Open Superstring Star as a Continuous Moyal Product

I.Ya. Aref'eva and A.A. Giryavets

Steklov Mathematical Institute,
Gubkin st. 8,
Moscow, 119991, Russia
arefeva@mi.ras.ru

Department of Physics,
Moscow State University,
Moscow, 119899, Russia
alexgir@mail.ru

Abstract

By diagonalizing the three-string vertex and using a special coordinate representation the matter part of the open superstring star is identified with the continuous Moyal product of functions of anti-commuting variables. We show that in this representation the identity and sliver have simple expressions. The relation with the half-string fermionic variables in continuous basis is given.

Contents

1 Introduction

During the last year it has been realized that the vacuum string field theory (VSFT) equations of motion [1]-[5] are similar to the non-commutative soliton equations in the large non-commutativity limit [6]. Soliton equations in this limit are just projector-like equations. Solutions of these equations are obtained immediately using an identification of fields with symbols of operators so that a non-local Moyal product of two fields corresponds to an operator product. To find solutions of VSFT matter equation of motion

$$|\Phi\rangle * |\Phi\rangle = |\Phi\rangle,$$

where $*$ is the Witten star product [7], it is also useful to identify the string field functionals with the symbols of operators or with matrices (see [8] for a review). Half-string formalism is rather suitable for this purpose [3],[9]. The Witten $*$ and Moyal $*$ were first identified by Bars using the half-string formalism [10],[11].

Recently, it has been shown by Douglas, Liu, Moore and Zwiebach [12] that the open string star product in the zero momentum sector can be described as a continuous tensor product of mutually commuting two-dimensional Moyal star products. Non-zero momentum case has been studied in [13]. A continuous parameter κ parameterizing components of the continuous tensor product is nothing but a parameter specifying the eigenvalues of the Neumann matrices of the three-string vertex operator. The three-string Neumann matrices can be diagonalized [14], [15] and there is a basis of oscillators [12] in which the three-string vertex has a simple form

$$|V\rangle_{123} = \exp\Big[-\int_0^\infty d\kappa\, \big(\tfrac{1}{2}\mu(\kappa)(o_\kappa^{1\dagger}o_\kappa^{1\dagger} + e_\kappa^{1\dagger}e_\kappa^{1\dagger}) + \mu_s(\kappa)(o_\kappa^{1\dagger}o_\kappa^{2\dagger} + e_\kappa^{1\dagger}e_\kappa^{2\dagger}) \\ + i\mu_a(\kappa)(e_\kappa^{1\dagger}o_\kappa^{2\dagger} - o_\kappa^{1\dagger}e_\kappa^{2\dagger}) + \text{cyc. per.}\big)\Big]|0\rangle_{123}. \quad (1.1)$$

A crucial observation [12] is that each factor in (1.1) is a two-dimensional Moyal product in the oscillator representation with a parameter of noncommutativity depending on κ. Indeed, the Moyal product with a parameter of noncommutativity θ of functions f and g of two-dimensional variables x

$$(f * g)(x^3) \equiv \int dx^1 dx^2\; K(x^1, x^2, x^3)\, f(x^1) g(x^2),$$

can be rewritten in the oscillator basic as $|f * g\rangle_3 = {}_1\langle f|_2\langle g|V_3^{\mathrm{M}}\rangle_{123}$, with $|V_3^{\mathrm{M}}\rangle_{123}$ given by

$$|V_3^{\mathrm{M}}\rangle_{123} = \exp\Big[-\tfrac{1}{2}m(\theta)(o^{1\dagger}o^{1\dagger} + e^{1\dagger}e^{1\dagger}) - m_s(\theta)(o^{1\dagger}o^{2\dagger} + e^{1\dagger}e^{2\dagger}) \\ - im_a(\theta)(o^{1\dagger}e^{2\dagger} - e^{1\dagger}o^{2\dagger}) + \text{cyc. per.}\Big]|0\rangle_{123}. \quad (1.2)$$

Here $m(\theta)$, $m_s(\theta)$, and $m_a(\theta)$ are given functions of the parameter of noncommutativity and $a^i = (e^i, o^i)$ are oscillators corresponding to two-dimensional Moyal coordinates $x^i = (q^i, p^i)$ $(i = 1, 2, 3)$. One gets this representation using a standard formula for an eigenvector of the operator of coordinate in the oscillator representation

$$\langle x| = \langle 0| \exp[-\tfrac{1}{2}x \cdot x + i\sqrt{2}\, a \cdot x + \tfrac{1}{2}a \cdot a]. \quad (1.3)$$

In [12] it has been observed that the components of the three-string vertex (1.1) have the form (1.2) for each κ, since $\mu(\kappa(\theta)) = m(\theta)$, $\mu_s(\kappa(\theta)) = m_s(\theta)$ and $\mu_a(\kappa(\theta)) = m_a(\theta)$ for $\theta(\kappa) = 2\tanh(\frac{\pi\kappa}{4})$.

The purpose of the present paper is a generalization of this result to the case of the matter sector of the Neveu-Schwarz superstring. The Neveu-Schwarz superstring star algebra and its projectors have been studied in [16]-[20]. The spectroscopy problem of Neumann matrices has been solved by Marino and Schiappa [17], see also appendix of [19] for a discussion of orthonormality and completeness of Neumann matrices eigenvectors. Note, that it is reasonable from the very beginning to expect a modification of an analog of formula (1.3) specifying a suitable coordinate representation for definition of the star product as a Moyal product, since supersting overlaps have a more complicated form in comparison with the bosonic string overlaps (due to half-integer weights).

We begin in section 2 by reminding the Moyal product for the functions of anti-commuting variables (see for example [21], [22]). In section 3 we diagonalize the three-string matter vertex. In section 4 we identify the matter part of open superstring star with a continuous Moyal product. An essential difference from the bosonic case is a non-trivial dependence of the coordinate representation on κ parameter. Namely, the coordinate representation is

$$\langle\eta| \equiv \langle 0|\exp\Big[\int_0^\infty d\kappa\ (\frac{1}{4}\tilde{f}(\kappa)\eta_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\eta_{\kappa,\beta} + \eta_{\kappa,\alpha}\,c_{\alpha\beta}\,\psi_{\kappa,\beta} + \frac{1}{2}\,\tau(\kappa)\psi_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\psi_{\kappa,\beta})\Big], \tag{1.4}$$

where $\eta_{\kappa,\alpha} = (\eta_{\kappa,e}, \eta_{\kappa,o})$ and $\psi_{\kappa,\alpha} = (e_\kappa, o_\kappa)$ are two-dimensional vectors of anti-commuting variables, $\tilde{f}(\kappa)$ and $\tau(\kappa)$ are functions of κ and $\epsilon_{\alpha\beta}$ and $c_{\alpha\beta}$ are defined in (3.20) and (3.13). In section 5 we rewrite the Neveu-Schwarz matter identity and sliver in this coordinate representation. In section 6 we give a relationship between the Moyal basis and the half-string fermionic variables. In section 7 we finish by giving an identification of the Moyal structures for some special Neveu-Schwarz superghost vertices.

