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QUIVER VARIETIES AND HILBERT SCHEMES

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ABSTRACT. In this note we give an explicit geometric description of some of the Nakajima's quiver varieties. More precisely, if $X = \mathbb{C}^2$, $\Gamma \subset \mathrm{SL}(\mathbb{C}^2)$ is a finite subgroup, and X_Γ is a minimal resolution of X/Γ, we show that $X^{\Gamma[n]}$ (the Γ-equivariant Hilbert scheme of X), and $X_\Gamma^{[n]}$ (the Hilbert scheme of X_Γ) are quiver varieties for the affine Dynkin graph corresponding to Γ via the McKay correspondence with the same dimension vectors but different parameters ζ (for earlier results in this direction see works by M. Haiman, M. Varagnolo and E. Vasserot, and W. Wang). In particular, it follows that the varieties $X^{\Gamma[n]}$ and $X_\Gamma^{[n]}$ are diffeomorphic. Computing their cohomology (in the case $\Gamma = \mathbb{Z}/d\mathbb{Z}$) via the fixed points of a $(\mathbb{C}^* \times \mathbb{C}^*)$-action we deduce the following combinatorial identity: the number $UCY(n, d)$ of Young diagrams consisting of nd boxes and uniformly colored in d colors equals the number $CY(n, d)$ of collections of d Young diagrams with the total number of boxes equal to n.

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1. Introduction

The quiver varieties defined by Nakajima in [9] are of fundamental importance in algebraic and differential geometry, theory of representations and other branches of mathematics. They provide a rich source of examples of hyperkähler manifolds with very interesting geometry. For example, the quiver variety $\mathfrak{M}_\zeta(\boldsymbol{v}^0, 0)$ corresponding to an affine Dynkin graph with the dimension vector $\boldsymbol{v}^0$ given by the minimal positive imaginary root of the corresponding affine root lattice and with any generic parameter ζ, is diffeomorphic to the minimal resolution X_Γ of the simple singularity X/Γ, where $X = \mathbb{C}^2$ and Γ is a finite subgroup in $\mathrm{SL}(\mathbb{C}^2)$ corresponding to the graph

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via the McKay correspondence (see [7]). Furthermore, Kronheimer and Nakajima proved in [8] that for some dimension vectors $\boldsymbol{v}$, $\boldsymbol{w}$ and for generic ζ the regular locus $\mathfrak{M}_\zeta^{\mathrm{reg}}(\boldsymbol{v}, \boldsymbol{w})$ of the quiver variety $\mathfrak{M}_\zeta^{\mathrm{reg}}(\boldsymbol{v}, \boldsymbol{w})$ is diffeomorphic to the framed moduli space of instantons on the 1-point compactification of $\mathfrak{M}_{-\zeta}(\boldsymbol{v}^0, 0)$. So, it would be natural to hope that one can give an algebraic description of $\mathfrak{M}_\zeta(\boldsymbol{v}, \boldsymbol{w})$ as a moduli space of coherent sheaves on $\mathfrak{M}_{-\zeta}(\boldsymbol{v}^0, 0)$. The main result of this note is a description of this kind in the case $\boldsymbol{v} = n\boldsymbol{v}^0$, $\boldsymbol{w} = \boldsymbol{w}^0$ — the simple root of the extending vertex 0 of the graph, and $\zeta = (0, \zeta_\mathbb{R})$ with $\zeta_\mathbb{R}$ lying in one of the following two cones in the space $\mathbb{R}^d$:

$$\mathbf{C}_+ = \{\zeta_\mathbb{R} \in \mathbb{R}^d \mid \zeta_\mathbb{R}^k > 0,\ 0 \leqslant k \leqslant d-1\},$$
$$\mathbf{C}_-(n) = \left\{\zeta_\mathbb{R} \in \mathbb{R}^d \;\middle|\; \frac{1}{n}\zeta_\mathbb{R}^k > \sum_{i=0}^{d-1} \zeta_\mathbb{R}^i \boldsymbol{v}_i^0 > 0,\ 1 \leqslant k \leqslant d-1\right\}.$$

We prove that for $\zeta_\mathbb{R} \in \mathbf{C}_+$ the quiver variety $\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$ is isomorphic to the Γ-equivariant Hilbert scheme of points on the plane X. This fact is well known, see e. g. [13], [12]; we include a proof only for the sake of completeness. We also prove that for $\zeta_\mathbb{R} \in \mathbf{C}_-(n)$ the quiver variety $\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$ is isomorphic to the Hilbert scheme of points on X_Γ. After this paper was written H. Nakaijma kindly informed me that this fact was known to him (see [13] and [3]), but his arguments are different (see Remark 46). Our proof is based on an interpretation of quiver varieties as moduli spaces of representations of the corresponding double quivers suggested by Crawley-Boevey in [2].

The paper is organized as follows. In Section 2 we recollect the necessary background: the definition of quiver varieties, representations of quivers and the construction of Crawley-Boevey. In Section 3 we reproduce in a short form a geometric version of the McKay correspondence based on investigation of X_Γ. We also prove here a generalization of the result of Kapranov and Vasserot [5]. More precisely, the authors of *loc. cit.* have constructed equivalences of (bounded) derived categories

$$\mathcal{D}^b(\mathrm{Coh}_\Gamma(X)) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{D}^b(\mathrm{Coh}(X_\Gamma)),$$

where Coh stands for the category of coherent sheaves, and Coh_Γ stands for the category of Γ-equivariant coherent sheaves. We show that there is a whole family of equivalences

$$\mathcal{D}^b(\mathrm{Coh}_\Gamma(X)) \underset{\Psi_{\zeta_\mathbb{R}}}{\overset{\Phi_{\zeta_\mathbb{R}}}{\rightleftarrows}} \mathcal{D}^b(\mathrm{Coh}(X_\Gamma))$$

which differ when parameters $\zeta_\mathbb{R}$ lie in distinct chambers of $\mathbb{R}^d$ with respect to roots hyperplanes. In Section 4 we prove the main results of this paper by constructing isomorphisms

$$\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0) \cong \begin{cases} X^{\Gamma[n]}, & \text{for } \zeta_\mathbb{R} \in \mathbf{C}_+,\\ X_\Gamma^{[n]}, & \text{for } \zeta_\mathbb{R} \in \mathbf{C}_-(n),\end{cases}$$

where $X^{\Gamma[n]}$ is the Γ-equivariant Hilbert scheme of X, and $X_\Gamma^{[n]}$ is the Hilbert scheme of X_Γ. It follows immediately from the general properties of quiver varieties that $X^{\Gamma[n]}$ and $X_\Gamma^{[n]}$ are diffeomorphic. We also prove a generalization of the first isomorphism in the Calogero–Moser context: we check that for an arbitrary $\tau \neq 0$ and $\zeta_{\mathbb{R}} \in \mathbf{C}_+$ we have

$$\mathfrak{M}_{((\tau,\dots,\tau),\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0) \cong (\mathcal{CM}_{nN})^\Gamma_{\mathrm{reg}},$$

where $N = |\Gamma|$, $\mathcal{CM}_{nN}$ is the Calogero–Moser space and $(\mathcal{CM}_{nN})^\Gamma_{reg}$ is the connected component of the set of Γ-fixed points, where the fiber of the tautological bundle is a multiple of the regular representation.

Finally, in Section 5 we consider $(\mathbb{C}^* \times \mathbb{C}^*)$-actions on $X^{\Gamma[n]}$ and $X_\Gamma^{[n]}$ for cyclic $\Gamma \cong \mathbb{Z}/d\mathbb{Z}$. We show that these actions have only a finite number of fixed points and that the number of fixed points equals the dimension of the cohomology. Furthermore, we check that the fixed points on $X^{\Gamma[n]}$ are in a bijection with the set of uniformly colored in d colors Young diagrams with dn boxes, and that the fixed points on $X_\Gamma^{[n]}$ are in a bijection with the set of collections of d Young diagrams with total number of boxes equal n. Recalling that $X^{\Gamma[n]}$ and $X_\Gamma^{[n]}$ are diffeomorphic we obtain the following combinatorial identity (see [4] for a combinatorial proof)

$$UCY(n,\, d) = CY(n,\, d),$$

where UCY and CY denote the number of uniformly colored diagrams and the number of collections of diagrams respectfully.

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2. Quiver Varieties

In this section we recall the definition and the basic properties of quiver varieties. We follow the notation of [9].

Definition. Choose an arbitrary finite graph and let H be the set of pairs consisting of an edge together with an orientation on it. Let $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) denote the incoming (resp. outgoing) vertex of $h \in H$. Let $\bar{h}$ denote the edge h with the reverse orientation. Furthermore, we choose an orientation of the graph, that is a subset $\Omega \subset H$ such that $\Omega \cup \overline{\Omega} = H$, $\Omega \cap \overline{\Omega} = \varnothing$. We define $\varepsilon(h) = 1$ for $h \in \Omega$ and $\varepsilon(h) = -1$ for $h \in \overline{\Omega}$. Finally, we identify the set of vertices of our graph with the set $\{0,\, 1,\, \dots,\, d-1\}$.

Choose a pair of Hermitian vector spaces V_k, W_k for each vertex of the graph and let

$$\boldsymbol{v} = (\dim_{\mathbb{C}} V_0,\, \dots,\, \dim_{\mathbb{C}} V_{d-1}), \;\; \boldsymbol{w} = (\dim_{\mathbb{C}} W_0,\, \dots,\, \dim_{\mathbb{C}} W_{d-1}) \in \mathbb{Z}^d$$

be their dimension vectors. We consider a partial order on the lattice $\mathbb{Z}^d$ of dimension vectors defined by

$$\boldsymbol{v}' < \boldsymbol{v} \quad \text{if} \quad \boldsymbol{v}'_k < \boldsymbol{v}_k \quad \text{for all } 0 \leqslant k \leqslant d-1.$$

The complex vector space

$$\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{v},\, \boldsymbol{w}) = \left(\bigoplus_{h\in H} \operatorname{Hom}(V_{\operatorname{out}(h)},\, V_{\operatorname{in}(h)})\right) \bigoplus \left(\bigoplus_{k=0}^{d-1} \big[\operatorname{Hom}(W_k,\, V_k) \oplus \operatorname{Hom}(V_k,\, W_k)\big]\right)$$

can be identified with the cotangent bundle of the Hermitian vector space

$$\boldsymbol{M}_\Omega(\boldsymbol{v},\, \boldsymbol{w}) = \left(\bigoplus_{h\in \Omega} \operatorname{Hom}(V_{\operatorname{out}(h)},\, V_{\operatorname{in}(h)})\right) \oplus \left(\bigoplus_{k=0}^{d-1} \operatorname{Hom}(W_k,\, V_k)\right).$$

In particular, $\boldsymbol{M}$ can be considered as a flat hyperkähler manifold.

Note that the group $G_{\boldsymbol{v}} = \prod_{k=0}^{d-1} U(V_k)$ acts on $\boldsymbol{M}$:

$$g = (g_k)_{k=0}^{n-1} \colon (B_h,\, i_k,\, j_k) \mapsto (g_{\operatorname{in}(h)} B_h g_{\operatorname{out}(h)}^{-1},\, g_k i_k,\, j_k g_k^{-1}),$$

where $B_h \in \operatorname{Hom}(V_{\operatorname{out}(h)},\, V_{\operatorname{in}(h)})$, $i_k \in \operatorname{Hom}(W_k,\, V_k)$, $j_k \in \operatorname{Hom}(V_k,\, W_k)$. This action evidently preserves the hyperkähler structure. The corresponding moment map $\mu = (\mu_{\mathbb{R}},\, \mu_{\mathbb{C}})$ is given by the following explicit formulas:

$$\mu_{\mathbb{C}}(B,\, i,\, j) = \left(\sum_{\operatorname{in}(h)=k} \varepsilon(h) B_h B_{\bar h} + i_k j_k\right) \in \bigoplus_{k=0}^{d-1} \mathfrak{gl}(V_k) = \mathfrak{g}_{\boldsymbol{v}} \otimes \mathbb{C},$$

$$\mu_{\mathbb{R}}(B,\, i,\, j) = \frac{i}{2}\left(\sum_{\operatorname{in}(h)=k} B_h B_h^\dagger - B_{\bar h}^\dagger B_{\bar h} + i_k i_k^\dagger - j_k^\dagger j_k\right) \in \bigoplus_{k=0}^{d-1} \mathfrak{u}(V_k) = \mathfrak{g}_{\boldsymbol{v}},$$

where $\mathfrak{g}_{\boldsymbol{v}}$ is the Lie algebra of $G_{\boldsymbol{v}}$ which is identified with its dual space $\mathfrak{g}_{\boldsymbol{v}}^*$ via the canonical Hermitian inner product.

