Noncommutative Field Theories

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1 Introduction.

It is well known that the gauge invariance principle is a basic principle in construction of modern field theories. Both general relativity and Yang-Mills theory use the gauge invariance. Another example of gauge invariant theory is string field theory [1]. To formulate this theory it is useful to employ notions of noncommutative differential geometry.

Noncommutative geometry has appeared in physics in the works of the founders of quantum mechanics. Heisenberg and Dirac have argued that the phase space of quantum mechanics must be noncommutative and it should be described by quantum algebra. After works of von Neumann and more recently by Connes mathematical and physical investigations in noncommutative geometry became very intensive.

Noncommutative geometry uses a generalization of the known duality between a space and the algebra of functions on it [2, 3]. One can take $\mathcal{A}$ to be an abstract noncommutative associative algebra $\mathcal{A}$ with a multiplication $\star$ and interpret $\mathcal{A}$ as an "algebra of functions" on a (nonexisting) noncommutative space. One can introduce many geometrical notions in this setting. A basic set consists of an external derivative Q (or an algebra $\mathcal{G}$ of derivatives) and an integral, which satisfy ordinary axioms. In Section 2 we will present several examples of noncommutative spaces.

Let us sketch motivations/applications of models of noncommutative differential geometry (NCDG) in physics:

- Noncommutative space-time;
- Large N reduced models (quenched Yang-Mills theory);
- D-brane physics, (M)atrix theory;
- Description of the behavior of models in external fields;
- Noncommutative Solitons and Instantons;
- Smoothing of singularities;
- String Field Theory (SFT).

Noncommutative Space-Time

▶ Uncertainty principle in gravity

There is an absolute limitation on length measurements in quantum gravity

$$\ell \geqslant \ell_{Pl}. \tag{1.1}$$

The reason is the following. If we want to locate a particle we need to have energy greater than the Planck mass $m_{Pl} = (\hbar c G^{-1})^{\frac{1}{2}}$ (G is Newton's constant). The corresponding gravitational field will have a horizon at $r = 2Gm_{Pl}c^{-2} = 2\ell_{Pl}$, shielding whatever happens inside the Schwarzschild radius and therefore the smallest "quanta" of space and time exists. It is tempting to relate the uncertainty δx in the length measurement (1.1) to the noncommutativity of coordinates

$$[x^\mu, x^\nu] = i\hbar\theta^{\mu\nu}. \tag{1.2}$$

The idea of quantization of space-time using noncommutative coordinates x_μ is an old one [4]-[10]. Snyder [5, 6] has proposed to use the noncommutative space to improve the UV behavior in quantum field theory. An extension of the Snyder's idea was considered in [11, 12] where the field theory with the momentum in the de Sitter space was considered. A proposal for a general (pseudo)Riemannian geometry in the momentum space was discussed in [13].

Deriving relation (1.1) from (1.2) by analogy with the derivation of the Heisenberg uncertainty inequality one gets the problem because (1.1) has to hold even for one-dimensional space. In [14] it was suggested that (1.1) indicates that at the Planck scale one should use non-Archimedean geometry. For recent discussion of space-time noncommutativity see [15]-[19]. The space-time stringy uncertainty has been discussed in [16, 20].

▶ UV divergences in QFT

One of earlier motivations to consider field theories on noncommutative space-time (noncommutative field theories) was a hope that it would be possible to avoid quantum field theory divergences [15, 16, 3, 22, 18, 23, 21]. Now a commonly accepted belief is that a theory on a noncommutative space is renormalizable if (and only if) the corresponding commutative theory is renormalizable. Results on the one-loop renormalizability of noncommutative gauge theory [24]-[26] as well as the two-loop renormalizability of the noncommutative scalar ϕ^4_4 theory [27] support this belief. However the renormalizability [28] does not guarantee that the theory is well-defined. There is a mixing of the UV and the IR divergences [29, 27]. For further developments see [30]-[78] and refs. therein.

Large N reduced models

It has been found that the large N limit in QCD can be described by a reduced quenched model [79, 80]. This model is equivalent to a noncommutative gauge model. A special large N matrix model was proposed to describe strings [81, 82] (see [83] in the context of noncommutative gauge models). It occurs that the master fields in the large N matrix models are the subject of the noncommutative plane [84].