2 Moyal product of functions of anti-commuting variables

Here we describe the Moyal product for functions of anti-commuting variables r and s, $\{r, s\} = 0$. Let us consider a pair of operators $\hat{q}$, $\hat{p}$, satisfying the anti-commutation relations

$$\{\hat{q}, \hat{p}\} = \theta, \quad \{\hat{q}, \hat{q}\} = 0, \quad \{\hat{p}, \hat{p}\} = 0, \tag{2.1}$$

and the Weyl operators depending on the anti-commuting variables r and s

$$U(r, s) = \exp(ir\hat{q} + is\hat{p}). \tag{2.2}$$

These operators satisfy the identity

$$U(r^1, s^1)U(r^2, s^2) = U(r^1 + r^2, s^1 + s^2)\exp\Big[\frac{\theta}{2}(s^1r^2 + r^1s^2)\Big]. \tag{2.3}$$

If a function $f(q, p)$ of two anti-commuting variables q, p is given in terms of its Fourier transform

$$f(q, p) = \int dsdr\ \exp(-iqr - ips)\tilde{f}(r, s), \tag{2.4}$$

then one can associate it with an operator $\hat{f}$ by the following formula

$$\hat{f} = \int dsdr\, U(r, s)\tilde{f}(r, s). \tag{2.5}$$

This procedure represents the Weyl quantization and the function $f(q, p)$ is the symbol of the operator $\hat{f}$ (see for example [22],[23]). One has a correspondence

$$\hat{f} \longleftrightarrow f = f(q, p) \tag{2.6}$$

If two operators $\hat{f}_1$ and $\hat{f}_2$ are given by symbols $f_1(q,p)$ and $f_2(q,p)$ then the symbol of product $\hat{f}_1\hat{f}_2$ is given by the Moyal product $(f_1 * f_2)(q,p)$. One has

$$\hat{f}_1(\hat{p},\hat{q})\hat{f}_2(\hat{p},\hat{q}) = \int ds^1 dr^1 ds^2 dr^2 \, U(r^1,s^1)U(r^2,s^2)\tilde{f}_1(r^1,s^1)\tilde{f}_2(r^2,s^2)$$
$$\equiv \int ds^3 dr^3 \, U(r^3,s^3)(\widetilde{f_1 * f_2})(r^3,s^3). \quad (2.7)$$

Define $k=(r,s)$, $dk = dsdr$ and $\Theta_{\alpha\beta} = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$. In terms of these notations one finds

$$(\widetilde{f_1 * f_2})(k^3) = \int dk^4 \; \exp\Big[\frac{1}{2}\Theta_{\alpha\beta}\, k^4_\alpha k^3_\beta\Big]\, \tilde{f}_1(k^4 + \frac{1}{2}k^3)\tilde{f}_2(-k^4+\frac{1}{2}k^3). \quad (2.8)$$

Introducing notations $x=(q,p)$ and $dx = dpdq$ and taking into account that

$$\tilde{f}(k) = \int dx \; \exp(ixk)f(x), \qquad f(x) = \int dk \; \exp(-ixk)\tilde{f}(k), \quad (2.9)$$

one can rewrite (2.8) in terms of Fourier components

$$(f_1 * f_2)(x^3) = \int dk^3 \; \exp(-ix^3k^3)(\widetilde{f_1 * f_2})(k^3) \equiv \int dx^1 dx^2 \; K(x^1,x^2,x^3) f_1(x^1) f_2(x^2). \quad (2.10)$$

Integrating one gets the following expression for the kernel K:

$$K(x^1,x^2,x^3) = \frac{\theta^2}{4}\exp[2(x^2-x^3)_\alpha \, \Theta^{-1}_{\alpha\beta}\,(x^3-x^1)_\beta]. \quad (2.11)$$

Notice a similarity of this formula with a corresponding bosonic formula [12]. One finds

$$\{q,p\}_* = \theta. \quad (2.12)$$

Also one has $\lim_{\theta\to 0} f * g = fg$.

3 Open superstring star in the continuous oscillator basis

3.1 Review of spectroscopy of the vertices

The commuting real symmetric matrices F_{rs} and $(C\tilde{F})_{rs}$ (twist matrix $C_{rs} = (-1)^{r+\frac{1}{2}}\delta_{rs}$, $r,s\geq\frac{1}{2}$) that specify the Neveu-Schwarz vertices have the common set of eigenvectors $v_s(\kappa)$ which are labelled by a continuous parameter $\kappa\in(-\infty,\infty)$,

$$\sum_{s\geq\frac{1}{2}} F_{rs}v_s(\kappa) = f(\kappa)v_r(\kappa), \qquad \sum_{s\geq\frac{1}{2}} (C\tilde{F})_{rs}v_s(\kappa) = \tilde{f}(\kappa)v_r(\kappa). \quad (3.1)$$

and are given by the generating function [17]

$$f_\kappa(z) = \sum_{r\geq\frac{1}{2}} v_r(\kappa) z^{r+\frac{1}{2}} = \mathcal{N}(\kappa)^{-\frac{1}{2}} \frac{z}{\sqrt{1+z^2}} \exp(-\kappa \arctan(z)), \tag{3.2}$$

with normalization factor $\mathcal{N}(\kappa) = 2\cosh\left(\frac{\pi\kappa}{2}\right)$ [19]. The eigenvalues are given by [17]

$$f(\kappa) = -\frac{1}{\cosh(\frac{\pi\kappa}{2})} = -\frac{1-\tau^2(\kappa)}{1+\tau^2(\kappa)}, \qquad \tilde{f}(\kappa) = -\tanh\left(\frac{\pi\kappa}{2}\right) = -\frac{2\tau(\kappa)}{1+\tau^2(\kappa)}, \tag{3.3}$$

where $\tau(\kappa) = \tanh(\frac{\pi\kappa}{4})$. The twist matrix C_{rs} acts on eigenvectors as follows

$$\sum_{s\geq\frac{1}{2}} C_{rs} v_s(\kappa) = -v_r(-\kappa). \tag{3.4}$$

and therefore odd and even components of the eigenvectors satisfy the relations

$$v_{r_o}(-\kappa) = v_{r_o}(\kappa), \qquad v_{r_e}(-\kappa) = -v_{r_e}(\kappa). \tag{3.5}$$

Here we defined even $r_e = 2n - \frac{1}{2}$ and odd $r_o = 2n - \frac{3}{2}$ indices ($n \geq 1$). The eigenvectors $v_r(\kappa)$ are orthogonal and complete [19]

$$\sum_{r\geq\frac{1}{2}} v_r(\kappa_1) v_r(\kappa_2) = \delta(\kappa_1 - \kappa_2), \qquad \int_{-\infty}^{\infty} d\kappa\; v_r(\kappa) v_s(\kappa) = \delta_{rs}. \tag{3.6}$$

3.2 Diagonalization of the vertex

As in the bosonic case [12],[13], instead of usual Neveu-Schwarz matter oscillators ($\psi_r, \psi_r^\dagger = \psi_{-r}$, $r \geq \frac{1}{2}$) it is convenient to introduce the continuous oscillators (Lorentz index is omitted)

$$\tilde{\psi}_\kappa = \sum_{r\geq\frac{1}{2}} v_r(\kappa)\psi_r, \qquad \tilde{\psi}_\kappa^\dagger = \sum_{r\geq\frac{1}{2}} v_r(\kappa)\psi_r^\dagger. \tag{3.7}$$

Due to anti-commutation relations $\{\psi_r, \psi_s^\dagger\} = \delta_{rs}$ the continuous oscillators satisfy the following anti-commutation relation

$$\{\tilde{\psi}_\kappa, \tilde{\psi}_{\kappa'}^\dagger\} = \delta(\kappa - \kappa'). \tag{3.8}$$

The inverse of relations (3.7) are

$$\psi_r = \int_{-\infty}^{\infty} d\kappa\, v_r(\kappa)\tilde{\psi}_\kappa, \qquad \psi_r^\dagger = \int_{-\infty}^{\infty} d\kappa\, v_r(\kappa)\tilde{\psi}_\kappa^\dagger. \tag{3.9}$$