Let $Z_{\boldsymbol{v}} \subset \mathfrak{g}_{\boldsymbol{v}}$ denote the center of the Lie algebra $\mathfrak{g}_{\boldsymbol{v}}$. For any element $\zeta = (\zeta_{\mathbb{C}},\, \zeta_{\mathbb{R}}) \in (Z_{\boldsymbol{v}} \otimes \mathbb{C}) \oplus Z_{\boldsymbol{v}}$, the corresponding quiver variety $\mathfrak{M}_\zeta$ is defined as a hyperkähler quotient

$$\mathfrak{M}_\zeta = \mathfrak{M}_\zeta(\boldsymbol{v},\, \boldsymbol{w}) = \{(B,\, i,\, j) \mid \mu(B,\, i,\, j) = -\zeta\}/G_{\boldsymbol{v}}.$$

In general, $\mathfrak{M}_\zeta$ has singularities, however its open subset

$$\mathfrak{M}_\zeta^{\mathrm{reg}} = \left\{ (B,\, i,\, j) \in \mu^{-1}(-\zeta) \;\middle|\; \begin{array}{l}\text{the stabilizer of } (B,\, i,\, j) \text{ in} \\ G_{\boldsymbol{v}} \text{ is trivial}\end{array} \right\} \Big/ G_{\boldsymbol{v}}$$

is a smooth hyperkähler manifold (possibly empty).

Roots and genericity. The center $Z_{\boldsymbol{v}}$ of $\mathfrak{g}_{\boldsymbol{v}} = \bigoplus_{k=0}^{d-1} \mathfrak{u}(V_k)$ coincides with the set $\{(\zeta_0,\, \dots,\, \zeta_{d-1}) \in \mathfrak{g}_{\boldsymbol{v}}\}$, where each ζ_k is an imaginary scalar matrix and $\zeta_k = 0$ if $\boldsymbol{v}_k = 0$. So, $Z_{\boldsymbol{v}}$ is naturally identified with (a subspace of) $\mathbb{R}^d$.

Let A denote the adjacency matrix of the graph and let $C = 2I - A$ be the generalized Cartan matrix. Then we consider the set of positive roots

$$R_+ = \{\theta = (\theta_k) \in (\mathbb{Z}_{\geqslant 0})^d \mid {}^t\theta C \theta \leqslant 2\} \setminus \{0\}.$$

Further we put

$$R_+(\boldsymbol{v}) = \{\theta \in R_+ \mid \theta_k \leqslant \boldsymbol{v}_k \text{ for all } k = 0,\, \dots,\, d-1\},$$

and for any positive root θ we consider a hyperplane

$$D_\theta = \{x = (x_k) \in \mathbb{R}^d \mid \sum x_k\theta_k = 0\} \subset \mathbb{R}^d.$$

The element $\zeta \in (Z_{\boldsymbol{v}} \otimes \mathbb{C}) \oplus Z_{\boldsymbol{v}} = \mathbb{R}^3 \otimes Z_{\boldsymbol{v}} \subset \mathbb{R}^3 \otimes \mathbb{R}^d$ is called generic (with respect to $\boldsymbol{v}$) if for any $\theta \in R_+(\boldsymbol{v})$ we have

$$\zeta \notin \mathbb{R}^3 \otimes D_\theta \subset \mathbb{R}^3 \otimes \mathbb{R}^d \supset \mathbb{R}^3 \otimes Z_{\boldsymbol{v}} = (Z_{\boldsymbol{v}} \otimes \mathbb{C}) \oplus Z_{\boldsymbol{v}}.$$

The importance of generic parameters ζ is explained by the following theorem of Nakajima (see [9] and [2]).

Theorem 1. *For any ζ, $\boldsymbol{v}$, and $\boldsymbol{w}$ the quiver variety $\mathfrak{M}_\zeta(\boldsymbol{v}, \boldsymbol{w})$ is either empty or nonemty and connected. Furthermore, if ζ is generic then $\mathfrak{M}_\zeta$ is smooth and there is a canonical map $\pi_0\colon \mathfrak{M}_{(\zeta_\mathbb{C},\zeta_\mathbb{R})} \to \mathfrak{M}_{(\zeta_\mathbb{C},0)}$ which is a resolution of singularities, provided $\mathfrak{M}^{\mathrm{reg}}_{(\zeta_\mathbb{C},0)}$ is nonempty. Finally, if both ζ and ζ' are generic then the varieties $\mathfrak{M}_\zeta(\boldsymbol{v}, \boldsymbol{w})$ and $\mathfrak{M}_{\zeta'}(\boldsymbol{v}, \boldsymbol{w})$ are diffeomorphic.*

Representations of quivers. Recall that a quiver Q is a finite oriented graph. Let $I(Q)$ denote the set of vertices and $A(Q)$ denote the set of arrows of Q. For any arrow of the quiver $\alpha \in A(Q)$ we denote by $\mathrm{in}(\alpha)$ and $\mathrm{out}(\alpha)$ the incoming and outgoing vertices of α. A representation ρ of a quiver Q is the following data:

$$\rho = (V, B),$$

where $V = (V_i)_{i\in I(Q)}$ is a collection of vector spaces for each vertex of the quiver and $B = (B_\alpha)_{\alpha\in A(Q)}$ is a collection of linear maps $B_\alpha\colon V_{\mathrm{out}(\alpha)} \to V_{\mathrm{in}(\alpha)}$ for each arrow of the quiver. A morphism of representations

$$\phi\colon \rho = (V, B) \to \rho' = (V', B')$$

is a collection of linear maps $\phi_i\colon V_i \to V_i'$ for each vertex of the quiver such that for any arrow α we have

$$\phi_{\mathrm{in}(\alpha)} B_\alpha = B'_\alpha \phi_{\mathrm{out}(\alpha)}.$$

The dimension of a representation $\rho = (V, B)$ is the collection of dimensions of the vector spaces V_i:

$$\dim \rho = (\dim V_i)_{i\in I(Q)}.$$

For any dimension vector $\boldsymbol{v} = (\boldsymbol{v}_i)_{i\in I(Q)}$ let

$$\mathrm{Rep}_Q(\boldsymbol{v}) = \bigoplus_{\alpha\in A(Q)} \mathrm{Hom}(\mathbb{C}^{\boldsymbol{v}_{\mathrm{out}(\alpha)}}, \mathbb{C}^{\boldsymbol{v}_{\mathrm{in}(\alpha)}})$$

be the space of $\boldsymbol{v}$-dimensional representations of Q. The group

$$\mathrm{GL}(\boldsymbol{v}, \mathbb{C}) = \prod_{i\in I(Q)} \mathrm{GL}(\boldsymbol{v}_i, \mathbb{C})$$

acts on $\mathrm{Rep}_Q(\boldsymbol{v})$ by conjugation:

$$g = (g_i)_{i\in I(Q)}\colon (B_\alpha)_{\alpha\in A(Q)} \mapsto (g_{\mathrm{in}(\alpha)} B_\alpha g^{-1}_{\mathrm{out}(\alpha)}).$$

This action evidently factors through the quotient group

$$\mathrm{PGL}(\boldsymbol{v}, \mathbb{C}) = \mathrm{GL}(\boldsymbol{v}, \mathbb{C})/\mathbb{C}^*,$$

where the embedding $\mathbb{C}^* \to \mathrm{GL}(\boldsymbol{v}, \mathbb{C})$ maps $\lambda \in \mathbb{C}^*$ to the element $(\mathrm{diag}(\lambda))_{i \in I(Q)}$. Moreover, it is clear that the set of $\mathrm{PGL}(\boldsymbol{v}, \mathbb{C})$-orbits in $\mathrm{Rep}_Q(\boldsymbol{v})$ is the set of isomorphism classes of $\boldsymbol{v}$-dimensional representations of Q.

Let $\chi\colon I(Q) \to \mathbb{R}$, $\chi(i) = \chi^i$ be a function on the set of vertices (a polarization). For any dimension vector $\boldsymbol{v}$ we define

$$\chi(\boldsymbol{v}) = \sum_{i \in I(Q)} \chi^i \boldsymbol{v}_i.$$

A representation ρ of a quiver Q is called χ-stable (resp. χ-semistable) if $\chi(\dim \rho) = 0$ and for any subrepresentation $\rho' \subset \rho$ such that $0 \neq \rho' \neq \rho$ we have $\chi(\dim \rho') > 0$ (resp. $\chi(\dim \rho') \geqslant 0$). Representations ρ and ρ' are called S-equivalent with respect to a polarization χ if both ρ and ρ' are χ-semistable and admit filtrations

$$0 = \rho_0 \subset \rho_1 \subset \cdots \subset \rho_n = \rho \quad \text{and} \quad 0 = \rho'_0 \subset \rho'_1 \subset \cdots \subset \rho'_n = \rho'$$

such that $\chi(\dim \rho_i) = \chi(\dim \rho'_i) = 0$ for all i, and

$$\bigoplus_{i=1}^{n} \rho_i/\rho_{i-1} \cong \bigoplus_{i=1}^{n} \rho'_i/\rho'_{i-1}.$$

The following theorem was proved in [6] and [10].

Theorem 2. *For any quiver Q, dimension vector $\boldsymbol{v}$, and polarization χ such that $\chi(\boldsymbol{v}) = 0$, there exists a coarse moduli space $\mathcal{M}od_Q(\boldsymbol{v}, \chi)$ of $\boldsymbol{v}$-dimensional χ-semistable representations of Q. Furthermore, if every χ-semistable $\boldsymbol{v}$-dimensional representation is χ-stable and the dimension vector $\boldsymbol{v}$ is indivisible, then $\mathcal{M}od_Q(\boldsymbol{v}, \chi)$ is a fine moduli space.*

Quivers with relations. Recall that the path algebra $\mathbb{C}[Q]$ of a quiver Q is an algebra with a basis given by (oriented) paths in the quiver and with multiplication given by concatenation of paths. It is clear that a representation of a quiver $\rho = (V, B)$ is the same as a structure of a right $\mathbb{C}[Q]$-module on the vector space $\bigoplus_{i \in I(Q)} V_i$.

A quiver with relations is a pair (Q, J), where Q is a quiver and J is a two-sided ideal $J \subset \mathbb{C}[Q]$ in its path algebra. A representation of a quiver with relations (Q, J) is a representation ρ of its underlying quiver Q such that the ideal J acts by zero in the corresponding right $\mathbb{C}[Q]$-module. In the other words, it is just a right $\mathbb{C}[Q]/J$-module. We denote by $\mathrm{Rep}_{Q,J}(\boldsymbol{v})$ the space of $\boldsymbol{v}$-dimensional representations of the quiver with relations (Q, J). It is clear that $\mathrm{Rep}_{Q,J}(\boldsymbol{v})$ is a closed algebraic subset in the vector space $\mathrm{Rep}_Q(\boldsymbol{v})$.

It is easy to see that an analogue of Theorem 2 is true for quivers with relations.

Theorem 3. *For any quiver with relations (Q, J), dimension vector $\boldsymbol{v}$ and polarization χ such that $\chi(\boldsymbol{v}) = 0$, there exists a coarse moduli space $\mathcal{M}od_{Q,J}(\boldsymbol{v}, \chi)$ of $\boldsymbol{v}$-dimensional χ-semistable representations of (Q, J). Furthermore, if every χ-semistable $\boldsymbol{v}$-dimensional representation is χ-stable and the dimension vector $\boldsymbol{v}$ is indivisible, then $\mathcal{M}od_{Q,J}(\boldsymbol{v}, \chi)$ is a fine moduli space.*

Remark 4. Fix the quiver (Q, J) and the dimension vector $\boldsymbol{v}$, and let χ vary in the space $D_{\boldsymbol{v}} = \{\chi \mid \chi(\boldsymbol{v}) = 0\}$. Then it is easy to see that the notion of χ-(semi)stability can change only when the polarization χ crosses a hyperplane $D_{\boldsymbol{v}'} \cap D_{\boldsymbol{v}} \subset D_{\boldsymbol{v}}$ for some $0 < \boldsymbol{v}' < \boldsymbol{v}$. Thus the space $D_{\boldsymbol{v}}$ acquires a chamber structure with hyperplanes $D_{\boldsymbol{v}'} \cap D_{\boldsymbol{v}}$ being the walls.

Remark 5. The moduli space $\mathcal{M}od_{Q,J}(\boldsymbol{v}, \chi)$ can also be constructed via symplectic reduction. Namely, we can consider the representations space $\mathrm{Rep}_{Q,J}(\boldsymbol{v})$ as a symplectic variety with respect to the Kähler form (corresponding to the flat Kähler metric on the vector space $\mathrm{Rep}_Q(\boldsymbol{v})$). Then the compact algebraic group

$$PU(\boldsymbol{v}) = U(\boldsymbol{v})/U(1) = \Big(\prod_{i \in I(Q)} U(\boldsymbol{v}_i)\Big)/U(1)$$

(a real form of $\mathrm{PGL}(\boldsymbol{v}, \mathbb{C})$) acts on $\mathrm{Rep}_{Q,J}(\boldsymbol{v})$ preserving the symplectic structure. Let $\mu\colon \mathrm{Rep}_{Q,J}(\boldsymbol{v}) \to \mathfrak{pu}(\boldsymbol{v})^*$ denote the corresponding moment map. The polarization χ can be thought of as an element of $\mathfrak{pu}(\boldsymbol{v})^*$. Then one can check (see e. g. [6]) that

$$\mathcal{M}od_{Q,J}(\boldsymbol{v}, \chi) = \mu^{-1}(-\chi)/PU(\boldsymbol{v}).$$

Modular description of quiver varieties. The following construction relating quiver varieties to the moduli spaces of representation of quivers was suggested in [2]. Assume that we are interested in a quiver variety $\mathfrak{M}_\zeta(\boldsymbol{v}, \boldsymbol{w})$ corresponding to a graph with the set of vertices $\{0, \ldots, d-1\}$ and with the set of oriented edges H. We define a quiver $Q = Q(\boldsymbol{w})$ (depending on the dimension vector $\boldsymbol{w}$) with the set of vertices

$$I(Q) = \{*\} \cup \{0, \ldots, d-1\}$$

and with the following set of arrows. First, we consider every oriented edge $h \in H$ as an arrow in the quiver with the same outgoing and incoming vertices. Second, for every $0 \leqslant i \leqslant d-1$ we draw $\boldsymbol{w}_i$ arrows from the vertex $*$ to the vertex i and backwards.

