setminus \{0\}$ these triangles simplify to isomorphisms between the restrictions of $\mathcal{S}_{\mathcal{X}}$ and $\mathcal{S}_{\mathcal{Y}}$, and $\mathcal{U}_{\mathcal{X}}$ and $\mathcal{U}_{\mathcal{Y}}$, respectively. On the other hand the restrictions to the central fiber $\mathcal{X}_0 \cong X$ can be identified as

$$\mathcal{S}_{\mathcal{X}}|_X \cong \mathbb{U} \quad \text{and} \quad \mathcal{U}_{\mathcal{X}}|_X \cong \widehat{\mathbb{S}},$$

where the right-hand sides were defined in [1], Propositions 8.4 and 8.7 and Lemma 8.12.

One can also prove that there is a $C$-linear semiorthogonal decomposition

$$\mathbf{D}^{\mathrm{b}}(\mathcal{X}) = \langle \mathcal{A}, \mathcal{A}(H), \mathcal{A}(2H), \mathcal{A}(3H) \rangle,$$

where $H$ is the relative hyperplane class for $\mathcal{X}$ over $C$ and

$$\mathcal{A} = \langle \mathcal{S}_{\mathcal{X}} \otimes \mathbf{D}^{\mathrm{b}}(C), \mathcal{U}_{\mathcal{X}} \otimes \mathbf{D}^{\mathrm{b}}(C), \mathcal{O}_{\mathcal{X}} \otimes \mathbf{D}^{\mathrm{b}}(C) \rangle.$$

Moreover, after the base changes to $\{0\}$ and $C \setminus \{0\}$ (see [4]) these decompositions coincide with the corresponding decompositions of $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y \times (C \setminus \{0\}))$.

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