Using the equations (A.3), (3.9) and the completeness relations (3.6) the three-string vertex [24]

$$|V_3\rangle_{123} = \exp\Big[\frac{1}{2}\sum_{a,b=1}^{3}\sum_{r,s\geq\frac{1}{2}} \psi_r^{a\,\dagger}(CM^{ab})_{rs}\psi_s^{b\,\dagger}\Big]|0\rangle_{123} \tag{3.10}$$

can be rewritten in the continuous basis

$$|V_3\rangle_{123} = \exp\Big[\frac{1}{2}\sum_{a,b=1}^{3}\int_{-\infty}^{\infty} d\kappa\, \mu^{ab}(\kappa) C\tilde{\psi}_\kappa^{\dagger\, a}\,\tilde{\psi}_\kappa^{\dagger\, b}\Big]|0\rangle_{123}. \tag{3.11}$$

Here the twist operator C acts in the continuous basis as follows: $C\tilde{\psi}_\kappa^\dagger = -\tilde{\psi}_{-\kappa}^\dagger$. Let us introduce even and odd continuous oscillators $e_\kappa^\dagger$ and $o_\kappa^\dagger$ with respect to C conjugation

$$e_\kappa^\dagger = \frac{1}{\sqrt{2}}(\tilde{\psi}_\kappa^\dagger + C\tilde{\psi}_\kappa^\dagger) = \sqrt{2}\sum_{r_e} v_{r_e}(\kappa)\,\psi_{r_e}^\dagger, \qquad \psi_{r_e}^\dagger = \sqrt{2}\int_0^\infty d\kappa\, v_{r_e}(\kappa)\, e_\kappa^\dagger, \tag{3.12a}$$

$$o_\kappa^\dagger = \frac{1}{\sqrt{2}}(\tilde{\psi}_\kappa^\dagger - C\tilde{\psi}_\kappa^\dagger) = \sqrt{2}\sum_{r_o} v_{r_o}(\kappa)\,\psi_{r_o}^\dagger, \qquad \psi_{r_o}^\dagger = \sqrt{2}\int_0^\infty d\kappa\, v_{r_o}(\kappa)\, o_\kappa^\dagger. \tag{3.12b}$$

In this basis the twist C matrix has a diagonal form

$$c_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.13}$$

Define the two dimensional variable $\psi_{\kappa,\alpha}^\dagger = (e_\kappa^\dagger, o_\kappa^\dagger)$. In these notations the anti-commutation relation (3.8) takes the form

$$\{\psi_{\kappa,\alpha}, \psi_{\kappa',\beta}^\dagger\} = \delta_{\alpha\beta}\delta(\kappa-\kappa'), \qquad \kappa,\kappa' > 0. \tag{3.14}$$

Note that $C\psi_{\kappa,\alpha} = c_{\alpha\beta}\psi_{\kappa,\beta}$. The BPZ conjugation of ψ_{-r} and ψ_r are given by ($\mathrm{bpz}^2 = 1$)

$$\mathrm{bpz}(\psi_{-r}) = (-1)^{-r+\frac{1}{2}}\psi_r, \qquad \mathrm{bpz}(\psi_r) = (-1)^{r-\frac{1}{2}}\psi_{-r}, \qquad r \geq \tfrac{1}{2}. \tag{3.15}$$

The BPZ conjugation does not change ordering $\mathrm{bpz}(\psi_{-r}\psi_{-s}) = \mathrm{bpz}(\psi_{-r})\mathrm{bpz}(\psi_{-s})$. Using (3.15) one finds the following BPZ conjugation of the continuous oscillators

$$\mathrm{bpz}(\psi_{\kappa,\alpha}^\dagger) = -c_{\alpha\beta}\psi_{\kappa,\beta}, \qquad \mathrm{bpz}(\psi_{\kappa,\alpha}) = -c_{\alpha\beta}\psi_{\kappa,\beta}^\dagger. \tag{3.16}$$

Using the relations $e_{-\kappa}^\dagger = -e_\kappa^\dagger$ and $o_{-\kappa}^\dagger = o_\kappa^\dagger$ one can rewrite the three-string vertex (3.11) in terms of oscillators $\psi_{\kappa,\alpha}^\dagger = (e_\kappa^\dagger, o_\kappa^\dagger)$ as

$$|V_3\rangle_{123} = \exp\Big[\frac{1}{2}\int_0^\infty d\kappa\; \psi_{\kappa,\alpha}^{a\,\dagger} V_{\kappa,\alpha\beta}^{ab}\psi_{\kappa,\beta}^{b\,\dagger}\Big]|0\rangle_{123}, \tag{3.17}$$

where $V^{ab}_{\kappa,\alpha\beta}$ is 6×6 matrix defined by

$$V^{ab}_{\kappa,\alpha\beta}=\mu(\kappa)\,\epsilon_{\alpha\beta}\otimes\delta^{ab}+\mu_a(\kappa)\,c_{\alpha\beta}\otimes\chi^{ab}+\mu_s(\kappa)\,\epsilon_{\alpha\beta}\otimes\varepsilon^{ab}; \tag{3.18}$$

$$\mu\equiv\mu^{11}=\tau\frac{1-\tau^2}{1+3\tau^2},\quad \mu_s\equiv\frac{1}{2}(\mu^{12}+\mu^{21})=-\tau\frac{1+\tau^2}{1+3\tau^2},\quad \mu_a\equiv\frac{1}{2}(\mu^{12}-\mu^{21})=\frac{1+\tau^2}{1+3\tau^2}. \tag{3.19}$$

Also we use the notations of [28]

$$\chi^{ab}=\begin{pmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{pmatrix},\qquad \varepsilon^{ab}=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix},\qquad \epsilon_{\alpha\beta}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{3.20}$$

Here and further the summation over $a,b=1,2,3$ is assumed.

3.3 Diagonalization of the identity and sliver

For the identity and sliver one finds the following expressions in the continuous basis

$$|I\rangle=\exp\Big[\frac{1}{2}\sum_{r,s\geq\frac{1}{2}}\psi^\dagger_r I_{rs}\psi^\dagger_s\Big]|0\rangle=\exp\Big[-\int_0^\infty d\kappa\;\tau(\kappa)e^\dagger_\kappa o^\dagger_\kappa\Big]|0\rangle, \tag{3.21}$$

$$|\Xi\rangle=\exp\Big[\frac{1}{2}\sum_{r,s\geq\frac{1}{2}}\psi^\dagger_r (CT)_{rs}\psi^\dagger_s\Big]|0\rangle=\exp\Big[\int_0^\infty d\kappa\;T(\kappa)e^\dagger_\kappa o^\dagger_\kappa\Big]|0\rangle, \tag{3.22}$$

where $T(\kappa)=\exp(-\frac{1}{2}\pi\kappa)=\dfrac{1-\tau(\kappa)}{1+\tau(\kappa)}$. $\mathcal{N}_I$ and $\mathcal{N}_\Xi$ are defined to be the norms of the surface states

$$\mathcal{N}_I\equiv\langle I|I\rangle=(\det(1+(CI)^2))^{\frac{1}{2}}\quad\text{and}\quad \mathcal{N}_\Xi\equiv\langle\Xi|\Xi\rangle=(\det(1+T^2))^{\frac{1}{2}}. \tag{3.23}$$

Actually these norms are infinite and the inclusion of ghost sector will apparently cancel the singularities. A similar problem in the bosonic case is discussed for example in [28].

4 Identification of Moyal structures

In this section we identify the Moyal structures in the three-string vertex by finding an appropriate coordinate representation. In the first subsection we introduce the coordinate representation and find the partition of unity. In the second subsection we use this coordinate representation for identification of the three-string star with the Moyal product of functions of anti-commuting variables introduced in section 2.