D-branes and M(atrix) theory

Connes, Douglas and Schwartz [85] (see also [88]) have shown that supersymmetric gauge theory on a noncommutative torus is naturally related to a compactification of the M(atrix) theory [86, 87]. M(atrix) theory is a maximally supersymmetric quantum mechanics with the action:

$$S = \int dt \ \mathrm{Tr} \sum_{i=1}^{9} (D_t X)^2 - \sum_{i<j} [X^i, X^j]^2 + \chi^\dagger (D_t + \Gamma_i X^i)\chi. \tag{1.3}$$

Here $D_t = \partial/\partial t + iA_0$, and variation over A_0 yields a Gauss law. This action is also known as a regularized form of the supermembrane action [89]. Equations of motion for the action (1.3) admit the following static solutions

$$[X^\mu, X^\nu] = i\hbar\theta^{\mu\nu} \tag{1.4}$$

that are nothing but commutation relations (1.2) defining noncommutative space-time. About spectral properties of the corresponding Hamiltonian see [90] and [91] and references therein.

Compactification on torus [92]

$$U_i^{-1} X^j U_i = X^j + \delta_i^j 2\pi R_i.$$

leads to the relations

$$U_i U_j = e^{i\theta_{ij}} U_j U_i,$$

which define the algebra of functions on a noncommutative torus with the noncommutativity parameter θ. For reviews and further developments see [93]-[123].

One of the natural questions is whether noncommutative gauge theory can be used to describe more general compactifications in string theory. In order to have progress in this direction one has to develop the theory of noncommutative quantum gauge fields not only on the torus, but also on more general manifolds [105]-[108]. A framework for a noncommutative gauge theory on Poisson manifolds by using recently developed deformation quantization [109, 110] has been discussed in [21].

Models in external fields

- Dipole in a magnetic field

It is well known that the position operators in quantum mechanics in the presence of a magnetic field does not commute. A model consisting of a pair of opposite charges moving in a strong magnetic field can be described as a particle moving on the noncommutative 2-plane [112].

▶ Quantum Hall effect

The model describing electrons in a magnetic field projected to the lowest Landau level can be naturally treated as a noncommutative field theory. Thus noncommutative field theory is relevant to the theory of the quantum Hall effect [112, 113]. It has been proposed that a good description of the fractional quantum Hall effect can be obtained using noncommutative Chern-Simons theory [114, 115].

▶ Strings in a magnetic field

Noncommutative geometry appears in string theory with a nonzero B-field [116] (for earlier considerations see [117]-[120])

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \left(g_{ij}\partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j \right). \tag{1.5}$$

Seiberg and Witten have found a limit in which the entire string dynamics is described by a supersymmetric gauge theory on a noncommutative space and shown an equivalence between ordinary gauge fields and noncommutative gauge fields. See [124]-[127] for further developments of these ideas.

Noncommutative Solitons and Instantons

In commutative scalar field theories there is Derrick's theorem, which prohibits the existence of finite energy solitons in two and more spatial dimensions. The proof is based on a simple scaling argument according to which no finite size minimum can exist, since upon shrinking all length scales both kinetic and potential energies decrease. This argument does not work in the presence of a length scale $\sqrt{\theta}$. It has been shown in [128], that for sufficiently large θ stable solitons exist in the noncommutative scalar field theory. For example, for a cubic potential solitons are defined as solutions of the following equation

$$\phi = \phi \star \phi. \tag{1.6}$$

While in a commutative theory this equation would admit only constant solutions, in a noncommutative theory there are countable many solutions, see [129, 130] for more details.

As for noncommutative instantons it was found that the ADHM construction admits a generalization to the noncommutative case [131]. Noncommutative instantons in the limit

$\theta \to 0$ are pure $U(1)$ gauge configurations with a singularity at the origin. For nonzero θ instantons are nonsingular objects with a size of order $\sqrt{\theta}$.

Smoothing of singularities

Solutions of a local field theory usually can be extended to the solutions of noncommutative theory. As a rule in such extensions any singularities disappear due to the position-space uncertainty. In particular, the noncommutative rank 1 theory has non-singular instanton, monopole and vortex solutions.