Then it is easy to see that a choice of bases in spaces $W_0, \ldots, W_{d-1}$ gives an identification

$$\boldsymbol{M}(\boldsymbol{v}, \boldsymbol{w}) = \mathrm{Rep}_{Q(\boldsymbol{w})}((1, \boldsymbol{v})).$$

Furthermore, the expression $\mu_{\mathbb{C}} + \zeta_{\mathbb{C}}$ can be considered as an element of the path algebra $\mathbb{C}[Q(\boldsymbol{w})]$ of the quiver $Q(\boldsymbol{w})$. We denote by $J(\zeta_{\mathbb{C}})$ the two-sided ideal in $\mathbb{C}[Q(\boldsymbol{w})]$ generated by $\mu_{\mathbb{C}}+\zeta_{\mathbb{C}}$. Then the algebraic subvariety $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \subset \boldsymbol{M}(\boldsymbol{v}, \boldsymbol{w})$ gets identified with the algebraic subvariety $\mathrm{Rep}_{Q(\boldsymbol{w}),J(\zeta_{\mathbb{C}})}((1, \boldsymbol{v})) \subset \mathrm{Rep}_{Q(\boldsymbol{w})}((1, \boldsymbol{v}))$:

$$\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) = \mathrm{Rep}_{Q(\boldsymbol{w}),J(\zeta_{\mathbb{C}})}((1, \boldsymbol{v})). \tag{6}$$

Further, note that the quiver variety

$$\mathfrak{M}_\zeta(\boldsymbol{v}, \boldsymbol{w}) = \mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) \cap \mu_{\mathbb{R}}^{-1}(-\zeta_{\mathbb{R}})/G_{\boldsymbol{v}}$$

is just the symplectic reduction of the variety $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}}) = \mathrm{Rep}_{Q(\boldsymbol{w}),J(\zeta_{\mathbb{C}})}((1, \boldsymbol{v}))$ with respect to its canonical Kähler form. Finally, the group $G_{\boldsymbol{v}}$ acting on $\mu_{\mathbb{C}}^{-1}(-\zeta_{\mathbb{C}})$

is canonically isomorphic to the group $PU((1,\,\boldsymbol{v}))$ acting on $\mathrm{Rep}_{Q(\boldsymbol{w}),J(\zeta_{\mathbb{C}})}((1,\,\boldsymbol{v}))$:

$$G_{\boldsymbol{v}} = \prod_{i=0}^{d-1} U(\boldsymbol{v}_i) \cong \left(U(1) \times \prod_{i=0}^{d-1} U(\boldsymbol{v}_i) \right) / U(1),$$

and under this identification the element $\zeta_{\mathbb{R}} \in \mathfrak{g}_{\boldsymbol{v}}$ corresponds to the polarization

$$\chi_{\boldsymbol{v}}(\zeta_{\mathbb{R}}) = (-\zeta_{\mathbb{R}}(\boldsymbol{v}),\,\zeta_{\mathbb{R}}) \in \mathfrak{pu}^*((1,\,\boldsymbol{v})). \tag{7}$$

Summing up and taking into account Remark 5 we obtain the following proposition.

Proposition 8 (cf. [2]). *For any $(\zeta_{\mathbb{C}},\,\zeta_{\mathbb{R}})$ we have an isomorphism of algebraic varieties*

$$\mathfrak{M}_{(\zeta_{\mathbb{C}},\zeta_{\mathbb{R}})}(\boldsymbol{v},\,\boldsymbol{w}) = \mathcal{M}od_{Q(\boldsymbol{w}),J(\zeta_{\mathbb{C}})}((1,\,\boldsymbol{v}),\,\chi_{\boldsymbol{v}}(\zeta_{\mathbb{R}})). \tag{9}$$

Remark 10. The arguments of Remark 4 show that $\mathfrak{M}_{\zeta}(\boldsymbol{v},\,\boldsymbol{w})$ depend on $\zeta_{\mathbb{R}}$ as follows. The space $\mathbb{R}^d$ of all $\zeta_{\mathbb{R}}$ has a chamber structure. Whenever $\zeta_{\mathbb{R}}$ varies within a chamber, the complex structure of $\mathfrak{M}_{(\zeta_{\mathbb{C}},\zeta_{\mathbb{R}})}(\boldsymbol{v},\,\boldsymbol{w})$ does not change at all (but the hyperkähler metric does), while when $\zeta_{\mathbb{R}}$ crosses a wall, $\mathfrak{M}_{(\zeta_{\mathbb{C}},\zeta_{\mathbb{R}})}(\boldsymbol{v},\,\boldsymbol{w})$ undergoes a certain (usually birational) transformation.

We finish this section with the following useful lemma.

Lemma 11. *If ζ is generic then $\mathcal{M}od_{Q(\boldsymbol{w}),J(\zeta_{\mathbb{C}})}((1,\,\boldsymbol{v}),\,\chi_{\boldsymbol{v}}(\zeta_{\mathbb{R}}))$ is a fine moduli space.*

Proof. Let $\chi = \chi_{\boldsymbol{v}}(\zeta_{\mathbb{R}})$. Since the dimension vector $(1,\,\boldsymbol{v})$ is evidently indivisible it suffices to check that every χ-semistable $(1,\,\boldsymbol{v})$-dimensional representation ρ is χ-stable. So assume that $\rho' \subset \rho$ is a χ-stable subrepresentation such that $0 \neq \rho' \neq \rho$ and $\chi(\dim \rho') = 0$. We have two possibilities: either $\dim \rho' = (0,\,\boldsymbol{v}')$ or $\dim \rho' = (1,\,\boldsymbol{v}')$ for some $\boldsymbol{v}'$.

Assume for example that $\dim \rho' = (0,\,\boldsymbol{v}')$. Then we have

$$0 = \chi(\dim \rho') = (-\zeta_{\mathbb{R}}(\boldsymbol{v}),\,\zeta_{\mathbb{R}})(0,\,\boldsymbol{v}') = \zeta_{\mathbb{R}}(\boldsymbol{v}').$$

Without loss of generality we can assume that ρ' is χ-stable. It follows from Theorem 0.2 of [2] that $(0,\,\boldsymbol{v}')$ is a positive root for the quiver $Q(\boldsymbol{w})$, hence $\boldsymbol{v}' \in R_+(\boldsymbol{v})$, hence $\zeta_{\mathbb{R}} \in D_{\boldsymbol{v}'}$. On the other hand, by Proposition 8 the representation ρ' corresponds to a point $(B',\,0,\,0)$ of the quiver variety $\mathfrak{M}_{\zeta}(\boldsymbol{v}',\,\boldsymbol{w})$, hence

$$\begin{aligned}\zeta_{\mathbb{C}}(\boldsymbol{v}') = \sum_{k=0}^{n-1} \mathrm{Tr}(\zeta_{\mathbb{C}}^k) &= \sum_{k=0}^{n-1} \mathrm{Tr}(-\mu_{\mathbb{C}}(B',\,0,\,0)) \\ &= -\mathrm{Tr}\left(\sum_{h\in H} \varepsilon(h) B'_h B'_{\bar{h}}\right) = -\mathrm{Tr}\left(\sum_{h\in\Omega} \left[B'_h,\,B'_{\bar{h}}\right]\right) = 0.\end{aligned}$$

Thus $\zeta \in \mathbb{R}^3 \otimes D_{\boldsymbol{v}'}$, a contradiction with the assumtion that ζ is generic.

Similarly, assume that $\dim \rho' = (1,\,\boldsymbol{v}')$. Consider the quotient representation $\rho'' = \rho/\rho'$. Then $\dim \rho'' = (0,\,\boldsymbol{v}'')$ with $\boldsymbol{v}'' = \boldsymbol{v} - \boldsymbol{v}'$, and without loss of generality we can assume that ρ'' is χ-stable. Repeating the above argument for ρ'' and $\boldsymbol{v}''$ instead of ρ' and $\boldsymbol{v}'$ we again deduce that ζ is not generic.

Thus we have proved that every χ-semistable representation is χ-stable, hence the moduli space is fine. ∎

3. The McKay Correspondence

The McKay correspondence associates to every (conjugacy class of a) finite subgroup of the group $\mathrm{SL}(\mathbb{C}^2)$ an affine Dynkin graph of type ADE. The graph corresponding to a subgroup $\Gamma \subset \mathrm{SL}(\mathbb{C}^2)$ can be constructed as follows. Let R_0, …, R_{d-1} be the set of all (isomorphism classes of) irreducible representations of the group Γ, and assume that R_0 is the trivial representation. Let $L = \mathbb{C}^2$ be the tautological representation. Furthermore, let $a_{k,l}$ be the multiplicities in the decomposition of the tensor product $R_k \otimes L$ into the sum of irreducible representations:

$$R_k \otimes L \cong \bigoplus_{l=0}^{d-1} R_l^{\oplus a_{k,l}}. \tag{12}$$

Then the graph with the set of vertices $\{0, \ldots, d-1\}$ and with $a_{k,l}$ edges between the vertices k and l is the corresponding affine Dynkin graph.

Quiver varieties for affine Dynkin graphs. From now on we will consider quiver varieties corresponding to an affine Dynkin graph. Let Γ be the corresponding subgroup of $\mathrm{SL}(\mathbb{C}^2)$. Let d denote the number of irreducible representations of Γ (which is equal to the number of vertices in the graph), and let N be the order of the group Γ.

Let V and W be representations of Γ. Then

$$V = \bigoplus_{k=0}^{d-1} V_k \otimes R_k, \quad W = \bigoplus_{k=0}^{d-1} W_k \otimes R_k. \tag{13}$$

The dimension vectors

$$\boldsymbol{v} = (\dim V_0, \ldots, \dim V_{d-1}), \quad \boldsymbol{w} = (\dim W_0, \ldots, \dim W_{d-1}).$$

can be thought of as classes of V and W in the Grothendieck ring $K_0(\Gamma)$. Consider a triple

$$(B, i, j) \in \mathrm{Hom}_\Gamma(V \otimes L, V) \oplus \mathrm{Hom}_\Gamma(W, V) \oplus \mathrm{Hom}_\Gamma(V, W),$$

where Hom_Γ denotes the space of all Γ-equivariant linear maps. Then from (13) and from

$$V \otimes L = \bigoplus_{k=0}^{d-1} V_k \otimes R_k \otimes L = \bigoplus_{k,l=0}^{d-1} V_k \otimes R_l^{\oplus a_{k,l}} = \bigoplus_{k,l=0}^{d-1} V_k^{\oplus a_{k,l}} \otimes R_l$$

it follows that a choice of (B, i, j) is equivalent to a choice of a collection of triples $(B_h, i_k, j_k) \in \boldsymbol{M}(\boldsymbol{v}, \boldsymbol{w})$. Thus,

$$\boldsymbol{M}(\boldsymbol{v}, \boldsymbol{w}) = \mathrm{Hom}_\Gamma(V \otimes L, V) \oplus \mathrm{Hom}_\Gamma(W, V) \oplus \mathrm{Hom}_\Gamma(V, W).$$

Further, it is easy to check that

$$\mu_{\mathbb{C}}(B, i, j) = [B, B] + ij, \quad \mu_{\mathbb{R}}(B, i, j) = BB^\dagger - B^\dagger B + ii^\dagger - j^\dagger j,$$

where $[B, B]$, $BB^\dagger$, and $B^\dagger B$ are defined as the compositions

$$[B, B]\colon V \xrightarrow{\lambda_L^{-1}} V\otimes L\otimes L \xrightarrow{B\otimes 1} V\otimes L \xrightarrow{B} V,$$
$$BB^\dagger\colon V \xrightarrow{h_V} V^* \xrightarrow{B^*} V^*\otimes L^* \xrightarrow{h_V^{-1}\otimes h_L^{-1}} V\otimes L \xrightarrow{B} V,$$
$$B^\dagger B\colon V \xrightarrow{B} V\otimes L^* \xrightarrow{h_V\otimes h_L^{-1}} V^*\otimes L \xrightarrow{B^*} V^* \xrightarrow{h_V^{-1}} V,$$

and λ_L, h_L and h_V stand for Γ-invariant symplectic form and Hermitian inner products on L and V respectively. Furthermore, we can consider the parameters $\zeta_{\mathbb{C}}$ and $\zeta_{\mathbb{R}}$ as elements of $Z(\mathbb{C}[\Gamma])$ and $Z(\mathbb{R}[\Gamma])$, the centers of the group algebras of Γ over $\mathbb{C}$ and $\mathbb{R}$ respectively (the element $\zeta_{\mathbb{C}}$ acts in R_k as $\zeta_{\mathbb{C}}^k$-multiplication). Finally, it is easy to see that $G_{\boldsymbol{v}} = U_\Gamma(V)$ and thus we obtain the following lemma.