4.1 Coordinate representation

Introduce the following continuous coordinate representation

$$\langle\eta| \equiv \langle 0|\exp\Big[\int_0^\infty d\kappa\,(\frac{1}{4}\tilde{f}(\kappa)\eta_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\eta_{\kappa,\beta} + \eta_{\kappa,\alpha}\,c_{\alpha\beta}\,\psi_{\kappa,\beta} + \frac{1}{2}\,\tau(\kappa)\psi_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\psi_{\kappa,\beta})\Big], \tag{4.1}$$

where $\eta_{\kappa,\alpha} = (\eta_{\kappa,e}, \eta_{\kappa,o})$ are anti-commuting variables. Actually these variables can be rescaled. We choose a fixed scale. The BPZ conjugated state $\mathrm{bpz}(\langle\eta|) \equiv |\eta\rangle$ is given by (use (3.16))

$$|\eta\rangle = \exp\Big[\int_0^\infty d\kappa\,(\frac{1}{4}\tilde{f}(\kappa)\eta_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\eta_{\kappa,\beta} - \eta_{\kappa,\alpha}\,\delta_{\alpha\beta}\,\psi^\dagger_{\kappa,\beta} - \frac{1}{2}\,\tau(\kappa)\psi^\dagger_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\psi^\dagger_{\kappa,\beta})\Big]|0\rangle. \tag{4.2}$$

The states $\langle\eta|$ are eigenstates of the operators

$$\hat{\eta}_{\kappa,e} = e^\dagger_\kappa + \tau(\kappa)o_\kappa, \qquad \hat{\eta}_{\kappa,o} = \tau(\kappa)e_\kappa - o^\dagger_\kappa, \tag{4.3a}$$

with eigenvalues $\eta_{\kappa,e}$ and $\eta_{\kappa,o}$. Also

$$\{\hat{\eta}_{\kappa,\alpha}, \hat{\eta}^\dagger_{\kappa',\beta}\} = \theta(\kappa)\delta_{\alpha\beta}\,\delta(\kappa-\kappa'), \tag{4.4}$$

where $\theta(\kappa) = 1 + \tau^2(\kappa)$. To understand a meaning of the coordinates $\eta_{\kappa,\alpha}$ it is useful to rewrite them in the discrete basis:

$$\tilde{\eta}_\kappa = \sum_{r\geq\frac{1}{2}} v_r(\kappa)\eta_r, \tag{4.5a}$$

$$\eta_{\kappa,e} = \frac{1}{\sqrt{2}}(\tilde{\eta}_\kappa + C\tilde{\eta}_\kappa) = \sqrt{2}\sum_{r_e} v_{r_e}(\kappa)\eta_{r_e}, \tag{4.5b}$$

$$\eta_{\kappa,o} = \frac{1}{\sqrt{2}}(\tilde{\eta}_\kappa - C\tilde{\eta}_\kappa) = \sqrt{2}\sum_{r_o} v_{r_o}(\kappa)\eta_{r_o}. \tag{4.5c}$$

In terms of these discrete variables the coordinate representation (4.2) takes the form

$$|\eta\rangle = \exp\Big[\frac{1}{4}\sum_{r,s\geq\frac{1}{2}} \eta_r\,\tilde{F}_{rs}\,\eta_s - \sum_{r\geq\frac{1}{2}} \eta_r\,\psi^\dagger_r + \frac{1}{2}\sum_{r,s\geq\frac{1}{2}} \psi^\dagger_r\,I_{rs}\,\psi^\dagger_s\Big]|0\rangle. \tag{4.6}$$

One sees that $|\eta\rangle$ has an interpretation of a coherent state over the identity surface state. One finds

$$\langle\eta|\lambda\rangle = \mathcal{N}_I\exp\Big[\int_0^\infty d\kappa\,\frac{1}{\theta(\kappa)}\,\eta_{\kappa,\alpha}\,c_{\alpha\beta}\,\lambda_{\kappa,\beta}\Big] = \mathcal{N}_I\exp\Big[\frac{1}{2}\sum_{r,s\geq\frac{1}{2}} \eta_r\,((1-F)C)_{rs}\,\lambda_s\Big]. \tag{4.7}$$

This provides the following partition of unit operator in continuous and discrete basis:

$$1 = \mathcal{N}_I \int \mathbf{d}\eta\, \mathbf{d}\lambda\, \exp\Big[\int_0^\infty d\kappa\, \frac{1}{\theta(\kappa)}\, \eta_{\kappa,\alpha}\, c_{\alpha\beta}\, \lambda_{\kappa,\beta}\Big] |\eta\rangle\langle\lambda|; \tag{4.8a}$$

$$1 = \mathcal{N}_I \int \mathbf{d}\eta\, \mathbf{d}\lambda\, \exp\Big[\frac{1}{2}\sum_{r,s\geq\frac{1}{2}} \eta_r\, ((1-F)C)_{rs}\, \lambda_s\Big] |\eta\rangle\langle\lambda|, \tag{4.8b}$$

where $\mathbf{d}\eta \equiv \prod_{r\geq\frac{1}{2}} d\eta_r$.

4.2 Open superstring star as a continuous Moyal product

In this subsection we insert two partitions of unity (4.8) in the three-string star product and identify it with the Moyal product of functions of anti-commuting variables introduced in section 2. The string field functional corresponding to string field $|\Psi\rangle$ is given by $\Psi(\eta) \equiv \langle\eta|\Psi\rangle = \langle\Psi|\eta\rangle$. Inserting two partitions of unity (4.8) in the three-string star product

$$|\Psi^1 * \Psi^2\rangle_3 = {}_1\langle\Psi^1|_2\langle\Psi^2|V_3\rangle_{123} \tag{4.9}$$

and multiplying it with $\langle\eta^3|$ one obtains

$$\begin{aligned}\langle\eta^3|\Psi^1 * \Psi^2\rangle = \mathcal{N}_I^2 \int \prod_{a=1,2} \mathbf{d}\eta^a\, \mathbf{d}\lambda^a\, \langle\Psi^a|\eta^a\rangle \exp\Big[\int_0^\infty d\kappa\, \frac{1}{\theta(\kappa)}\, \eta^a_{\kappa,\alpha}\, c_{\alpha\beta}\, \lambda^a_{\kappa,\beta}\Big]\ K(\lambda^1,\lambda^2,\eta^3)\\ = \int \mathbf{d}\eta^1\, \mathbf{d}\eta^2\, \Psi^1(\eta^1)\Psi^2(\eta^2)\tilde{K}(\eta^1,\eta^2,\eta^3).\end{aligned} \tag{4.10}$$

Here the kernel $K(\eta^1,\eta^2,\eta^3)$ is given by

$$K(\eta^1,\eta^2,\eta^3) \equiv {}_1\langle\eta^1|_2\langle\eta^2|_3\langle\eta^3|V_3\rangle_{123} = \mathcal{N}_K \exp\Big[\frac{1}{2}\int_0^\infty d\kappa\, \frac{1}{\theta(\kappa)}\, \eta^a_{\kappa,\alpha}\, c_{\alpha\beta}\otimes\chi^{ab}\, \eta^b_{\kappa,\beta}\Big], \tag{4.11}$$

where normalization $\mathcal{N}_K$ is of the form

$$\mathcal{N}_K = \left[\det\left(\tfrac{1}{4}(1-F)^2(2+F)\right)\right]^{-\frac{1}{2}}. \tag{4.12}$$