Solitons in noncommutative theories can be stable even when their commutative counterparts are not, so noncommutativity provides a natural mechanism of stabilization for objects of the size $\sqrt{\theta}$. This behavior is naturally related to nonlocality. In other words, noncommutativity inevitably leads to nonlocality.

String Field Theory

One of the main motivations to construct string field theory (SFT) — an off-shell formulation of a string theory — was a hope to study non-perturbative phenomena in string theory [1]. The Witten SFT action for open bosonic string is

$$S = \frac{1}{2}\int \Phi \star Q\Phi + \frac{1}{3}\int \Phi \star \Phi \star \Phi. \tag{1.7}$$

Open bosonic string has a tachyon that leads to a classical instability of the perturbative vacuum. In the early works by Kostelecky and Samuel [138] it was proposed to use SFT to describe condensation of the tachyon to a stable vacuum. By using the level truncation scheme they have shown that the tachyon potential in open bosonic string has a nontrivial minimum. Further [139], the level truncation method has been applied to study an effective potential of auxiliary fields in the cubic superstring field theory (SSFT) [140, 141, 142]. It was found that some of the low-lying auxiliary scalar fields acquire non-zero vacuum expectation values that was considered as an indication of supersymmetry breaking in this vacuum.

Later on, Sen has proposed [143] to interpret the tachyon condensation as a decay of unstable D-brane. In the framework of this interpretation the vacuum energy of the open bosonic string in the Kostelecky and Samuel vacuum cancels the tension of the unstable space-filling 25-brane. This cancellation has been checked with high accuracy [147, 148].

The cubic open string field theory around the tachyon vacuum, or the vacuum string field theory (VSFT) was proposed in [157] and it is investigated now very intensively [158]-[181]. This VSFT has the same form as (1.7) but with a new differential operator, which after restriction to the space of string fields can be written in the following form

$$Q_{new} = Q + \{\Phi_{vac}, \ \cdot \ \}.$$

The triviality of the cohomology of the new BRST operator Q_{new} around the tachyon vacuum is expected [143]. This means that this theory should not contain physical open string excitations. The evidence in favor of this conjecture was also found by demonstrating the absence of kinetic term for the tachyon field in nonperturbative vacuum [**?**, **?**]. VSFT is conjectured to describe the appearance of closed strings after the open string tachyon condensation. Moreover, it was suggested [157] that after some (maybe singular) field redefinition Q_{new} can be written as a pure ghost operator $\mathcal{Q}$. If this conjecture is true it is instructive to search solutions of VSFT equation of motion

$$\mathcal{Q}\Phi + \Phi \star \Phi = 0 \tag{1.8}$$

in the factorized form $\Phi = \Phi_{matter} \otimes \Phi_{ghost}$, thus obtaining a projector-like equation for matter sector:

$$\Phi_{matter} = \Phi_{matter} \star \ \Phi_{matter}. \tag{1.9}$$

An equation similar to (1.9) has appeared in construction of solitonic solutions in noncommutative field theories in the large coupling limit [128].

As an example of the projector-like solution, there is a state called sliver, $|\Xi\rangle$. This projector was constructed in the oscillator formalism by Kostelecky and Potting [156]. In fact a wedge state as a candidate to solve (1.9) was appeared first in the CFT framework [150]. Then it was at first identified with the sliver numerically [158] and later on by direct calculations [170]. The sliver state is special in a sense that it can be defined in an arbitrary boundary CFT [150].

The projector-like form of eq. (1.9) gave a new impulse for the development of the half-string formalism [1],[190]-[194], which drastically simplifies Witten's $\star$-product. To construct projections more systematically, it is useful to find a matrix representation of open string fields. To see matrix representation in a more transparent way it is fruitful to use the so-called "comma" form of the overlap vertex [191, 192],[165],[170]. The Bogoliubov transformation that makes the sliver to be a new vacuum requires the 3-string vertex to take a comma form [156]. The Bogoliubov transformation relating the sliver and initial Fock vacuum cannot be defined as an operator in the Fock space [177]. This is not surprising since we deal with infinite number of degrees of freedom. One can use the dressing transformation technique [182] to give to the sliver a meaning of a state in the Hilbert space. Just due to the infinite dimensional nature of the star product appear associativity anomaly [136, 137, 213] and twist anomaly [176].