Lemma 14. *We have*

$$\mathfrak{M}_{(\zeta_{\mathbb{C}},\zeta_{\mathbb{R}})}(\boldsymbol{v}, \boldsymbol{w}) = \left\{(B, i, j)\ \middle|\ \begin{matrix}[B, B]+ij=-\zeta_{\mathbb{C}}\\ [B, B^\dagger]+ii^\dagger-j^\dagger j=-\zeta_{\mathbb{R}}\end{matrix}\right\}\Big/ U_\Gamma(V).$$

Let $X=\mathbb{C}^2$ with the tautological action of Γ; identify L with the space of linear functions on X.

Lemma 15 (cf. [5], Pr. 3.4)**.** *For any point $(B, i, j)\in\mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(\boldsymbol{v}, \boldsymbol{w})$ such that $j=0$, the map B induces the structure of a Γ-equivariant $\mathbb{C}[X]$-module on V (where the linear functions on X act via B).*

Proof. Since we have

$$[B, B] = \mu_{\mathbb{C}}(B, i, j) - ij = 0,$$

the actions of linear functions on X commute. Therefore, they induce an action of $\mathbb{C}[X]$. The resulting $\mathbb{C}[X]$-module structure on V is evidently Γ-equivariant. ☐

Combining this with (6), we obtain the following.

Corollary 16. *There is an isomorphism*

$$\{(V_{\bullet}, B, i, j)\in \operatorname{Rep}_{Q(\boldsymbol{w}),J(0)}((1, \boldsymbol{v}))\mid j=0\}\cong \left\{\begin{matrix}\Gamma\text{-equivariant } \mathbb{C}[X]\text{-module structures on } V=\bigoplus_{k=0}^{d-1} V_k\otimes R_k\\ \text{with a framing } i\in \operatorname{Hom}_\Gamma(W, V)\end{matrix}\right\}.$$

Abusing the notation we will denote by $V_{\bullet}$ the representation of the quiver $(Q(\boldsymbol{w}), J(0))$ corresponding to a Γ-equivariant $\mathbb{C}[X]$-module V and Γ-morphism $i\colon W\to V$. Vice versa, the Γ-equivariant $\mathbb{C}[X]$-module corresponding to a representation ρ of the quiver $(Q(\boldsymbol{w}), J(0))$ will be denoted by $\operatorname{Tot}(\rho)$.

Remark 17. In fact, one can reformulate Corollary 16 in terms of Morita equivalence. Let $e_*\in\mathbb{C}[Q(\boldsymbol{w})]$ denote the idempotent of the vertex $*$ (the path of length 0 with $*$ being the outgoing (and incoming) vertex). Then the algebra $\mathbb{C}[Q(\boldsymbol{w})]/\langle J(0), e_*\rangle$ is Morita-equivalent to the smash product algebra $\mathbb{C}[X]\,\#\,\Gamma$.

Resolutions of simple singularities. We fix a pair of dimension vectors:

$$\boldsymbol{v}^0 = (\dim R_0, \dim R_1, \ldots, \dim R_{d-1}) \quad \text{and} \quad \boldsymbol{w}^0 = (1, 0, \ldots, 0).$$

Note that

$$N = \sum_{k=0}^{d-1} (\boldsymbol{v}_i^0)^2 = |\Gamma|. \tag{18}$$

We fix also the complex parameter $\zeta_{\mathbb{C}} = 0$ but let the real parameter $\zeta_{\mathbb{R}}$ vary. The quiver varieties $\mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(\boldsymbol{v}^0, \boldsymbol{w}^0)$ are described by the following theorem.

Theorem 19 (see [7])**.** (i) *The quiver variety* $\mathfrak{M}_{(0,0)}(\boldsymbol{v}^0, \boldsymbol{w}^0)$ *is isomorphic to the quotient variety* X/Γ.

(ii) *For generic* $\zeta_{\mathbb{R}}$ *the quiver variety* $X_\Gamma(\zeta_{\mathbb{R}}) = \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(\boldsymbol{v}^0, \boldsymbol{w}^0)$ *is a minimal resolution of the quotient variety* X/Γ *via the canonical map* $\pi_0 \colon \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(\boldsymbol{v}^0, \boldsymbol{w}^0) \to \mathfrak{M}_{(0,0)}(\boldsymbol{v}^0, \boldsymbol{w}^0)$.

Remark 20. Results of [7] concern varieties $\mathfrak{M}_\zeta(\boldsymbol{v}^0, 0)$ rather than $\mathfrak{M}_\zeta(\boldsymbol{v}^0, \boldsymbol{w}^0)$. However, it is easy to see that these varieties are canonically isomorphic. Indeed, arguing like in the proof of Lemma 25 below, one can check that for any point of $\mathfrak{M}_\zeta(\boldsymbol{v}^0, \boldsymbol{w}^0)$ we have

$$j_0 i_0 = \sum \zeta_{\mathbb{C}}^k \boldsymbol{v}_k^0, \quad i_0 i_0^\dagger - j_0^\dagger j_0 = -2\sqrt{-1} \sum \zeta_{\mathbb{R}}^k \boldsymbol{v}_k^0.$$

It follows that the map $\boldsymbol{M}(\boldsymbol{v}^0, \boldsymbol{w}^0) \supset \mu^{-1}(\zeta) \to \mu^{-1}(\zeta) \subset \boldsymbol{M}(\boldsymbol{v}^0, 0)$ induced by the canonical projection $\boldsymbol{M}(\boldsymbol{v}^0, \boldsymbol{w}^0) \to \boldsymbol{M}(\boldsymbol{v}^0, 0)$ (forgetting of i_0 and j_0) is a principal $U(1)$-bundle. Moreover, this map is $G_{\boldsymbol{v}^0}$-equivariant and the action of $U(1)$ on fibers is nothing but the action of a subgroup $U(1) \subset G_{\boldsymbol{v}^0}$. Hence the quotients modulo $G_{\boldsymbol{v}^0}$ are canonically isomorphic.

Remark 21. In dimension 2 any two minimal resolutions necessarily coincide. Thus, we have a canonical identification $X_\Gamma(\zeta_{\mathbb{R}}) \cong X_\Gamma(\zeta'_{\mathbb{R}})$ for any generic $\zeta_{\mathbb{R}}$ and $\zeta'_{\mathbb{R}}$, compatible with the projection to X/Γ. Thus, we can (and will) write X_Γ instead of $X_\Gamma(\zeta_{\mathbb{R}})$ without a risk of misunderstanding.

Theorem 8 implies that for any $\zeta_{\mathbb{R}}$ we have an isomorphism

$$X_\Gamma = \mathcal{M}od_{Q(\boldsymbol{w}),J(0)}((1, \boldsymbol{v}^0), \chi_{\boldsymbol{v}^0}(\zeta_{\mathbb{R}})),$$

while Lemma 11 implies that X_Γ is a fine moduli space whenever $\zeta_{\mathbb{R}}$ is generic. This means that for any generic $\zeta_{\mathbb{R}}$ we have a universal representation $\underline{\rho}^{\zeta_{\mathbb{R}}}$ of the quiver $(Q(\boldsymbol{w}^0), J(0))$ over X_Γ, that is, a collection of vector bundles $\underline{V}_k^{\zeta_{\mathbb{R}}}$ over X_Γ, $k \in \{*\} \cup \{0, \ldots, d-1\}$ the ranks of which are given by

$$\mathrm{r}(\underline{V}_*^{\zeta_{\mathbb{R}}}) = 1, \quad \mathrm{r}(\underline{V}_k^{\zeta_{\mathbb{R}}}) = \boldsymbol{v}_k^0 \quad \text{for } k \in \{0, \ldots, d-1\}, \tag{22}$$

and morphisms

$$\underline{B}_h^{\zeta_{\mathbb{R}}} : \underline{V}_{\mathrm{out}(h)}^{\zeta_{\mathbb{R}}} \to \underline{V}_{\mathrm{in}(h)}^{\zeta_{\mathbb{R}}}, \quad \underline{i}_0^{\zeta_{\mathbb{R}}} : \underline{V}_*^{\zeta_{\mathbb{R}}} \to \underline{V}_0^{\zeta_{\mathbb{R}}}, \quad \underline{j}_0^{\zeta_{\mathbb{R}}} : \underline{V}_0^{\zeta_{\mathbb{R}}} \to \underline{V}_*^{\zeta_{\mathbb{R}}}$$

(recall the choice of $\boldsymbol{w}^0$) satisfying the equations

$$\begin{aligned} \sum_{\mathrm{in}(h)=k} \varepsilon(h)\underline{B}_h^{\zeta_\mathbb{R}}\underline{B}_{\bar h}^{\zeta_\mathbb{R}} &= 0 \quad \text{for } k = 1, \dots, d-1, \\ \underline{i}_0^{\zeta_\mathbb{R}}\underline{j}_0^{\zeta_\mathbb{R}} + \sum_{\mathrm{in}(h)=0} \varepsilon(h)\underline{B}_h^{\zeta_\mathbb{R}}\underline{B}_{\bar h}^{\zeta_\mathbb{R}} &= 0 \end{aligned} \tag{23}$$

and such that for any $x \in X_\Gamma$ the corresponding representation

$$\underline{\rho}(x)^{\zeta_\mathbb{R}} = \left(\underline{V}^{\zeta_\mathbb{R}}_{|x}, \underline{B}^{\zeta_\mathbb{R}}(x), \underline{i}_0^{\zeta_\mathbb{R}}(x), \underline{j}_0^{\zeta_\mathbb{R}}(x)\right)$$

of the quiver $Q(\boldsymbol{w}^0)$ is $\chi_{\boldsymbol{v}^0}(\zeta_\mathbb{R})$-stable.

Remark 24. Actually, the family $\rho^{\zeta_\mathbb{R}}$ does not change when $\zeta_\mathbb{R}$ varies within a chamber.

Note that the universal representation is defined up to a twist by a line bundle, so without a loss of generality we may (and will) assume that $\underline{V}_*^{\zeta_\mathbb{R}} = \mathcal{O}$.

Lemma 25. *For any* $(1, \boldsymbol{v})$*-dimensional representation* (V, B_h, i_0, j_0) *of the quiver* $(Q(\boldsymbol{w}^0), J(0))$ *we have* $j_0 i_0 = 0$.

Proof. Since $\dim V_* = 1$ it follows that

$$j_0 i_0 = \operatorname{Tr} j_0 i_0 = \operatorname{Tr} i_0 j_0.$$

On the other hand, summing up equations (23) and taking the trace we see that

$$\operatorname{Tr} i_0 j_0 = \sum_{h \in \Omega} \operatorname{Tr}[B_h, B_{\bar h}] + \operatorname{Tr} i_0 j_0 = \operatorname{Tr} \mu_\mathbb{C}(B_h, i_0, j_0) = 0. \qquad \square$$

Since $\mathrm{r}(\underline{V}_*^{\zeta_\mathbb{R}}) = \mathrm{r}(\underline{V}_0^{\zeta_\mathbb{R}}) = 1$ by (22), it follows from Lemma 25 that for any $x \in X_\Gamma$ we either have $\underline{j}_0^{\zeta_\mathbb{R}}(x) = 0$, or $\underline{i}_0^{\zeta_\mathbb{R}}(x) = 0$. Assume that

$$\zeta_\mathbb{R}(\boldsymbol{v}) > 0. \tag{26}$$

In this case if $\underline{i}_0^{\zeta_\mathbb{R}}(x) = 0$ then the $\chi_{\boldsymbol{v}^0}(\zeta_\mathbb{R})$-stability would be violated because then $\underline{\rho}^{\zeta_\mathbb{R}}(x)$ would admit a $(1, 0, \dots, 0)$-dimensional subrepresentation

$$\chi_{\boldsymbol{v}^0}(\zeta_\mathbb{R})(1, 0, \dots, 0) = (-\zeta_\mathbb{R}(\boldsymbol{v}^0), \zeta_\mathbb{R})(1, 0, \dots, 0) = -\zeta_\mathbb{R}(\boldsymbol{v}^0) < 0.$$

So it follows that when (26) holds we have $\underline{j}_0^{\zeta_\mathbb{R}} = 0$, and that $\underline{i}_0^{\zeta_\mathbb{R}}$ is an embedding of line bundles, and hence an isomorphism. Thus we proved the following lemma.

Lemma 27. *For any generic* $\zeta_\mathbb{R}$ *satisfying* (26) *there exists a universal* $\chi_{\boldsymbol{v}^0}(\zeta_\mathbb{R})$*-stable family on* X_Γ *such that*

$$\underline{V}_*^{\zeta_\mathbb{R}} \cong \underline{V}_0^{\zeta_\mathbb{R}} \cong \mathcal{O}_{X_\Gamma}, \quad \underline{j}_0^{\zeta_\mathbb{R}} = 0, \quad \textit{and} \quad \underline{i}_0^{\zeta_\mathbb{R}} = \mathrm{id}.$$

Remark 28. This universal family appeared earlier in [8].