Integration over λ^1 and λ^2 in (4.10) yields

$$\begin{aligned}\tilde{K}(\eta^1,\eta^2,\eta^3) = \mathcal{N}_K\mathcal{N}_I^2 \int \mathbf{d}\lambda^1\, \mathbf{d}\lambda^2\, \exp\Big[\int_0^\infty d\kappa\, \frac{1}{\theta(\kappa)}\, \big(\sum_{a=1,2} \eta^a_{\kappa,\alpha}\, c_{\alpha\beta}\, \lambda^a_{\kappa,\beta}\big)\Big]\\ \times \exp\Big[\int_0^\infty d\kappa\, \frac{1}{\theta(\kappa)}\, (\lambda^1_{\kappa,\alpha}\, c_{\alpha\beta}\, \lambda^2_{\kappa,\beta} + \lambda^2_{\kappa,\alpha}\, c_{\alpha\beta}\, \eta^3_{\kappa,\beta} + \eta^3_{\kappa,\alpha}\, c_{\alpha\beta}\, \lambda^1_{\kappa,\beta})\Big] = K(-\eta^1,\eta^2,\eta^3).\end{aligned} \tag{4.13}$$

Finally one finds

$$(\Psi^1 * \Psi^2)(\eta^3) = \int \mathbf{d}\eta^1\, \mathbf{d}\eta^2\, \Psi^1(\eta^1)\Psi^2(\eta^2) K(-\eta^1, \eta^2, \eta^3)$$
$$= \int \mathbf{d}\eta^1\, \mathbf{d}\eta^2\, \Psi^1(-\eta^1)\Psi^2(\eta^2) K(\eta^1, \eta^2, \eta^3). \quad (4.14)$$

Note that there is a minus sign in the argument of functional Ψ^1 in the last line of (4.14). This is a mild (up to a sign) associativity anomaly which was found in [25] and has been recently discussed in [17]. Note that this anomaly exist only in the GSO$-$ sector. The star product is associative in the GSO$+$ sector.

For the identification with the Moyal product obtained in section 2 instead of $\eta_{\kappa,\alpha} = (\eta_{\kappa,e}, \eta_{\kappa,o})$ it is relevant to use the anti-commuting variables $x_\kappa = (q_\kappa, p_\kappa)$

$$q_\kappa = \frac{1}{2}(\eta_{\kappa,e} + \eta_{\kappa,o}), \qquad p_\kappa = \frac{1}{2}(\eta_{\kappa,e} - \eta_{\kappa,o}). \quad (4.15)$$

In terms of these variables the kernel (4.11) takes the form

$$K(x^1, x^2, x^3) \equiv K(\eta^1, \eta^2, \eta^3) = \exp\Big[\int_0^\infty d\kappa\, \frac{2}{\theta(\kappa)}\, q^a_\kappa\, \chi^{ab}\, p^b_\kappa\Big]. \quad (4.16)$$

Up to normalization this coincides for every κ with the Moyal kernel (2.11). Further we will also use the notation $\langle x|$ instead of $\langle \eta|$ ($\langle x| \equiv \langle \eta|$). Note also that

$$\{q_\kappa, p_{\kappa'}\}_* = \theta(\kappa)\delta(\kappa - \kappa'). \quad (4.17)$$

5 Sliver, identity and wedge states

Now we are going to rewrite the well known star algebra projectors, the identity (3.21) and sliver (3.22), in the $x_\kappa = (p_\kappa, q_\kappa)$ basis. In the coordinate representation the sliver is given by

$$\langle x|\Xi\rangle = (\det(1 + CIT))^{\frac{1}{2}} \exp\Big[\int_0^\infty d\kappa\, \frac{2}{\theta(\kappa)}\, q_\kappa p_\kappa\Big]. \quad (5.1)$$

One expects this answer since $f(x) = \frac{1}{2}\exp(\,\frac{2}{\theta}\, qp\,)$ is a projector with respect to the Moyal product of functions of anti-commuting variables (2.10). In the coordinate representation the identity is given by

$$\langle x|I\rangle = \mathcal{N}_I. \quad (5.2)$$

This is also an expected answer since $f(x) = 1$ is a projector of the Moyal product (2.10). So we found that up to normalization the sliver and identity in the coordinate representation are the projectors with respect to the Moyal product.

A few comment on wedge states. It is known that the star algebra has a subalgebra of the wedge states $|n\rangle$, $n \geq 1$ with the multiplication rule [2] [1]

$$|n\rangle * |m\rangle = |n+m-1\rangle, \tag{5.3}$$

where $|1\rangle$ corresponds to the identity $|I\rangle$ and $|2\rangle$ corresponds to the Neveu-Schwarz superstring vacuum $|0\rangle$. The wedge states subalgebra has a natural interpretation in terms of the Moyal product of wedge states in coordinate representation. The ground state and the identity functions are $f^1(x) \equiv \langle x|0\rangle$ and $f^0(x) \equiv \langle x|I\rangle$, respectively. It is known that the wedge states $|n\rangle$ can be obtained by multiplying the vacuum $n-1$ times, i.e. $|n\rangle = (|0\rangle)_*^{n-1}$. In the functional language $f^{n-1}(x) \equiv \langle x|n\rangle$ and one gets a correspondence

$$\underbrace{|0\rangle * |0\rangle * ... * |0\rangle}_{n-1} = |n\rangle \longleftrightarrow f^{n-1}(x) = \underbrace{f^1 * f^1 * ... * f^1}_{n-1}(x) \tag{5.4}$$

for $n \geq 2$. This means that there is a subalgebra of algebra with a Moyal product. Basis elements of this subalgebra are $f^n(x)$; $f^0(x)$ is the identity and the multiplication rule is given by

$$(f^n * f^m)(x) = f^{n+m}(x). \tag{5.5}$$

6 Relation with half-string fermionic variables

In this section we give a relationship between the continuous Moyal basis and the half-string fermionic variables. To work out half-sting fermionic variables it is more convenient to use $x_\kappa = (q_\kappa, p_\kappa)$ basis. In this basis (4.14) takes the form

$$(\Psi^1 * \Psi^2)(q^3, p^3) = \mathcal{N}_K \int \mathbf{d}q^1 \, \mathbf{d}p^1 \, \mathbf{d}q^2 \, \mathbf{d}p^2 \; \Psi^1(-q^1, -p^1)\Psi^2(q^2, p^2) \exp\Big[\int_0^\infty d\kappa \, \frac{2}{\theta(\kappa)} \, q_\kappa^a \, \chi^{ab} \, p_\kappa^b\Big]. \tag{6.1}$$

Often for simplicity we will omit index κ in the coordinates. Performing Fourier transformation in p_κ variable

$$\Psi(q, u) \equiv \int \mathbf{d}p \; \exp\Big[\int_0^\infty d\kappa \; p_\kappa u_\kappa\Big] \Psi(q, p), \tag{6.2}$$

where $\mathbf{d}p \equiv \prod_{r \geq \frac{1}{2}} dp_r$ for discrete basis and introducing the notation

$$\Psi(q, u) \equiv \tilde{\Psi}\Big(q_\kappa - \frac{\theta(\kappa)}{2} u_\kappa, q_\kappa + \frac{\theta(\kappa)}{2} u_\kappa\Big) \equiv \tilde{\Psi}(\psi^l, \psi^r) \tag{6.3}$$

[1] For algebraic construction of wedge states for Neveu-Schwarz string see [19].