Several tools are used to study VSFT:

1. the functional approach, where string fields are discussed in terms of left-right modes functionals, is developed in [161],[162],

2. the operator approach, where explicit Fock-space operators for left-right parts of the string are explored, is presented in [193].

Using these two descriptions the sliver states and their generalizations are studied in [163, 168, 169, 171, 172]. In these lectures (Section **??**) we will discuss the half-string formalism with main focus on the operator description of the matter sector.

In [157] a simple linear form for $\mathcal{Q} = c_o + \sum f_n(c_n + (-1)^n c_n^\dagger)$ was proposed. Under this assumption the ghost part of the equation admits an analytical investigation [175, 176, 177, 178, 180, 181]. Since an analytic tachyon vacuum solution is unknown, it is impossible to derive this form of VSFT action [157] from the initial Witten action (1.7). However, one can check if the proposed VSFT action passes some tests to be acceptable. In particular, in [173] a classical solutions of VSFT that reproduces the perturbative open string vacuum is constructed (see [176, 181]). Several attempts of getting closed strings from VSFT have been recently performed [179, 180].

This consideration concerns only bosonic cubic SFT. To study the same questions in the superstring case it is worth to work with a formalism which deals with equations of motions in the simple form (1.8) without any modifications. String fields in 0 picture [140], [141],[142] (see also [199]) give us a suitable formalism.

In this lecture notes we would like to explain and discuss some ideas concerning non-commutative geometry in general, as well as noncommutative field theories and string field theories. The material of the lectures is organized as follows. In Section 2, we explain basic mathematical notions of noncommutative differential geometry (NCDG) and give the most important examples of noncommutative spaces used in physical applications. In Section 3, we consider noncommutative quantum field theories emphasizing an issue of their renormalizability. Also we pay a special attention to the problem of the UV/IR mixing.

Let us say a few words about the style which we tried to follow in these lecture notes. We do not pretend to write complete notes textbook on noncommutative quantum filed theories, but would like to present key ideas about these subjects and show explicitly how to perform the calculations.

Sometimes the reader may find our exposition very technical but we try to be explicit and to keep track of instructive computations. However, not to make a text extremely cumbersome we will present in several cases just the results.

We do not pretend to put the complete list of the references on the subject, and apologize to those authors whose papers are not in the list.

2 Noncommutative Geometry. Examples.

In this section we consider the basic ideas of the noncommutative geometry [2] and give several examples of noncommutative spaces.

Noncommutative geometry uses a generalization of the known duality between a space and its algebra of functions, see [2, 3, 85]. If one knows the associative commutative algebra $\mathcal{A}(M)$ of complex-valued functions on topological space M then one can restore the space M. Therefore, all topological notions can be expressed in terms of algebraic properties of $\mathcal{A}(M)$. For example, the space of continuous sections of a vector bundle over M can be regarded as a projective $\mathcal{A}(M)$-module. (We are speaking about left modules. A projective module is a module that can be embedded into a free module as a direct summand). So, the vector bundle over a compact space M can be identified with projective modules over $\mathcal{A}(M)$.

2.1 Axioms of Noncommutative Differential Geometry

Let $\mathcal{A}$ be an abstract associative algebra with the multiplication $\star$

$$\star : \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \tag{2.1}$$

One can interpret $\mathcal{A}$ as an "algebra of functions" on a (nonexisting) noncommutative space.

Let us introduce some basic geometrical notions on this algebra $\mathcal{A}$: external derivative Q (or an algebra $\mathcal{G}$ of derivatives)

$$Q : \mathcal{A} \to \mathcal{A} \tag{2.2a}$$

and integral (linear map)

$$\int : \mathcal{A} \to \mathbb{C}. \tag{2.2b}$$

Noncommutative space is the set

$$\left(\mathcal{A}, Q, \int\right), \tag{2.2c}$$

which satisfies the following axioms:

- associativity

$$A \star (B \star C) = (A \star B) \star C, \quad A, B, C \in \mathcal{A}; \tag{2.2d}$$