Applying Corollary 16 we deduce

Corollary 29. *For any generic* $\zeta_\mathbb{R}$ *such that* (26) *holds,* $\operatorname{Tot}(\underline{\rho}^{\zeta_\mathbb{R}})$ *is a family of* Γ*-equivariant* $\mathbb{C}[X]$*-modules over* X_Γ.

From now on we will denote by $\underline{\rho}^{\zeta_{\mathbb{R}}}$ the universal representation normalized as in Lemma 27. Moreover, to simplify the notation we will omit the superscript $\zeta_{\mathbb{R}}$ when this cannot lead us to a confusion.

Let ρ_0 denote the unique $(1, 0, \ldots, 0)$-dimensional representation of the quiver $(Q(\boldsymbol{w}^0), J(\zeta_{\mathbb{C}}))$. It follows from Lemma 27 that for any x we have canonical surjective homomorphism $\underline{\rho}(x) \to \rho_0$. Denote its kernel by $\hat{\rho}(x)$.

Derived equivalences. For each generic $\zeta_{\mathbb{R}}$ we define the functors

$$\mathcal{D}^b(\mathrm{Coh}_\Gamma(X)) \underset{\Psi_{\zeta_{\mathbb{R}}}}{\overset{\Phi_{\zeta_{\mathbb{R}}}}{\rightleftarrows}} \mathcal{D}^b(\mathrm{Coh}(X_\Gamma)) ,$$

where $\mathrm{Coh}_\Gamma(X)$ denotes the category of Γ-equivariant coherent sheaves on X, $\mathrm{Coh}(X_\Gamma)$ denotes the category of coherenet sheaves on X_Γ, and $\mathcal{D}^b$ stands for the bounded derived category. To do this we consider the family $\mathrm{Tot}(\underline{\rho}^{\zeta_{\mathbb{R}}})$ of Γ-equivariant $\mathbb{C}[X]$-modules as a sheaf on the product $X_\Gamma \times X$. Denoting the projections to X_Γ and to X by p_1 and p_2 respectfully, we put

$$\Phi_{\zeta_{\mathbb{R}}}(F) = (Rp_{1*}R\mathcal{H}om(\mathrm{Tot}(\underline{\rho}^{\zeta_{\mathbb{R}}}), p_2^*F))^G, \quad \text{and}$$
$$\Psi_{\zeta_{\mathbb{R}}}(F) = Rp_{2*}(p_1^*F \otimes^L \mathrm{Tot}(\underline{\rho}^{\zeta_{\mathbb{R}}})).$$

Theorem 30. *For any generic $\zeta_{\mathbb{R}}$, the functors $\Phi_{\zeta_{\mathbb{R}}}$ and $\Psi_{\zeta_{\mathbb{R}}}$ are mutually inverse equivalences of categories.*

This theorem is just a slight generalization of Theorem 1.4 of Kapranov and Vasserot (see [5]). In fact, the equivalences Φ and Ψ from *loc. cit.* are isomorphic to the functors $\Phi_{\zeta_{\mathbb{R}}}$ and $\Psi_{\zeta_{\mathbb{R}}}$ with

$$\zeta_{\mathbb{R}} \in \{\zeta_{\mathbb{R}} \in \mathbb{R}^n \mid \zeta_{\mathbb{R}}^i < 0 \text{ for all } 0 \leqslant i \leqslant n-1\}.$$

The proof of this theorem can be done by the same argument as in *loc. cit.*

Remark 31. Actually, the equivalences $\Phi_{\zeta_{\mathbb{R}}}$ and $\Psi_{\zeta_{\mathbb{R}}}$ do not change when $\zeta_{\mathbb{R}}$ varies within a chamber (cf. Remark 24).

Remark 32. For any pair of generic $\zeta_{\mathbb{R}}$ and $\zeta'_{\mathbb{R}}$, the compositions of functors $\Phi_{\zeta_{\mathbb{R}}} \cdot \Psi_{\zeta'_{\mathbb{R}}}$ and $\Psi_{\zeta'_{\mathbb{R}}} \cdot \Phi_{\zeta_{\mathbb{R}}}$ are autoequivalences of the derived categorie $\mathcal{D}^b(\mathrm{Coh}(X_\Gamma))$ and $\mathcal{D}^b(\mathrm{Coh}_\Gamma(X))$ respectively. One can check that these equivalences generate the action of the affine braid group described in [11].

4. Interpretation of Quiver Varieties

Symmetric powers of X/Γ. From now on we will be concerned with an explicit geometric description of quiver varieties $\mathfrak{M}_\zeta(n\boldsymbol{v}^0, \boldsymbol{w}^0)$ for $\zeta_{\mathbb{C}} = 0$, $\boldsymbol{v}^0$ and $\boldsymbol{w}^0$ as above, and various n and $\zeta_{\mathbb{R}}$. We begin with the simplest case $\zeta_{\mathbb{R}} = 0$. In fact, this is well known, but we put here a proof for the sake of completeness.

Proposition 33. *There is a bijection $\mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0, \boldsymbol{w}^0) = S^n(X/\Gamma)$, where S^n stands for the symmetric power.*

Remark 34. Actually, it is natural to expect that the bijection constructed in the proof is an isomorphism of algebraic varieties. However, it is not so easy to prove since varieties $\mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$ and $S^n(X/\Gamma)$ are not fine moduli spaces.

Proof. The case $n = 1$ follows from the first assertion of Theorem 19. So assume that $n > 1$.

It follows from Proposition 8 that it suffices to check that

$$\mathcal{M}od_{Q(\boldsymbol{w}),J(0)}((1,\, n\boldsymbol{v}^0),\, 0) = S^n(X/\Gamma).$$

Fix a generic $\zeta_{\mathbb{R}}$. For a collection of points $(x_1, \dots, x_n)$, $x_i \in X/\Gamma$ we put

$$g_0(x_1, \dots, x_n) = \rho_0 \oplus \hat\rho(\tilde x_1) \oplus \dots \hat\rho(\tilde x_n),$$

where $\tilde x_i \in \pi_0^{-1}(x_i) \in X_\Gamma(\zeta_{\mathbb{R}})$ are arbitrary lifts of the points x_i. Since every representation is semistable with respect to the trivial polarization $\chi = 0$, it follows that g_0 induces a map

$$g_0\colon S^n(X/\Gamma) \to \mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1,\, n\boldsymbol{v}^0),\, 0).$$

To construct the inverse we need the following lemma.

Lemma 35. *Any* $(1, \boldsymbol{v})$*-dimensional representation of* $(Q(\boldsymbol{w}^0), J(0))$ *is* S*-equivalent* (*with respect to* $\chi = 0$) *to a representation with* $j = 0$.

Proof. Let $\rho = (V_*, V_\bullet)$ be a $(1, \boldsymbol{v})$-dimensional representation of $(Q(\boldsymbol{w}^0), J(0))$. Let $B\colon V \otimes L \to V$, $i\colon W \to V$, and $j\colon V \to W$ be the corresponding Γ-equivariant morphisms. Let U be the minimal subspace of V such that $i(W) \subset U$ and $B(U \otimes L) \subset U$. Then U is invariant under the action of Γ, hence $U = \bigoplus U_l \otimes R_k$. Moreover, $(V_*, U_\bullet)$ is a subrepresentation of $(V_*, V_\bullet)$, thus

$$(V_*, V_\bullet) \overset{0}{\sim} (V_*, U_\bullet) \oplus (0, V_\bullet/U_\bullet),$$

where $\overset{0}{\sim}$ denotes S-equivalence with respect to $\chi = 0$. On the other hand, the arguments of [10], Lemma 2.8 show that $j = 0$ in the first summand in the RHS, and in the second summand $j = 0$ for trivial reasons. □

Now we need to construct a map inverse to g_0. Take an arbitrary element $\rho \in \mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1,\, n\boldsymbol{v}^0),\, 0)$. By Lemma 35 we can assume that $j = 0$ in ρ. Then $\operatorname{Tot}(\rho)$ is a Γ-equivariant $\mathbb{C}[X]$-module. Regarding it as a Γ-equivariant coherent sheaf on X we define

$$f_0(\rho) = \operatorname{supp}(\operatorname{Tot}(\rho)) \in (S^{nN}X)^\Gamma = S^n(X/\Gamma).$$

Note that the map f_0 is well defined. Indeed, assume that $\rho' \overset{0}{\sim} \rho$ and $j = 0$ in ρ'. Then ρ and ρ' admit filtrations with isomorphic associated factors (up to a permutation). These filtrations induce filtrations of sheaves $\operatorname{Tot}(\rho)$ and $\operatorname{Tot}(\rho')$ with isomorphic associated factors. But then $\operatorname{supp}(\operatorname{Tot}(\rho)) = \operatorname{supp}(\operatorname{Tot}(\rho'))$.

Now we have to check that the maps g_0 and f_0 are mutually inverse. More precisely, one has to check that

$$g_0(f_0(\rho)) \overset{0}{\sim} \rho \quad \text{and} \quad f_0(g_0(x_1, \dots, x_n)) = (x_1, \dots, x_n). \tag{36}$$

Before we begin the proof let us note that from the explicit construction of isomorphisms in Theorem 19 it follows that for any point $\tilde{x} \in X_\Gamma$ we have

$$\operatorname{supp}\mathrm{Tot}(\hat{\rho}(\tilde{x})) = \pi_0(\tilde{x}).$$

Now, take $\rho \in \mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1,\, n\boldsymbol{v}^0),\, 0)$. By Lemma 35 we can assume that $j = 0$ in ρ, hence $\rho \overset{0}{\sim} \rho_0 \oplus \hat{\rho}$. Now consider the sheaf $\mathrm{Tot}(\hat{\rho})$. It is a Γ-equivariant sheaf of finite length on X, hence it admits a filtration with associated factors being either the structure sheaves of a Γ-orbit corresponding to a point $0 \neq x_i \in X/\Gamma$ or length 1 skyscraper sheaves supported at zero. This filtration induces a filtration on $\hat{\rho}$ with associated factors being either $\hat{\rho}(\tilde{x}_i)$, with $\tilde{x}_i = \pi_0^{-1}(x_i)$, or ρ_k, where ρ_k is the unique representation with $\dim(\rho_k) = \boldsymbol{w}^k = (0,\, \dots,\, 0,\, 1,\, 0,\, \dots,\, 0)$ (1 stands on the k-th position). Thus,

$$\rho \overset{0}{\sim} \rho_0 \oplus \left(\bigoplus_{i=1}^{m} \hat{\rho}(\tilde{x}_i) \right) \oplus \left(\bigoplus_{k=0}^{d-1} (\rho_k) \right)^{\oplus (n-m)\boldsymbol{v}_k^0}. \tag{37}$$

Then

$$\begin{aligned} g_0(f_0(\rho)) = g_0(\operatorname{supp}\mathrm{Tot}(\rho)) = g_0(x_1,\, \dots,\, x_m,\, 0,\, \dots,\, 0) \\ = \rho_0 \oplus \left(\bigoplus_{i=1}^{m} \hat{\rho}(\tilde{x}_i) \right) \oplus \hat{\rho}(\tilde{x}_0)^{\oplus (n-m)\boldsymbol{v}_k^0}, \end{aligned}$$

where $\tilde{x}_0 \in \pi_0^{-1}(0)$. But decomposition (37) for the representation $\rho_0 \oplus \hat{\rho}(\tilde{x}_0)^{\oplus (n-m)\boldsymbol{v}_k^0}$ gives

$$\hat{\rho}(\tilde{x}_0)^{\oplus (n-m)\boldsymbol{v}_k^0} \overset{0}{\sim} \left(\bigoplus_{k=0}^{d-1} (\rho_k) \right)^{\oplus (n-m)\boldsymbol{v}_k^0},$$

hence $g_0(f_0(\rho)) \overset{0}{\sim} \rho$, the first equality of (36) is proved.

Now, take $x_1,\, \dots,\, x_n \in X/\Gamma$. Then

$$\begin{aligned} f_0(g_0(x_1,\, \dots,\, x_n)) = f_0(\rho_0 \oplus \hat{\rho}(x_1) \oplus \cdots \oplus \hat{\rho}(x_n)) \\ = \operatorname{supp}(\mathrm{Tot}(\hat{\rho}(x_1)) \oplus \cdots \oplus \mathrm{Tot}(\hat{\rho}(x_n))) = \{x_1,\, \dots,\, x_n\}. \end{aligned}$$

This completes the proof of the proposition. □

Γ-equivariant Hilbert scheme. The action of the group Γ on X induces an action of Γ on the Hilbert scheme $X^{[nN]}$ of nN-tuples of points on X. Let $(X^{[nN]})^\Gamma$ denote the Γ-invariant locus. Since the group Γ acts on the tautological bundle $\underline{V}$ on $X^{[nN]}$, the restriction of $\underline{V}$ to $(X^{[nN]})^\Gamma$ decomposes into the direct sum of locally free sheaves indexed by the irreducible representations of Γ:

$$\underline{V}_{|(X^{[nN]})^\Gamma} = \bigoplus_{k=0}^{d-1} \underline{V}_k \otimes R_k. \tag{38}$$

Let $X^{\Gamma[n]}$ denote the locus of $(X^{[nN]})^\Gamma$, where the rank of every $\underline{V}_k$ equals $n\boldsymbol{v}_k^0$ (or, in other words, where $\underline{V}$ is a multiple of the regular representation). It is clear that

$X^{\Gamma[n]}$ is a union of connected components of $(X^{[nN]})^{\Gamma}$. We will reffer to $X^{\Gamma[n]}$ as the Γ-equivariant Hilbert scheme of X.