one rewrites (6.1) as

$$(\widetilde{\Psi^1 * \Psi^2})(\psi^l,\psi^r) = \int \mathbf{d}p\ \exp\Big[\int_0^\infty d\kappa\ p_\kappa u_\kappa\Big](\Psi^1 * \Psi^2)(q,p) = \mathcal{N}_H \int \mathbf{d}\xi\ \tilde{\Psi}^1(-\psi^l,-\xi)\tilde{\Psi}^2(\xi,\psi^r), \tag{6.4}$$

where $\mathcal{N}_H$ is normalization constant. The operators corresponding to the left and right fermionic variables ψ^l and ψ^r in the oscillator representation are given by

$$\hat{\psi}^l_\kappa = \hat{q}_\kappa - \frac{\theta(\kappa)}{2}\hat{u}_\kappa = \frac{1}{\sqrt{2}}((\tau(\kappa)-1)\tilde{\psi}_\kappa + (\tau(\kappa)+1)C\tilde{\psi}^\dagger_\kappa), \tag{6.5a}$$

$$\hat{\psi}^r_\kappa = \hat{q}_\kappa + \frac{\theta(\kappa)}{2}\hat{u}_\kappa = \frac{1}{\sqrt{2}}((\tau(\kappa)+1)\tilde{\psi}_\kappa - (\tau(\kappa)-1)C\tilde{\psi}^\dagger_\kappa). \tag{6.5b}$$

Here $\hat{u}_\kappa = \frac{1}{\theta(\kappa)}(\hat{\eta}^\dagger_{\kappa,e} - \hat{\eta}^\dagger_{\kappa,o})$. In discrere basis (6.5) takes the form

$$\hat{\psi}^l_\kappa = -\frac{1}{\sqrt{2}\tilde{f}(\kappa)}((C\psi^\dagger + F\psi + \tilde{F}\psi^\dagger)_s + C_{sr}(C\psi + F\psi^\dagger + \tilde{F}\psi)_r)v_s(\kappa), \tag{6.6a}$$

$$\hat{\psi}^r_\kappa = -\frac{1}{\sqrt{2}\tilde{f}(\kappa)}((-C\psi^\dagger + F\psi + \tilde{F}\psi^\dagger)_s - C_{sr}(-C\psi + F\psi^\dagger + \tilde{F}\psi)_r)v_s(\kappa). \tag{6.6b}$$

Formulae (6.6) suggest the following half-string fermionic variables

$$l_r = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma\ e^{-ir\sigma}\psi(\sigma), \qquad r_s = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma\ e^{-is\sigma} i\psi(\pi-\sigma), \tag{6.7}$$

such that

$$l_{r_e} = \psi_{r_e} + F_{r_e s_e}\psi^\dagger_{s_e} + \tilde{F}_{r_e s_o}\psi_{s_o}, \qquad l_{r_o} = \psi_{r_o} - F_{r_o s_o}\psi^\dagger_{s_o} - \tilde{F}_{r_o s_e}\psi_{s_e}; \tag{6.8a}$$

$$r_{r_e} = -\psi^\dagger_{r_e} + F_{r_e s_e}\psi_{s_e} + \tilde{F}_{r_e s_o}\psi^\dagger_{s_o}, \qquad r_{r_o} = \psi^\dagger_{r_o} + F_{r_o s_o}\psi_{s_o} + \tilde{F}_{r_o s_e}\psi^\dagger_{s_e}. \tag{6.8b}$$

In this case the left operator $\hat{\psi}^l_\kappa$ and the right operator $\hat{\psi}^r_\kappa$ are expressed in terms of the left and right fermionic oscillators (6.8)

$$\psi^l_\kappa = -\sum_{r,s\geq\frac{1}{2}} \frac{1}{\sqrt{2}\tilde{f}(\kappa)}(l_r + C_{rs}l^\dagger_s)v_r(\kappa) \quad \text{and} \quad \psi^r_\kappa = -\sum_{r,s\geq\frac{1}{2}} \frac{1}{\sqrt{2}\tilde{f}(\kappa)}(r_r - C_{rs}r^\dagger_s)v_r(\kappa). \tag{6.9}$$

This gives a precise relationship between the half-string fermionic variables and the continuous Moyal basis.

7 Superghost vertex as Moyal kernel

The ghost vertex for the bosonic string was identified with the Moyal product in the bosonized form in [26], [28] and in non-bosonized form in the Siegel gauge in [27]. The most direct way for an identification of superghost vertex with the Moyal product will be to use the bosonized form of superghosts and picture changing operator $Y_{-2}=Y(i)Y(-i)$. However, in this section we use the non-bosonized language to give a diagonalization of the minus one picture three-string vertex, which can be identified with the continuous Moyal product in the manner analogous to the one which was used in the Neveu-Schwarz matter case. Similarly one can define the three-string vertex in picture minus two, so that the string fields are in picture zero. If all three vacua in the three-string vertex are in picture minus two then we can define the continuous variables making a shift in creation and annihilation operators that corresponds to a shift of all vacua from one picture to another. Such diagonalization is alike the one presented in this section but more cumbersome.

7.1 Diagonalization of vertex

In this subsection we introduce the continuous oscillators and diagonalize the three-string superghost vertex in the minus one picture. The Neveu-Schwarz superghost oscillators satisfy the following commutation relation and hermitian conjugation properties

$$[\gamma_r,\beta_s]=\delta_{r+s,0},\qquad \gamma_r^\dagger=\gamma_{-r},\quad \beta_s^\dagger=-\beta_{-s}. \tag{7.1}$$

The minus one picture vacuum is defined so that

$$\gamma_r|-1\rangle=0,\quad r\geq\tfrac{1}{2},\qquad \beta_s|-1\rangle=0,\quad s\geq\tfrac{1}{2}, \tag{7.2}$$

and $\langle -1|-1\rangle=1$. For this vacuum the BPZ conjugation is ($\text{bpz}^2=1$)

$$\text{bpz}(\gamma_{-r})=(-1)^{-r-\frac{1}{2}}\gamma_r,\quad r\geq\tfrac{1}{2},\qquad \text{bpz}(\gamma_r)=(-1)^{r+\frac{1}{2}}\gamma_{-r},\quad r\geq\tfrac{1}{2}, \tag{7.3a}$$

$$\text{bpz}(\beta_{-s})=(-1)^{-s+\frac{3}{2}}\beta_s,\quad s\geq\tfrac{1}{2},\qquad \text{bpz}(\beta_s)=(-1)^{s-\frac{3}{2}}\beta_{-s},\quad s\geq\tfrac{1}{2}. \tag{7.3b}$$

Let us define oscillators in the continuous basis as follows

$$\tilde{\beta}_\kappa^\dagger=\sum_{r\geq\frac{1}{2}}v_r(\kappa)\beta_{-r},\qquad \tilde{\beta}_\kappa=-\sum_{r\geq\frac{1}{2}}v_r(\kappa)\beta_r, \tag{7.4a}$$

$$\tilde{\gamma}_\kappa^\dagger=\sum_{r\geq\frac{1}{2}}v_r(\kappa)\gamma_{-r},\qquad \tilde{\gamma}_\kappa=\sum_{r\geq\frac{1}{2}}v_r(\kappa)\gamma_r. \tag{7.4b}$$

The commutation relations (7.1) in the continuous basis become

$$[\tilde{\gamma}_\kappa,\tilde{\beta}_{\kappa'}^\dagger]=[\tilde{\beta}_\kappa,\tilde{\gamma}_{\kappa'}^\dagger]=\delta(\kappa-\kappa'). \tag{7.5}$$