- nilpotency

$$Q^2 = 0; \tag{2.2e}$$

- Leibnitz rule

$$Q(A \star B) = (QA) \star B) + A \star (QB), \quad A, B \in \mathcal{A}; \tag{2.2f}$$

- cyclicity property

$$\int (A \star B) = \int (B \star A); \tag{2.2g}$$

- "integration by part"

$$\int (QA) = 0. \tag{2.2h}$$

In the most applications the algebra $\mathcal{A}$ is also equipped with a natural $\mathbb{Z}_2$-grading: $|A \star B| = |A| + |B|$. For example, in Section 3 $\mathbb{Z}_2$-grading is usual boson/ferimon super-grading. One has to modify Leibnitz rule (2.2f) and cyclicity property (2.2g) to be in agreement with $\mathbb{Z}_2$-grading:

$$Q(A \star B) = (QA) \star B + (-1)^{|A|} A \star (QB), \quad A, B \in \mathcal{A}; \tag{2.2f$'$}$$

$$\int (A \star B) = (-1)^{|A||B|} \int (B \star A), \quad A, B \in \mathcal{A} \tag{2.2g$'$}$$

where $(-1)^{|A||B|} = -1$ if both A and B are odd elements and equal 1 otherwise.

If $\mathcal{A}$ admits a Lie algebra of derivatives one can develop a noncommutative bundles theory generalizing the standard theory. Let $\mathcal{G}$ be a Lie algebra of derivatives on $\mathcal{A}$ and $\alpha_1, \ldots, \alpha_d$ be generators of $\mathcal{G}$. Then one can generalize the notion of non-commutative space (2.2c) to the set

$$\left(\mathcal{A}, \mathcal{G}, \int \right), \tag{2.3a}$$

which satisfies axioms (2.2d) and (2.2g) as well as

$$\alpha_i (A \star B) = (\alpha_i A) \star B + A \star (\alpha_i B), \quad A, B \in \mathcal{A}, \quad \alpha_i \in \mathcal{G}; \tag{2.3b}$$

and

$$\int \alpha_i \, A = 0. \tag{2.3c}$$

If V is a projective module over $\mathcal{A}$ (i.e. a "vector bundle over $\mathcal{A}$") one defines a connection in V as the set of linear operators $\nabla_1, \ldots, \nabla_d$ on V satisfying

$$\nabla_i (a\phi) = a \nabla_i(\phi) + \alpha_i(a)\phi \tag{2.4}$$

for all $a \in \mathcal{A}$ and $\phi \in V$, $\quad i = 1, \ldots, d$. The curvature of connection ∇_i is

$$F_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i - f_{ij}^k \nabla_k \tag{2.5}$$

belongs to the algebra of endomorphisms of the $\mathcal{A}$-module V.

A profit from such kinds of constructions is that we can immediately write an action that is invariant under gauge transformations. If we deal with the set of axioms (2.2e)-(2.2f′) we can define the gauge transformation as

$$\delta A = Q\Lambda + A\star\Lambda - \Lambda\star A, \quad A,\Lambda\in\mathcal{A}. \tag{2.6}$$

The gauge invariant action is [1]

$$S[A] = \frac{1}{2}\int A\star QA + \frac{1}{3}\int A\star A\star A. \tag{2.7}$$

This is a generalization of the Chern-Simons action.

In the case of axioms (2.2d), (2.2g), (2.4), (2.3b) and (2.3c) one has a generalization of the Yang-Mills action

$$S = -\frac{1}{4}\int (F_{ij}F^{ij}) \tag{2.8}$$

with

$$F_{ij} = \alpha_i A_j - \alpha_j A_i + A_i\star A_j - A_j\star A_i. \tag{2.9}$$

The action is invariant under the following gauge transformations

$$\delta(A_i) = \alpha_i(\omega) + A_i\star\omega - \omega\star A_i, \tag{2.10}$$

where $\omega\in\mathcal{A}$ is a gauge parameter.

There are several examples of realization of axioms (2.2) and (2.3). We will discuss the following realizations of axioms (2.3):

- The d-dimensional noncommutative torus $\mathbb{T}^d_\theta$.
- The d-dimensional noncommutative Euclidian space $\mathbb{R}_\theta$.