Denote by $\mathbf{C}_+$ the positive octant of $\mathbb{R}^d$, that is

$$\mathbf{C}_+ = \{\zeta_{\mathbb{R}} \in \mathbb{R}^d \mid \zeta_{\mathbb{R}}^k > 0 \text{ for all } 0 \leqslant k \leqslant d-1\}.$$

Note that any ζ with $\zeta_{\mathbb{R}} \in \mathbf{C}_+$ is generic (because its real component $\zeta_{\mathbb{R}}$ is strictly positive) and any $\zeta_{\mathbb{R}} \in \mathbf{C}_+$ satisfies the restriction (26).

The following theorem is well known (see [13], [12]).

Theorem 39. *For any integer $n \geqslant 0$ and any parameter $\zeta_{\mathbb{R}} \in \mathbf{C}_+$ the quiver variety $\mathfrak{M}_{(0,\zeta_R)}(n \cdot \boldsymbol{v}^0, \boldsymbol{w}^0)$ is isomorphic to the Γ-equivariant Hilbert scheme $X^{\Gamma[n]}$.*

We begin with the following obvious lemma.

Lemma 40. *Let U be a Γ-equivariant finite dimensional $\mathbb{C}[X]$-module with a Γ-invariant vector $u \in U$ and let ρ be the corresponding $(1, \boldsymbol{u})$-dimensional representation of the quiver $(Q(\boldsymbol{w}^0), J(0))$. Then ρ is $\chi_{\boldsymbol{u}}(\zeta_{\mathbb{R}})$-stable with $\zeta_{\mathbb{R}} \in \mathbf{C}_+$ if and only if U has no proper Γ-equivariant $\mathbb{C}[X]$-submodule containing u.*

Proof. Assume that $u \in U' \subset U$ is a Γ-equivariant proper $\mathbb{C}[X]$-submodule. Let ρ' be the corresponding $(1, \boldsymbol{u}')$-dimensional representation of the quiver. Then ρ' is a subrepresentation of ρ and

$$\chi_{\boldsymbol{u}}(\zeta_{\mathbb{R}})(\rho') = (-\zeta_{\mathbb{R}}(\boldsymbol{u}), \zeta_{\mathbb{R}})(1, \boldsymbol{u}') = \zeta_R(\boldsymbol{u}') - \zeta_{\mathbb{R}}(\boldsymbol{u}) = -\zeta_{\mathbb{R}}(\boldsymbol{u} - \boldsymbol{u}') < 0$$

since $\boldsymbol{u} > \boldsymbol{u}'$ and $\zeta_{\mathbb{R}}$ is positive. Hence ρ is unstable.

Similarly, assume that ρ is unstable. Then it contains a subrepresentation ρ' such that

$$\chi_{\boldsymbol{u}}(\zeta_{\mathbb{R}})(\dim \rho') = (-\zeta_{\mathbb{R}}(\boldsymbol{u}), \zeta_{\mathbb{R}})(\boldsymbol{u}'_*, \boldsymbol{u}') = \zeta_R(\boldsymbol{u}') - \boldsymbol{u}'_*\zeta_{\mathbb{R}}(\boldsymbol{u}) < 0,$$

where $(\boldsymbol{u}'_*, \boldsymbol{u}') = \dim \rho'$. Since $\zeta_{\mathbb{R}}$ is positive the case $\boldsymbol{u}'_* = 0$ is impossible. Hence, $\boldsymbol{u}'_* = 1$. This means that the subspace $\mathrm{Tot}(\rho') \subset \mathrm{Tot}(\rho) = U$ contains the vector u. On the other hand, it is a proper Γ-equivariant $\mathbb{C}[X]$-submodule in U. □

Now we can prove the theorem.

Proof. Recall that $\underline{V}$ carries the structure of a family of quotient algebras of the algebra of functions $\mathbb{C}[X]$, and in particular, of a Γ-equivariant $\mathbb{C}[X]$-module. In particular, the unit of the algebra $\mathbb{C}[X]$ induces a Γ-equivariant morphism $i\colon \mathcal{O} \to \underline{V}$. Applying the relative analogue of Corollary 16 we see that for $\underline{V}_* = \mathcal{O}$ we obtain on $(\underline{V}_*, \underline{V}_k)$ a natural structure of a family of $(1, n\boldsymbol{v}^0)$-dimensional representations of the quiver $(Q(\boldsymbol{w}^0), J(0))$.

Since $\underline{V}$ is a family of quotient algebras of $\mathbb{C}[X]$ it follows that it is pointwise generated by the image of $1 \in \mathbb{C}[X]$ as a family of $\mathbb{C}[X]$-modules. In other words, it has no proper Γ-invariant $\mathbb{C}[X]$-submodules containig the image of 1. Hence Lemma 40 implies the $\chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})$-stability of the family $(\underline{V}_*, \underline{V}_k)$. Thus, we obtain a map

$$f_+\colon X^{\Gamma[n]} \to \mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1, n\boldsymbol{v}^0), \chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})) = \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0, \boldsymbol{w}^0).$$

Similarly, let $\underline{\rho} = (\underline{V}_*, \underline{V}_k)$ be a universal family on the moduli space

$$\mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1, n\boldsymbol{v}^0), \chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})).$$

Twisting it by the line bundle $(\underline{V}_*)^{-1}$ we can assume that $\underline{V}_* = \mathcal{O}$. Then Corollary 16 implies that $\operatorname{Tot}(\underline{\rho})$ is a family of Γ-equivariant $\mathbb{C}[X]$-modules endowed with a morphism $i\colon \mathcal{O} \to \operatorname{Tot}(\underline{\rho})$. Furthermore, $\chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})$-stability of the family $\underline{\rho}$ combined with Lemma 40 implies that $\operatorname{Tot}(\underline{\rho})$ is in fact a family of quotient algebras of $\mathbb{C}[X]$, where $1 \in \mathbb{C}[X]$ corresponds to $i(1) \in \operatorname{Tot}(\underline{\rho})$. The rank of this family is

$$\mathrm{r}(\operatorname{Tot}(\underline{\rho})) = \sum_{k=0}^{d-1} \mathrm{r}(\underline{V}_k) \dim R_k = \sum_{k=0}^{d-1} (n\boldsymbol{v}_k^0)\boldsymbol{v}_k^0 = n \sum_{k=0}^{d-1} (\boldsymbol{v}_k^0)^2 = nN;$$

hence we obtain a map

$$g_+\colon \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0) = \mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1,\, n\boldsymbol{v}^0),\, \chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})) \to X^{[nN]}.$$

Γ-equivariance of $\underline{V}$ implies that g_+ goes in fact to the Γ-invariant part of $X^{[nN]}$ and moreover to $X^{\Gamma[n]}$.

It remains to notice that the maps f_+ and g_+ are evidently mutually inverse. □

Remark 41. It is easy to see that the constructed isomorphism of varieties $\mathfrak{M}_{(0,\zeta_R)}(n\cdot \boldsymbol{v}^0,\, \boldsymbol{w}^0)$ and $X^{\Gamma[n]}$ is compatible with the bijection of $\mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0)$ and $S^n(X/\Gamma)$, constructed in Proposition 33, that is that the diagram

$$\begin{array}{ccc}
\mathfrak{M}_{(0,\zeta_R)}(n\cdot\boldsymbol{v}^0,\, \boldsymbol{w}^0) & \underset{f_+}{\overset{g_+}{\rightleftarrows}} & X^{\Gamma[n]} \\
\downarrow & & \downarrow \\
\mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0) & \underset{f_0}{\overset{g_0}{\rightleftarrows}} & S^n(X/\Gamma)
\end{array}$$

is commutative, where the vertical arrows stand for the natural projections. Since these projections are birational, it follows that the bijections f_0 and g_0 are birational isomorphisms.

The Hilbert scheme of X_Γ. Let $K_n,\, \bar{K}_n \subset \mathbb{Z}^d$ denote the following finite sets of dimension vectors:

$$K_n = \{\boldsymbol{v} \in \mathbb{Z}^d_{\geqslant 0} \mid \boldsymbol{v}_0\boldsymbol{v}^0 < \boldsymbol{v} < n\boldsymbol{v}^0\},$$
$$\bar{K}_n = \{\boldsymbol{v} \in \mathbb{Z}^d_{\geqslant 0} \mid \boldsymbol{v}_0\boldsymbol{v}^0 \leqslant \boldsymbol{v} \leqslant n\boldsymbol{v}^0\}.$$

Then

$$\mathbf{C}_-(n) = \left\{ \zeta_{\mathbb{R}} \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \zeta_{\mathbb{R}}(\boldsymbol{v}^0) > 0, \\ \zeta_{\mathbb{R}}(\boldsymbol{v}) > n\zeta_{\mathbb{R}}(\boldsymbol{v}^0), \quad \text{for all } \boldsymbol{v} \in K_n \end{array} \right\}$$

is a convex polyhedral cone.

Lemma 42. *We have*

$$\mathbf{C}_-(n) = \left\{ \zeta_{\mathbb{R}} \in \mathbb{R}^d \;\middle|\; \frac{1}{n} \min_{k=1}^{d-1} \zeta_{\mathbb{R}}^k - \sum_{k=1}^{d-1} \zeta_{\mathbb{R}}^k \boldsymbol{v}_k^0 > \zeta_{\mathbb{R}}^0 > -\sum_{k=1}^{d-1} \zeta_{\mathbb{R}}^k \boldsymbol{v}_k^0 \right\}.$$

In particular, the cone $\mathbf{C}_-(n)$ is nonempty and for any $\zeta_{\mathbb{R}} \in \mathbf{C}_-(n)$ we have $\zeta_{\mathbb{R}}^k > 0$ for all $1 \leqslant k \leqslant d-1$.

Proof. Easy. □

Theorem 43. *If $n \geqslant 0$ and $\zeta_{\mathbb{R}} \in \mathbf{C}_-(n)$, then the quiver variety $\mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$ is isomorphic to the Hilbert scheme $X_\Gamma^{[n]}$.*

Proof. Let $\underline{\rho} = \underline{\rho}^{\zeta_{\mathbb{R}}} = (\underline{V}_*, \underline{V}_k, \underline{B}_h, \underline{i}_0, \underline{j}_0)$ be the universal family on X_Γ normalized as in Lemma 27. Let $Z \subset X_\Gamma$ be a length n subscheme in X_Γ and let $\mathcal{O}_Z$ be its structure sheaf. We associate to Z the following representation $\rho(Z) = (V_*(Z), V_k(Z), B(Z), i_0(Z), j_0(Z))$ of the quiver $Q(\boldsymbol{w}^0)$. Put

$$V_k(Z) = \Gamma(X_\Gamma, \underline{V}_k \otimes \mathcal{O}_Z), \quad k = 0, \dots, d-1,$$
$$V_*(Z) = \Gamma(X_\Gamma, \mathcal{O}_{X_\Gamma}) \cong \mathbb{C}$$

with morphisms $B_h(Z)\colon V_{\mathrm{out}(h)}(Z) \to V_{\mathrm{in}(h)}(Z)$ induced by the maps $\underline{B}_h \otimes \mathrm{id}_{\mathcal{O}_Z}\colon \underline{V}_{\mathrm{out}(h)} \otimes \mathcal{O}_Z \to \underline{V}_{\mathrm{in}(h)} \otimes \mathcal{O}_Z$, while $j_0(Z) = 0$ and $i_0(Z)$ is induced by the composition

$$\mathcal{O}_{X_\Gamma} \cong \underline{V}_0 \to \underline{V}_0 \otimes \mathcal{O}_Z,$$

where the first is the isomorphism of Lemma 27 and the second morphism is obtained from the canonical projection $\mathcal{O}_{X_\Gamma} \to \mathcal{O}_Z$ by tensoring with $\underline{V}_0$.

Since Z is a length n subscheme of X_Γ and $\mathrm{r}(\underline{V}_k) = \boldsymbol{v}_k^0$, it follows that $V_k(Z)$ is an $n\boldsymbol{v}_k^0$-dimensional vector space for any $0 \leqslant k \leqslant d-1$. On the other hand, $V_*(Z)$ is 1-dimensional by definition. Hence $\rho(Z)$ is a $(1, n\boldsymbol{v}^0)$-dimensional reresentation of the quiver $Q(\boldsymbol{w}^0)$. Further, it follows from (23) and Lemma 25 that $\rho(Z)$ satisfies the relations $J(0)$. Now we want to check that $\rho(Z)$ is $\chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})$-stable for any $\zeta_{\mathbb{R}} \in \mathbf{C}_-(n)$. We begin with the following proposition.