The twist operator C acts in the continuous basis as follows: $C\tilde{\beta}_\kappa^\dagger=-\tilde{\beta}_{-\kappa}^\dagger$ and $C\tilde{\gamma}_\kappa^\dagger=-\tilde{\gamma}_{-\kappa}^\dagger$. Now introduce the even and odd under C conjugation oscillators

$$\beta_{\kappa,e}^\dagger=\frac{1}{\sqrt{2}}(\tilde{\beta}_\kappa^\dagger+C\tilde{\beta}_\kappa^\dagger),\qquad \beta_{\kappa,o}^\dagger=\frac{1}{\sqrt{2}}(\tilde{\beta}_\kappa^\dagger-C\tilde{\beta}_\kappa^\dagger), \tag{7.6a}$$

$$\gamma_{\kappa,e}^\dagger=\frac{1}{\sqrt{2}}(\tilde{\gamma}_\kappa^\dagger+C\tilde{\gamma}_\kappa^\dagger),\qquad \gamma_{\kappa,o}^\dagger=\frac{1}{\sqrt{2}}(\tilde{\gamma}_\kappa^\dagger-C\tilde{\gamma}_\kappa^\dagger). \tag{7.6b}$$

We use the following notations: $\beta^{\dagger}_{\kappa,\alpha} = (\beta^{\dagger}_{\kappa,e}, \beta^{\dagger}_{\kappa,o})$ and $\gamma^{\dagger}_{\kappa,\alpha} = (\gamma^{\dagger}_{\kappa,e}, \gamma^{\dagger}_{\kappa,o})$. Using (7.3) one finds the following BPZ conjugation of the continuous oscillators

$$\mathrm{bpz}(\beta^{\dagger}_{\kappa,\alpha}) = -c_{\alpha\beta}\beta_{\kappa,\beta}, \qquad \mathrm{bpz}(\gamma^{\dagger}_{\kappa,\alpha}) = c_{\alpha\beta}\gamma_{\kappa,\beta}. \tag{7.7}$$

The Neveu-Schwarz superghost three-string vertex in the minus one picture in discrete and continuous basis is given by

$$|\tilde{V}_3\rangle_{123} = \exp\Big[\sum_{a,b=1}^{3}\sum_{r,s\geq\frac{1}{2}} \beta^{a}_{-r}(C\tilde{M}^{ab})_{rs}\gamma^{b}_{-s}\Big]|-1\rangle_{123}; \tag{7.8}$$

$$|\tilde{V}_3\rangle_{123} = \exp\Big[\int_0^{\infty} d\kappa\; \beta^{\dagger a}_{\kappa,\alpha}\tilde{V}^{ab}_{\kappa,\alpha\beta}\gamma^{\dagger b}_{\kappa,\beta}\Big]|-1\rangle_{123}. \tag{7.9}$$

Here $\tilde{V}^{ab}_{\kappa,\alpha\beta}$ is 6×6 matrix of the form

$$\tilde{V}^{ab}_{\kappa,\alpha\beta} = \tilde{\mu}(\kappa)\,\epsilon_{\alpha\beta}\otimes\delta^{ab} + \tilde{\mu}_a(\kappa)\,c_{\alpha\beta}\otimes\chi^{ab} + \tilde{\mu}_s(\kappa)\,\epsilon_{\alpha\beta}\otimes\varepsilon^{ab}; \tag{7.10}$$

and

$$\tilde{\mu}\equiv\tilde{\mu}^{11} = -\tilde{\tau}\frac{1-\tilde{\tau}^2}{1+3\tilde{\tau}^2}, \quad \tilde{\mu}_s \equiv \frac{1}{2}(\tilde{\mu}^{12}+\tilde{\mu}^{21}) = \tilde{\tau}\frac{1+\tilde{\tau}^2}{1+3\tilde{\tau}^2}, \quad \tilde{\mu}_a \equiv \frac{1}{2}(\tilde{\mu}^{12}-\tilde{\mu}^{21}) = -\frac{1+\tilde{\tau}^2}{1+3\tilde{\tau}^2}, \tag{7.11}$$

where $\tilde{\tau}(\kappa) = \coth\left(\frac{\pi\kappa}{4}\right)$.

7.2 Identification of Moyal structures

In this subsection we introduce the coordinate representation in which the three-string vertex becomes the Moyal kernel. We find the partition of unity and identify the Moyal structures. Introduce the continuous coordinate representation as follows

$$\langle\eta| = \langle -1|\exp\Big[\int_0^{\infty} d\kappa\; (\frac{1}{2}\,\tilde{f}(\kappa)\eta^{\beta}_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\eta^{\gamma}_{\kappa,\beta} + \eta^{\gamma}_{\kappa,\alpha}\,c_{\alpha\beta}\,\beta_{\kappa,\beta} - \eta^{\beta}_{\kappa,\alpha}\,c_{\alpha\beta}\,\gamma_{\kappa,\beta} + \tilde{\tau}(\kappa)\beta_{\kappa,\alpha}\epsilon_{\alpha\beta}\gamma_{\kappa,\beta})\Big], \tag{7.12}$$

where $\eta^{\beta}_{\kappa,\alpha} = (\eta^{\beta}_{\kappa,e}, \eta^{\beta}_{\kappa,o})$ and $\eta^{\gamma}_{\kappa,\alpha} = (\eta^{\gamma}_{\kappa,e}, \eta^{\gamma}_{\kappa,o})$. The BPZ conjugated state $\mathrm{bpz}(\langle\tilde{\eta}|)\equiv|\tilde{\eta}\rangle$ is given by

$$|\eta\rangle = \exp\Big[\int_0^{\infty} d\kappa\; (\frac{1}{2}\,\tilde{f}(\kappa)\eta^{\beta}_{\kappa,\alpha}\,\epsilon_{\alpha\beta}\,\eta^{\gamma}_{\kappa,\beta} - \eta^{\gamma}_{\kappa,\alpha}\,\delta_{\alpha\beta}\,\beta^{\dagger}_{\kappa,\beta} - \eta^{\beta}_{\kappa,\alpha}\,\delta_{\alpha\beta}\,\gamma^{\dagger}_{\kappa,\beta} + \tilde{\tau}(\kappa)\beta^{\dagger}_{\kappa,\alpha}\epsilon_{\alpha\beta}\gamma^{\dagger}_{\kappa,\beta})\Big]|-1\rangle. \tag{7.13}$$

Let us rewrite this coordinate representation in discrete basis. Introduce the discrete variables

$$\tilde{\eta}^{\beta}_{\kappa} = \sum_{r\geq\frac{1}{2}} v_r(\kappa)\eta^{\beta}_r, \qquad \tilde{\eta}^{\gamma}_{\kappa} = \sum_{r\geq\frac{1}{2}} v_r(\kappa)\eta^{\gamma}_r, \tag{7.14a}$$

$$\eta^{\beta}_{\kappa,e} = \sqrt{2}\sum_{r_e} v_{r_e}(\kappa)\eta^{\beta}_{r_e}, \qquad \eta^{\gamma}_{\kappa,e} = \sqrt{2}\sum_{r_e} v_{r_e}(\kappa)\eta^{\gamma}_{r_e}, \tag{7.14b}$$

$$\eta^{\beta}_{\kappa,o} = \sqrt{2}\sum_{r_o} v_{r_o}(\kappa)\eta^{\beta}_{r_o}, \qquad \eta^{\gamma}_{\kappa,o} = \sqrt{2}\sum_{r_o} v_{r_o}(\kappa)\eta^{\gamma}_{r_o}. \tag{7.14c}$$

In terms of these discrete variables the coordinate representation (7.13) takes the form

$$|\eta\rangle = \exp\Big[\frac{1}{2}\sum_{r,s\geq\frac{1}{2}} \eta^{\beta}_r\,\tilde{F}_{rs}\,\eta^{\gamma}_s - \sum_{r\geq\frac{1}{2}} \eta^{\gamma}_r\,\beta_{-r} - \sum_{r\geq\frac{1}{2}} \eta^{\beta}_r\,\gamma_{-r} + \sum_{r,s\geq\frac{1}{2}} \beta_{-r}\,\widetilde{I}_{rs}\,\gamma_{-s}\Big]|-1\rangle. \tag{7.15}$$