String field theory (SFT) presents an example of axioms (2.2). There is also a realization of axioms (2.2) on matrices [**?**].

2.2 Examples.

2.2.1 Noncommutative torus $\mathbb{T}^d_\theta$.

The d-dimensional noncommutative torus is defined by its algebra $\mathcal{A}_\theta$ with generators $U_1,\dots,U_d$ satisfying the relations

$$U_iU_j = e^{2i\pi\theta_{ij}}U_jU_i$$

where $i,j=1,\ldots,d$ and $\theta=(\theta_{ij})$ ia a real antisymmetric matrix. The algebra $\mathcal{A}_\theta$ is equipped with an antilinear involution $*$ obeying $U_i^*=U_i^{-1}$ (i.e. $\mathcal{A}_\theta$ is a *-algebra). An element of $\mathcal{A}_\theta$ is a power series

$$f=\sum f(p_1,...,p_d)U_1^{p_1}...U_d^{p_d}$$

where $p=(p_1,...,p_d)\in Z^d$ and the sequence of complex coefficients $f(p_1,...,p_d)$ decreases faster than any power of $|p|=|p_1|+...+|p_d|$ when $|p|\to\infty$. The function $f(p)$ is called the symbol of the element f. We denote by U^p the product $U_1^{p_1}...U_d^{p_d}$. Then one has $U^pU^k=e^{2i\pi\varphi(p,k)}U^{p+k}$, where $\varphi(p,k)=\sum\varphi_{ij}p_ip_j$ and φ_{ij} is a matrix obtained from θ after deleting all its elements below the diagonal. To simplify the product rule we replace U^p by $e^{i\pi\varphi(p,q)}U^p$ so that we have

$$U^pU^k=e^{i\pi\theta(p,k)}U^{p+k}.$$

If f and g are two elements of $\mathcal{A}_\theta$,

$$f=\sum_p f(p)U^p,\quad g=\sum_k g(k)U^k$$

then the product

$$fg=\sum_{p,k}f(p)g(k)U^pU^k=$$

$$=\sum_{p,k}f(p)g(k)e^{2i\pi\theta(p,k)}U^{p+k}=\sum_q(f\star g)(q)U^q,$$

where the star-product $(f\star g)(q)$ of symbols $f(p)$ and $g(k)$ is

$$(f\star g)(q)=\sum_p f(p)g(q-p)e^{i\pi\theta(p,q-p)}.$$

The differential calculus on the noncommutative torus is introduced by means of the derivations ∂_j defined as

$$\partial_jU^p=ip_jU^p,\quad j=1,\ldots,d.$$

They satisfy the Leibnitz rule $\partial_j(fg)=\partial_jf\cdot g+f\cdot\partial_jg$ for any $f,g\in\mathcal{A}_\theta$.

The integral of $f=\sum f(p)U^p$ is defined as $\int f=f(0)$, which is in correspondence with the commutative case. The integral has the property of being the trace on the algebra $\mathcal{A}_\theta$, i.e. $\int fg=\int gf$ for any $f,g\in\mathcal{A}_\theta$. Moreover one has

$$\int\partial_jf\cdot g=-\int\partial_jg\cdot f.$$

The gauge field A_i on the noncommutative torus is defined as

$$A_i=\sum_{p\in Z^d}A_i(p)U^p,\quad i=1,\ldots,d.$$

Here $A_i(p)$ is a sequence of $N \times N$ complex matrices indexed by a space-time index. It corresponds to the Fourier representation of the ordinary gauge theory on commutative torus. The gauge field is antihermitian, $A_i^* = -A_i$, or $A_i(p)^* = -A_i(-p)$. A_i is an element of a matrix algebra with coefficients in $\mathcal{A}_\theta$ and its curvature is defined as

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j],$$

where ∂_i is a derivative on $\mathcal{A}_\theta$ defined above. The Yang-Mills action is

$$S = -\frac{1}{4}\int \mathrm{tr}(F_{ij}F^{ij}).$$

This action is invariant under gauge transformations

$$A_i \to \Omega A_i \Omega^{-1} + \Omega\partial_i\Omega^{-1}, \quad F_{ij} \to \Omega F_{ij}\Omega^{-1},$$

where Ω is a unitary element of the algebra of matrices over $\mathcal{A}_\theta$.