Proposition 44. *If $\rho' \subset \rho(Z)$ is a subrepresentation and $\rho' \neq \rho(Z)$ then either*

$$\dim \rho' = (0, \boldsymbol{v}) \text{ with } \boldsymbol{v} \in \bar{K}_n, \text{ or } \dim \rho' = (1, \boldsymbol{v}) \text{ with } \boldsymbol{v} \in K_n.$$

Proof. Choose a chain of subschemes

$$Z = Z_n \supset Z_{n-1} \supset \dots \supset Z_1 \supset Z_0 = \varnothing$$

such that Z_i is a length i subscheme. This chain of subschemes induces a chain of surjections of structure sheaves

$$\mathcal{O}_Z = \mathcal{O}_{Z_n} \to \mathcal{O}_{Z_{n-1}} \to \dots \to \mathcal{O}_{Z_1} \to \mathcal{O}_{Z_0} = 0,$$

and of representations of quivers

$$\rho(Z) = \rho(Z_n) \to \rho(Z_{n-1}) \to \dots \to \rho(Z_1) \to \rho(Z_0) = \rho_0. \tag{45}$$

Note that for any $i = 1, \dots, n$ we have an exact sequence of sheaves

$$0 \to \mathcal{O}_{x_i} \to \mathcal{O}_{Z_i} \to \mathcal{O}_{Z_{i-1}} \to 0$$

for some $x_i \in X_\Gamma$ which induces exact sequence of representations

$$0 \to \hat{\rho}(x_i) \to \rho(Z_i) \to \rho(Z_{i-1}) \to 0.$$

Assume now that ρ' is a subrepresentation of $\rho(Z)$. Let ρ'_i denote the image of ρ' in $\rho(Z_i)$ with respect to the surjection (45). Then (45) induces a chain of surjections

$$\rho' = \rho'_n \to \rho'_{n-1} \to \dots \to \rho'_1 \to \rho'_0 = \rho_0.$$

Let ρ''_i denote the kernel of the map $\rho'_i \to \rho'_{i-1}$. Then ρ''_i is a subrepresentation of $\hat{\rho}(x_i)$, hence $\dim \rho''_i = (0,\, \boldsymbol{u}^i)$ for some dimension vector $\boldsymbol{u}^i \leqslant \boldsymbol{v}^0$. Since $\boldsymbol{v}^0_0 = 1$, it follows that either $\boldsymbol{u}^i_0 = 0$ or $\boldsymbol{u}^i_0 = 1$.

Assume that $\boldsymbol{u}^i_0 = 1$. Then it is easy to see that we can extend the subrepresentation $\rho''_i \subset \hat{\rho}(x_i)$ to a $(1,\, \boldsymbol{u}^i)$-dimensional subrepresentation $\widetilde{\rho}''_i \subset \rho(x_i)$. Note now that if $\boldsymbol{u}^i < \boldsymbol{v}^0$ then since $\boldsymbol{u}^i_0 = \boldsymbol{v}^0_0$ and that, since $\zeta^k_{\mathbb{R}} > 0$ for all $1 \leqslant k \leqslant d-1$ by Lemma 42, it follows that

$$\chi_{\boldsymbol{v}^0}(\zeta_{\mathbb{R}})(1,\, \boldsymbol{u}^i) = (-\zeta_{\mathbb{R}}(\boldsymbol{v}^0),\, \zeta_{\mathbb{R}})(1,\, \boldsymbol{u}^i) < (-\zeta_{\mathbb{R}}(\boldsymbol{v}^0),\, \zeta_{\mathbb{R}})(1,\, \boldsymbol{v}^0) = 0,$$

which contradicts the $\chi_{\boldsymbol{v}^0}(\zeta_{\mathbb{R}})$-stability of $\rho(x_i)$. Thus we proved that if $\boldsymbol{u}^i_0 = 1$ then $\boldsymbol{u}^i = \boldsymbol{v}^0$.

On the other hand, if $\boldsymbol{u}^i_0 = 0$ then certainly $\boldsymbol{u}^i \geqslant 0$. Thus, in both cases we have

$$\boldsymbol{u}^i \geqslant \boldsymbol{u}^i_0 \boldsymbol{v}^0.$$

Summing up these inequalities from $i = 1$ to n we see that

$$\boldsymbol{v} = \sum_{i=1}^{n} \boldsymbol{u}^i \geqslant \sum_{i=1}^{n} \boldsymbol{u}^i_0 \boldsymbol{v}^0 = \boldsymbol{v}_0 \boldsymbol{v}^0,$$

hence $\boldsymbol{v} \in \bar{K}_n$.

It remains to check that if $\dim \rho' = (1,\, \boldsymbol{v})$ then $\boldsymbol{v} \in K_n$. The above argument shows that $\boldsymbol{v} \in \bar{K}_n$. So, if $\boldsymbol{v} \notin K_n$ then $\boldsymbol{v} = m\boldsymbol{v}_0$ for some $0 \leqslant m \leqslant n$. Let us check that the case $m < n$ is impossible.

For this we will use induction on n. The base of induction, $n = 0$, is trivial. Assume that $n > 0$, $Z \subset X_\Gamma$ is a length n subscheme, and that $\rho' \subset \rho(Z)$ is a $(1,\, m\boldsymbol{v}^0)$-dimensional subrepresentation with $m < n$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \rho''_n & \longrightarrow & \rho' & \longrightarrow & \rho'_{n-1} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \hat{\rho}(x_n) & \longrightarrow & \rho(Z) & \longrightarrow & \rho(Z_{n-1}) & \longrightarrow & 0, \end{array}$$

where ρ''_n and ρ'_{n-1} were defined above. Denote

$$\dim \rho''_n = (0,\, \boldsymbol{u}), \qquad \dim \rho'_{n-1} = (1,\, \boldsymbol{v}').$$

Then the above argument shows that

$$\boldsymbol{u} \geqslant \boldsymbol{u}_0 \boldsymbol{v}^0, \qquad \boldsymbol{v}' \geqslant \boldsymbol{v}'_0 \boldsymbol{v}^0.$$

However, from $\boldsymbol{u} + \boldsymbol{v}' = m\boldsymbol{v}^0 = (\boldsymbol{u}_0 + \boldsymbol{v}'_0)\boldsymbol{v}^0$ it follows that

$$\boldsymbol{u} = m''\boldsymbol{v}^0, \qquad \boldsymbol{v}' = m'\boldsymbol{v}^0, \qquad m' + m'' = m.$$

In particular, $\rho(Z_{n-1})$ contains a $(1,\, m'\boldsymbol{v}^0)$-dimensional subrepresentation. The induction hypothesis for Z_{n-1} implies then that $m' = n-1$. On the other hand, we have $m'' = 1$ or $m'' = 0$. In the first case we have $m = n$, a contradiction. In the second case, it follows that $\rho' = \rho(Z_{n-1})$, hence the exact sequence

$$0 \to \hat{\rho}(x_n) \to \rho(Z) \to \rho(Z_{n-1}) \to 0$$

splits. Comparing the definition of $\rho(Z)$ with the definition of the equivalence $\Psi_{\zeta_{\mathbb{R}}}$, we deduce from Theorem 30 that the splitting of the above sequence implies that the sequence

$$0 \to \mathcal{O}_{x_n} \to \mathcal{O}_Z \to \mathcal{O}_{Z_{n-1}} \to 0,$$

also splits, and that this splitting is compatible with the projections $\mathcal{O}_{X_\Gamma} \to \mathcal{O}_Z$ and $\mathcal{O}_{X_\Gamma} \to \mathcal{O}_{Z_{n-1}}$. But this means that

$$J_{Z_{n-1}} \cong J_Z \oplus \mathcal{O}_{x_n},$$

which is false. Thus we again come to a contradiction, and the Lemma is proved. □

Now we can continue the proof of the Theorem. The $\chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})$-stability of the representation $\rho(Z)$ follows immediately from the definition of the cone $\mathbf{C}_-(n)$ and from Proposition 44. Thus, $\rho(Z)$ forms a family of $(1,\, n\boldsymbol{v}^0)$-dimensional $\chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})$-stable representations of the quiver $(Q(\boldsymbol{w}^0),\, J(0))$ over the Hilbert scheme $X_\Gamma^{[n]}$. This family induces a regular map

$$g_-\colon X_\Gamma^{[n]} \to \mathcal{M}od_{Q(\boldsymbol{w}^0),J(0)}((1,\, n\boldsymbol{v}^0),\, \chi_{n\boldsymbol{v}^0}(\zeta_{\mathbb{R}})) = \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0).$$

Moreover, it is easy to see that the map fits into the commutative diagram

$$\begin{array}{ccc} X_\Gamma^{[n]} & \xrightarrow{g_-} & \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0) \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ S^n(X/\Gamma) & \xrightarrow{g_0} & \mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0), \end{array}$$

where the left vertical arrow is the composition of the Hilbert–Chow morphism $X_\Gamma^{[n]} \to S^n(X_\Gamma)$ with the map induced by the projection $\pi_0\colon X_\Gamma \to X/\Gamma$. Since both vertical maps are birational, and g_0 is an isomorphism by Proposition 33, it follows that g_- is a regular birational map. To check that this map is a biregular isomorphism we are going to check that both varieties $X_\Gamma^{[n]}$ and $\mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0)$ are holomorphically symplectic varieties and that the map g_- is compatible with the symplectic forms.

To do this we extend the above commutative diagram to

$$\begin{array}{ccc} X_\Gamma^{[n]} & \xrightarrow{g_-} & \mathfrak{M}_{(0,\zeta_{\mathbb{R}})}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0) \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ S^n(X/\Gamma) & \xrightarrow{g_0} & \mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0) \\ \uparrow & & \uparrow \\ (X_\Gamma)^n & = & (\mathfrak{M}_{(0,0)}(\boldsymbol{v}^0,\, \boldsymbol{w}^0))^n, \end{array}$$

where the bottom isomorphism is the n-fold product of the isomorphism of Theorem 19, the arrow $(X_\Gamma)^n \to S^n(X/\Gamma)$ is the composition $(X_\Gamma)^n \to S^n(X_\Gamma) \to S^n(X/\Gamma)$, and the arrow $(\mathfrak{M}_{(0,0)}(\boldsymbol{v}^0,\, \boldsymbol{w}^0))^n \to \mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0)$ is defined as follows. Given an n-tuple $(B^\alpha,\, i^\alpha,\, j^\alpha)_{\alpha=1}^n$ of points of $\mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0,\, \boldsymbol{w}^0)$ we map it to the point $(\bigoplus B^\alpha,\, \sum i^\alpha,\, \sum j^\alpha)$. Since $j_0^\alpha i_0^\alpha = 0$ for all α by Lemma 25 and $\boldsymbol{v}_0^0 = 1$, we have $i_0^\alpha j_0^\alpha = 0$, hence $[B^\alpha, B^\alpha] = 0$, and it follows that $(\bigoplus B^\alpha, \sum i^\alpha, \sum j^\alpha)$ is indeed a point of the quiver variety $\mathfrak{M}_{(0,0)}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$.

Consider now the canonical holomorphic symplectic form on $\mathfrak{M}_{(0,0)}(\boldsymbol{v}^0, \boldsymbol{w}^0)$ and use the isomorphism of Theorem 19 to induce a symplectic form on X_Γ. Furthermore, we provide $(X_\Gamma)^n$ and $(\mathfrak{M}_{(0,0)}(\boldsymbol{v}^0, \boldsymbol{w}^0))^n$ with the n-fold products of these symplectic forms, so that the bottom arrow of the diagram becomes a symplectic isomorphism. Then we use a classical result of Beauville [1] that the symplectic form on X_Γ induces a symplectic form on the Hilbert scheme $X_\Gamma^{[n]}$, and note that the maps in the left column of the diagram are compatible with the symplectic forms on certain open subsets of $X_\Gamma^{[n]}$ and $(X_\Gamma)^n$. On the other hand, it is easy to see that the maps in the right column of the diagram are also compatible with the symplectic forms on certain open subsets of $\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$ and $(\mathfrak{M}_{(0,0)}(\boldsymbol{v}^0, \boldsymbol{w}^0))^n$. We conclude that the map g_- is compatible with the symplectic forms on open subsets of $X_\Gamma^{[n]}$ and $\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0)$. Since this map is regular, it follows that it is compatible with the symplectic forms everywhere. □

Remark 46. H. Nakaijma indicated to me an outline of his argument. He constructs a map $\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0) \to X_\Gamma^{[n]}$ (note that the direction is opposite to that of the map g_-) using a certain complex and utilizing the stability condition to ensure that certain cohomology groups vanish.

Combining Theorems 39 and 43 with Theorem 1 we obtain the following corollary.