Notice that $|\eta\rangle$ has an interpretation of the coherent state over the identity surface state in the picture minus one. One finds in the continuous and discrete basis

$$\langle\eta|\lambda\rangle = \tilde{\mathcal{N}}_I \exp\Big[\int_0^{\infty} d\kappa\,\frac{1}{\tilde{\theta}(\kappa)}\,(\eta^{\beta}_{\kappa,\alpha}\,c_{\alpha\beta}\,\lambda^{\gamma}_{\kappa,\beta} - \eta^{\gamma}_{\kappa,\alpha}\,c_{\alpha\beta}\,\lambda^{\beta}_{\kappa,\beta})\Big], \tag{7.16a}$$

$$\langle\eta|\lambda\rangle = \tilde{\mathcal{N}}_I \exp\Big[\frac{1}{2}\sum_{r,s\geq\frac{1}{2}} (\eta^{\beta}_r\,((1+F)C)_{rs}\,\lambda^{\gamma}_s - \eta^{\gamma}_r\,((1+F)C)_{rs}\,\lambda^{\beta}_s)\Big], \tag{7.16b}$$

where $\tilde{\theta}(\kappa) = 1+\tilde{\tau}^2(\kappa)$ and $\tilde{\mathcal{N}}_I = \det(1+(C\tilde{I})^2)^{-1}$. The partitions of unity in the continuous and discrete basis are given by

$$1 = \tilde{\mathcal{N}}_I \int \mathbf{d}\eta^{\beta}\,\mathbf{d}\eta^{\gamma}\,\mathbf{d}\lambda^{\beta}\,\mathbf{d}\lambda^{\gamma}\,\exp\Big[-\int_0^{\infty} d\kappa\,\frac{1}{\tilde{\theta}(\kappa)}\,(\eta^{\beta}_{\kappa,\alpha}\,c_{\alpha\beta}\,\lambda^{\gamma}_{\kappa,\beta} - \eta^{\gamma}_{\kappa,\alpha}\,c_{\alpha\beta}\,\lambda^{\beta}_{\kappa,\beta})\Big]|\eta\rangle\langle\lambda|; \tag{7.17a}$$

$$1 = \tilde{\mathcal{N}}_I \int \mathbf{d}\eta^{\beta}\,\mathbf{d}\eta^{\gamma}\,\mathbf{d}\lambda^{\beta}\,\mathbf{d}\lambda^{\gamma}\,\exp\Big[-\frac{1}{2}\sum_{r,s\geq\frac{1}{2}} (\eta^{\beta}_r\,((1+F)C)_{rs}\,\lambda^{\gamma}_s - \eta^{\gamma}_r\,((1+F)C)_{rs}\,\lambda^{\beta}_s)\Big]|\eta\rangle\langle\lambda|. \tag{7.17b}$$

Here

$$\mathbf{d}\eta^{\beta} \equiv \prod_{r\geq\frac{1}{2}} d\eta^{\beta}_r\,, \qquad \mathbf{d}\eta^{\gamma} \equiv \prod_{r\geq\frac{1}{2}} d\eta^{\gamma}_r. \tag{7.18}$$

The functional corresponding to string field $|\Psi\rangle$ in the minus one picture is given by $\Psi(\eta) \equiv \langle\eta|\Psi\rangle = \langle\Psi|\eta\rangle$. Inserting two partitions of unity (7.17) in the three-string superghost star product

$$|\Psi^1 * \Psi^2\rangle_3 = {}_1\langle\Psi^1|_2\langle\Psi^2|\tilde{V}_3\rangle_{123} \tag{7.19}$$

one finally finds

$$(\Psi^1 * \Psi^2)(\eta^3) = \int \mathbf{d}\eta^{\beta\,1}\,\mathbf{d}\eta^{\gamma\,1}\,\mathbf{d}\eta^{\beta\,2}\,\mathbf{d}\eta^{\gamma\,2}\,\Psi^1(\eta^1)\Psi^2(-\eta^2)K(\eta^1,\eta^2,\eta^3). \tag{7.20}$$

Here the kernel corresponding to the three-string vertex (7.9) is given by

$$K(\eta^1,\eta^2,\eta^3) \equiv {}_1\langle\eta^1|_2\langle\eta^2|_3\langle\eta^3|\tilde{V}_3\rangle_{123} = \tilde{\mathcal{N}}_K \exp\Big[\int_0^\infty d\kappa\,\frac{1}{\tilde{\theta}(\kappa)}\,\eta^{\beta\,a}_{\kappa,\alpha}\,c_{\alpha\beta}\otimes\chi^{ab}\,\eta^{\gamma\,b}_{\kappa,\beta}\Big], \tag{7.21}$$

and

$$\tilde{\mathcal{N}}_K = \det\left(\tfrac{1}{4}(1+F)^2(2-F)\right). \tag{7.22}$$

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Appendices

A The three-string matter vertex

The Neveu-Schwarz three-string matter vertex and identity Neumann matrices are expressed in terms of the real symmetric matrices F and $C\tilde{F}$ as [24]

$$M^{11} = \frac{FC\tilde{F}}{(1-F)(2+F)},\ M^{12} = \frac{C\tilde{F}+(1-F)}{(1-F)(2+F)},\ M^{21} = \frac{C\tilde{F}-(1-F)}{(1-F)(2+F)},\ CI = \frac{C\tilde{F}}{1-F}. \tag{A.1}$$

They satisfy the following cyclic property

$$M^{a+1\,b+1} = M^{a\,b},\ \ \forall\ \ a,b,a+1,b+1(\bmod 3). \tag{A.2}$$

These matrices are commuting and have the same as F and $C\tilde{F}$ set of eigenvectors

$$\sum_{s\geq\frac{1}{2}} M^{ab}_{rs}v_s(\kappa) = \mu^{ab}(\kappa)v_r(\kappa), \qquad \sum_{s\geq\frac{1}{2}}(CI)_{rs}v_s(\kappa) = -\tau(\kappa)v_r(\kappa), \tag{A.3}$$

the eigenvalues (3.19) are given in [17]. Note also that $\mu^{ab}(-\kappa) = -\mu^{ba}(\kappa)$.

The hermitian matrices F_{rs} and $(C\tilde{F})_{rs}$ are defined by

$$F_{rs} = -\frac{2}{\pi}\frac{\imath^{r-s}}{r+s}, \quad r = s \bmod 2, \qquad \tilde{F}_{rs} = -\frac{2}{\pi}\frac{\imath^{r+s}}{r-s}, \quad r = s+1 \bmod 2. \tag{A.4}$$

They have the following properties

$$F^2-\tilde{F}^2 = 1, \quad [F,\tilde{F}] = 0, \quad [F,C] = 0, \quad F^T = F, \quad \{\tilde{F},C\} = 0, \quad \tilde{F}^T = -\tilde{F}. \tag{A.5}$$

B The three-string ghost vertex

The three-string Neveu-Schwarz superghost vertex in the minus one picture has the following Neumann matrices

$$\tilde{M}^{11} = \frac{FC\tilde{F}}{(1+F)(2-F)}, \quad \tilde{M}^{12} = \frac{-C\tilde{F}-(1+F)}{(1+F)(2-F)}, \quad \tilde{M}^{21} = \frac{-C\tilde{F}+(1+F)}{(1+F)(2-F)}, \quad C\tilde{I} = -\frac{C\tilde{F}}{1+F}. \tag{B.1}$$

They have the same set of eigenvectors as the matter vertex matrices. The property $\tilde{\mu}^{ab}(-\kappa) = -\tilde{\mu}^{ba}(\kappa)$ is obeyed.

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