2.2.2 Noncommutative flat space $\mathbb{R}_\theta$.

The main property of the noncommutative flat space R_θ^d space is the non-zero commutator of the coordinates

$$[x_i, x_j] = i\theta_{ij}$$

where θ is called the parameter of noncommutativity. We begin with the formulation of the Moyal product as one of the important realization of such multiplication law.

Let us consider one-dimensional quantum mechanics in the Hilbert space of square integrable functions on the real line $L^2(\mathbf{R})$ with ordinary canonical operators of position $\hat{q}$ and momentum $\hat{p}$,

$$[\hat{p}, \hat{q}] = -i\hbar,$$

acting as $\hat{q}\psi(x) = x\psi(x)$, $\hat{p}\psi(x) = -i\hbar d\psi(x)/dx$. If a function of two real variables $f(q,p)$ is given in terms of its Fourier transform

$$f(q,p) = \int e^{i(rq+sp)}\tilde{f}(r,s)drds,$$

then one can associate with it an operator $\hat{f}$ in $L^2(\mathbf{R})$ by the following formula

$$\hat{f} = \int e^{i(r\hat{q}+s\hat{p})}\tilde{f}(r,s)drds.$$

This procedure is called the Weyl quantization and the function $f(q,p)$ is called the *symbol* of the operator $\hat{f}$ [109, 110]. One has the correspondence

$$\hat{f} \longleftrightarrow f = f(q,p).$$

If $\hat{f}^*$ is the Hermitian adjoint to $\hat{f}$ then its symbol is $f^* = f^*(q,p)$

$$\hat{f}^* \longleftrightarrow \hat{f}^*(q,p) = \bar{f}(q,p).$$

If two operators $\hat{f}_1$ and $\hat{f}_2$ are given with symbols $f_1(q,p)$ and $f_2(q,p)$ then the symbol of product $\hat{f}_1\hat{f}_2$ is given by the Moyal product $f_1 \star f_2 = (f_1 \star f_2)(p,q)$ as [109, 110]

$$(f_1 \star f_2)(p,q) = \sum_{\alpha,\beta} \frac{(-1)^\beta}{\alpha!\beta!}(\frac{i\hbar}{2})^{\alpha+\beta}(\partial_q^\alpha \partial_p^\beta f_1(p,q)) \cdot (\partial_q^\beta \partial_p^\alpha f_2(p,q)) =$$

$$= e^{i\hbar L}(f_1(q_1,p_1)f_2(q_2,p_2))|_{q_1=q_2=q, \quad p_1=p_2=p},$$

where

$$L = \frac{1}{2}(\frac{\partial^2}{\partial q_1 \partial p_2} - \frac{\partial^2}{\partial q_2 \partial p_1}).$$

This is also can be written as (we assume $\hbar = 1$)

$$(f_1 \star f_2)(p,q) = \frac{1}{(2\pi)^2}\int e^{2i[(q-q_2)p_1+(q_1-q)p_2+(q_2-q_1)p]}f_1(q_1,p_1)f_2(q_2,p_2)dq_1dq_2dp_1dp_2.$$

Introducing the constant Poisson structure on the plain $\omega^{\mu\nu} = -\omega^{\nu\mu}$, $\omega^{12} = 1$ and notations $x = (q,p)$, $x_i = (q_i,p_i)$, $i = 1,2$, $x_1\omega x_2 = q_1p_2 - q_2p_1$ one obtains

$$(f_1 \star f_2)(x) = \frac{1}{(2\pi)^2}\int f_1(x_1)f_2(x_2)e^{2i(x\omega x_1+x_1\omega x_2+x_2\omega x)}dx_1dx_2.$$