Corollary 47. *The Γ-equivariant Hilbert scheme $X^{\Gamma[n]}$ is diffeomorphic to the Hilbert scheme $X_\Gamma^{[n]}$. In particular, we have an isomorphism of cohomology groups*

$$H^\bullet(X^{\Gamma[n]}, \mathbb{Z}) \cong H^\bullet(X_\Gamma^{[n]}, \mathbb{Z}).$$

Remark 48. It is also easy to deduce that for any

$$\zeta_\mathbb{R} \in \{\zeta_\mathbb{R} \mid \zeta_\mathbb{R}(\boldsymbol{v}^0) = 0, \text{ and } \zeta_\mathbb{R}^k > 0 \text{ for all } 1 \leqslant k \leqslant d-1\}$$

we have

$$\mathfrak{M}_{(0,\zeta_\mathbb{R})}(n\boldsymbol{v}^0, \boldsymbol{w}^0) \cong S^n(X_\Gamma).$$

Generalizations: the Calogero–Moser space. Let Γ be a finite subgroup in $\mathrm{SL}(2, \mathbb{C})$ and $\Gamma' \subset \Gamma$ its central subgroup. Let $\mathfrak{M}^\Gamma$ and $\mathfrak{M}^{\Gamma'}$ denote the quiver varieties corresponding to the affine Dynkin graphs of Γ and Γ' respectively.

Since Γ' is central in Γ, the embedding $\Gamma' \to \Gamma$ induces embeddings

$$\sigma_\mathbb{C}^*\colon Z(\mathbb{C}[\Gamma']) \to Z(\mathbb{C}[\Gamma]), \quad \sigma_\mathbb{R}^*\colon Z(\mathbb{R}[\Gamma']) \to Z(\mathbb{R}[\Gamma]).$$

Remark 49. The embedding σ^* can be described in terms of the Dynkin graphs as follows. Let I and I' be the sets (of isomorphism classes) of irreducible representations of Γ and Γ' respectively. In other words, I and I' are the sets of vertices of the corresponding affine Dynkin graphs. Consider an irreducible Γ-module R_i^Γ, $i \in I$. Since Γ' is central it follows from the Schur Lemma that the restriction $(R_i^\Gamma)_{|\Gamma'}$ is a multiple of an irreducible representation of the group Γ', say $R_{i'}^{\Gamma'}$, $i' \in I'$. So, associating this way to arbitrary vertex $i \in I$ the vertex $i' \in I'$ we obtain a map $\sigma\colon I \to I'$. The map σ^* is induced by σ.

Let $N = |\Gamma/\Gamma'|$ denote the index of Γ' in Γ. Choose an arbitrary generic $\zeta \in Z(\mathbb{C}[\Gamma']) \oplus Z(\mathbb{R}[\Gamma'])$. Let also W be an arbitrary representation of Γ, let W' be the restriction of W to Γ', and let V' be an arbitrary representation of Γ'. Let $\boldsymbol{v}'$, $\boldsymbol{w}'$ and $\boldsymbol{w}$ denote the classes of V', W' and W in the Grothendieck rings of Γ' and Γ respectively (i. e. their dimension vectors).

Recall that by Lemma 14 the quiver variety $\mathfrak{M} = \mathfrak{M}_\zeta^{\Gamma'}(\boldsymbol{v}', \boldsymbol{w}')$ coincides with the set of all triples

$$(B, i, j) \in \mathrm{Hom}_{\Gamma'}(V' \otimes L, V') \oplus \mathrm{Hom}_{\Gamma'}(W', V') \oplus \mathrm{Hom}_{\Gamma'}(V', W')$$

such that

$$[B, B] + ij = -\zeta_{\mathbb{C}}, \qquad [B, B^\dagger] + ii^\dagger - j^\dagger j = -\zeta_{\mathbb{R}}$$

modulo action of $U_{\Gamma'}(V')$. We define for every $\gamma \in \Gamma$ another triple

$$B^\gamma = B \cdot (1 \otimes \gamma) \colon V \otimes L \xrightarrow{1\otimes\gamma} V \otimes L \xrightarrow{B} V, \quad i^\gamma = i \cdot \gamma, \quad j^\gamma = \gamma^{-1} \cdot j.$$

Since Γ commutes with Γ', it follows that $(B^\gamma, i^\gamma, j^\gamma)$ gives another point of $\mathfrak{M}$, hence the correspondence $B \mapsto B^\gamma$ defines an action of the group Γ on the quiver variety $\mathfrak{M}$. Let $\mathfrak{M}^\Gamma$ denote the set of fixed points of Γ on $\mathfrak{M}$.

Take an arbitrary $(B, i, j) \in \mathfrak{M}^\Gamma$. Then (B, i, j) and $(B^\gamma, i^\gamma, j^\gamma)$ should be conjugate under the action of $U_{\Gamma'}(V')$. Hence there exists a $g_\gamma \in U_{\Gamma'}(V')$ such that

$$g_\gamma B g_\gamma^{-1} = B^\gamma, \quad g_\gamma i = i^\gamma, \quad j g_\gamma^{-1} = j^\gamma. \tag{50}$$

Moreover, when ζ is generic such g_γ is unique (because the action of $U_{\Gamma'}(V')$ is free in this case). It follows that $\gamma \mapsto g_\gamma$ defines an action of Γ on V' extending the action of Γ'.

Now let V be an arbitrary representation of Γ such that its restriction to Γ' is isomorphic to V'. Let $\boldsymbol{v}$ be its class in the Grothendieck ring of Γ. Let $\mathfrak{M}_{\boldsymbol{v}}^\Gamma$ denote the locus of the set $\mathfrak{M}^\Gamma$, where the defined above structure of a representation of Γ on V' is isomorphic to V. Then we have.

Theorem 51. *For any generic $\zeta \in Z(\mathbb{C}[\Gamma']) \oplus Z(\mathbb{R}[\Gamma'])$, $\boldsymbol{v}, \boldsymbol{w} \in K_0(\Gamma)$ let $\boldsymbol{v}' = \boldsymbol{v}_{|\Gamma'}$, $\boldsymbol{w}' = \boldsymbol{w}_{|\Gamma'}$. Then we have*

$$\left(\mathfrak{M}_\zeta^{\Gamma'}(\boldsymbol{v}', \boldsymbol{w}')\right)_{\boldsymbol{v}}^\Gamma = \mathfrak{M}_{\sigma^*\zeta}^\Gamma(\boldsymbol{v}, \boldsymbol{w}).$$

Proof. Note that the equations (50) mean that the triple (B, i, j) is Γ-equivariant with respect to the action $\gamma \to g_\gamma$ of the group Γ on V and its canonical actions on W and L. Choose an arbitrary Γ-equivariant isomorphism $V' \to V$. Then

$$(B, i, j) \in \mathrm{Hom}_\Gamma(V \otimes L, V) \oplus \mathrm{Hom}_\Gamma(W, V) \oplus \mathrm{Hom}_\Gamma(V, W).$$

Furthermore, it follows from the definition of σ^* that

$$[B, B] + ij = -\sigma^*_{\mathbb{C}}\zeta_{\mathbb{C}}, \quad [B, B^\dagger] + ii^\dagger - j^\dagger j = -\sigma^*_{\mathbb{R}}\zeta_{\mathbb{R}}.$$

Thus we obtain a map

$$\left(\mathfrak{M}_\zeta^{\Gamma'}(\boldsymbol{v}', \boldsymbol{w}')\right)_{\boldsymbol{v}}^\Gamma \to \mathfrak{M}_{\sigma^*\zeta}^\Gamma(\boldsymbol{v}, \boldsymbol{w}).$$

Similarly, forgetting the Γ-structure on V and W for any $(B, i, j) \in \mathfrak{M}_{\sigma^*\zeta}^\Gamma(\boldsymbol{v}, \boldsymbol{w})$ we can consider it as a point of $\mathfrak{M}_\zeta^{\Gamma'}(\boldsymbol{v}', \boldsymbol{w}')$. Thus we obtain the inverse map. ☐

Consider the case $\Gamma' = \{1\}$, $W = R_0$, $W' = \mathbb{C}$, $V' = \mathbb{C}^{nN}$. Then the quiver variety $\mathfrak{M}_\zeta^{\Gamma'}(\boldsymbol{v}', \boldsymbol{w}')$ coincides with the Hilbert scheme $X^{[nN]}$ when $\zeta_\mathbb{C} = 0$ and with the so-called Calogero–Moser space $\mathcal{CM}_{nN}$ when $\zeta_\mathbb{C} \neq 0$. Thus in the case $\zeta_\mathbb{C} = 0$, Theorem 51 specializes to Theorem 39, and in the case $\zeta_\mathbb{C} = \tau \neq 0$ we obtain the following.

Corollary 52. *For any $\tau \neq 0$ and $n \geqslant 0$ we have*

$$(\mathcal{CM}_{nN})^\Gamma_{n\boldsymbol{v}^0} = \mathfrak{M}^\Gamma_\zeta(n\boldsymbol{v}^0, \boldsymbol{w}^0),$$

where $\zeta_\mathbb{C} = (\tau, \ldots, \tau)$ and $\zeta_\mathbb{R}$ is arbitrary.

5. Combinatorial Applications

From now on assume that $\Gamma \cong \mathbb{Z}/d\mathbb{Z}$ (the $\widetilde{A}_{d-1}$-case). Then both X and X_Γ admit a Γ-equivariant action of the torus $\mathbb{C}^* \times \mathbb{C}^*$. The first action is the coordinatewise dilation, and the second is induced by the first one. These actions induce actions on the Hilbert schemes $X^{\Gamma[n]}$ and $X_\Gamma^{[n]}$. In both cases, there is only a finite number of fixed points. Now we give their combinatorial description.

Consider a Young diagram as a domain in the top right-hand octant of the coordinate plane. We associate to a box of a diagram the coordinates of its bottom left-hand corner. Thus the coordinates of any box are nonnegative integers. We denote by (p, q) the box with the coordinates (p, q). A Young diagram Δ is called *uniformly colored in d colors* if the integer

$$n_i(\Delta) = \#\{(p, q) \in \Delta \mid p - q \equiv i \pmod d\}, \quad 0 \leqslant i \leqslant d-1$$

(the number of boxes of the color i) doesn't depend on i. One can say that we color the diagonals of the digram Δ periodically in d colors and call the diagram uniformly colored if the number of boxes of each color is the same. Let $UCY(n, d)$ denote the number of uniformly colored in d colors Young diagrams with n boxes of each color.

Lemma 53. *The $\mathbb{C}^* \times \mathbb{C}^*$-fixed points on the Γ-equivariant Hilbert scheme $X^{\Gamma[n]}$ are in a $1-1$ correspondence with Young diagrams that are uniformely colored in d colors with n boxes of each color.*

Proof. Note that by definition $X^{\Gamma[n]}$ is a connected component of $(X^{[dn]})^\Gamma$. The fixed points on $X^{[dn]}$ are numbered by Young diagrams with dn boxes. All these points are Γ-invariant (since the action of Γ factors through the torus action), so it remains to understand which of these points lie in the component.

Note that the fiber of the tautological bundle over $X^{[dn]}$ at the $\mathbb{C}^* \times \mathbb{C}^*$-fixed point corresponding to a Young diagram Δ is isomorphic as a Γ-module to the representation $\bigoplus_{(p,q)\in\Delta} R_{p-q}$. In particular, it is a multiple of the regular representation if and only if Δ is uniformly colored in d colors. □

Let $CY(n, d)$ be the number of ordered collections $(\Delta_1, \ldots, \Delta_d)$ of Young diagrams such that

$$\sum_{k=1}^{d} |\Delta_k| = n,$$

where $|\Delta|$ is the number of boxes in Δ.

Lemma 54. *The $\mathbb{C}^* \times \mathbb{C}^*$-fixed points on the Hilbert scheme $X_\Gamma^{[n]}$ are in a $1-1$ correspondence with ordered collections $(\Delta_1, \ldots, \Delta_d)$ of Young diagrams such that $\sum_{k=1}^{d} |\Delta_k| = n$.*

Proof. First note that the number of $\mathbb{C}^* \times \mathbb{C}^*$-fixed points on X_Γ equals d. It follows, for example, from the evident fact that $X_\Gamma = X^{\Gamma,1}$ and $UCY(1, d) = d$. Furthermore, denoting these points by $x_1, \ldots, x_d$ it is easy to see that Z is a fixed point on $X_\Gamma^{[n]}$ if and only if it splits into a union $Z = Z_1 \cup \cdots \cup Z_d$, where Z_i is a $\mathbb{C}^* \times \mathbb{C}^*$-invariant length n_i-subscheme in X_Γ with support at x_i. Linearizing the action of the torus in a neighbourhood of $x_i \in X_\Gamma$ we see that such Z_i are numbered by Young diagrams Δ_i with $|\Delta_i| = n_i$. Finally, the condition $Z \in X_\Gamma^{[n]}$ is just $\sum_{i=1}^{d} n_i = n$, whence the lemma. □

Theorem 55. *For any n, $d > 0$ we have $UCY(n, d) = CY(n, d)$.*

Remark 56. See [4] for the combinatorial proof of this identity.

Proof. Using the argument of Nakajima (see [10], Chapter 5) one can check that the dimensions of the cohomolgy groups of the Hilbert schemes $X^{\Gamma[n]}$ and $X_\Gamma^{[n]}$ are equal to the number of the fixed points with respect to the torus action. Thus,

$$H^\bullet(X^{\Gamma[n]}, \mathbb{Z}) \cong \mathbb{Z}^{UCY(n,d)}, \quad H^\bullet(X_\Gamma^{[n]}, \mathbb{Z}) \cong \mathbb{Z}^{CY(n,d)},$$

and the theorem follows from Corollary 47. □

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