On the other hand the exponential factor can be rewritten as

$$2i(x\omega x_1 + x_1\omega x_2 + x_2\omega x) = 2i\int_\triangle pdq,$$

where $\triangle$ is the triangle on the plane with vertices $x = (p,q)$, $x_1 = (p_1,q_1)$ and $x_2 = (p_2,q_2)$ and there is the path integral representation

$$e^{2i\int_\triangle pdq} = \int e^{2i\int_\triangle pdq}\prod dx(\tau).$$

One integrates over trajectories $x = x(\tau)$, $0 \leqslant \tau \leqslant 1$ in the phase plane with the boundary conditions

$$x(0) = x(1) = x, \quad x(1/3) = x_1, \quad x(2/3) = x_2.$$

Therefore, we conclude that the Moyal product is represented in the form

$$(f \star g)(x) = \int K(x,x_1,x_2)f(x_1)g(x_2)dx_1dx_2$$

and the kernel $K(x, x_1, x_2)$ has a path integral representation

$$K(x, x_1, x_2) = \int e^{iS} \prod dx(\tau),$$

where $S = 2\int pdq$ and the path integral is taken over the trajectories $x = x(\tau)$, $0 \leqslant \tau \leqslant 1$ subject to the boundary conditions specified above. About a connection with deformation quantization see [21] and refs. therein.

2.2.3 Noncommutative sphere S^2_θ.

The last example of the noncommutative space we would like to describe in this section is the noncommutative sphere S^2_θ introduced by Madore [3] (see also [105]). This space is associated with an algebra of operators x^i, $i = 1, 2, 3$ satisfying the relations

$$[x^i, x^j] = i\theta\epsilon^{ijk}x^k$$

and

$$\sum_{i=1}^{3}(x^i)^2 = R^2.$$

The derivative and curvature can be defined in the following form:

$$\partial_i f = [X^i, f]$$

and

$$F_{ij} = i[\partial_i + A_i, \partial_j + A_j] - i[\partial_i, \partial_j].$$

3 Noncommutative Field Theory Models.

In this section we discuss main features of field theory models on R_θ. We pay the most attention to the problem of the renormalizability of these theories and the UV/IR mixing.

3.1 Noncommutative φ^4-model.

The simplest nontrivial model which can demonstrate the most important properties of the noncommutative field theories is a non-commutative φ^4-model [22].

The theory is defined by the action

$$S = S_0 + S_{int} = \int d^dx\,[\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}(\varphi\star\varphi\star\varphi\star\varphi)(x)], \tag{3.1}$$

where $\star$ is a Moyal product

$$(f\star g)(x) = e^{i\xi\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu} f(x)\otimes g(x),$$

ξ is a deformation parameter, $\theta^{\mu\nu}$ is a non-degenerate skew-symmetric real constant matrix, $\theta^2 = -1$, d is even. In this section we deal with four dimensional Euclidean space. Also, we introduce the convenient notation

$$p_1\wedge p_2 \equiv \xi p_1\theta p_2.$$

Let us rewrite the interaction term in the Fourier representation

$$S_{int} = \frac{g}{4!(2\pi)^d}\int dp_1dp_2dp_3dp_4\, e^{-ip_1\wedge p_2 - ip_3\wedge p_4}\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\delta(p_1+p_2+p_3+p_4). \tag{3.2}$$

There are the following distinguished properties of the deformed theory as compared with standard local φ^4_d model:

- There are non-local phase factors in the vertex.
- These factors provide regularization for some loop integrals but not for all [24, 25, 26, 21, 112].
- To have renormalizability the sum of divergences in each order of perturbation theory must have a phase factor already present in the action.

To single out phase factors it is convenient to use the 't Hooft double-line graphs and a notion of planar graphs. For planar graphs the phase factors do not affect the Feynman integrations at all (see [22] for details). In particular the planar graphs have exactly the same divergences as in the commutative theory [112]. There are no superficial divergences

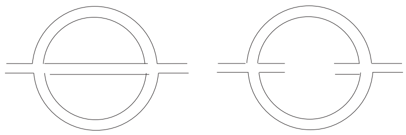


Figure 1: Planar graph with a nonplanar subgraph.

in nonplanar graphs since they are regulated by the phase factors [24, 25, 26, 21, 112, 28]. Moreover, oscillating phases regulate also divergent subgraphs, unless they are not divergent planar subgraphs.

So, at first sight it seems that the proof of renormalizability is rather trivial. One has divergences only in planar graphs, so one can use the fact that planar theory is renormalizable if its scalar counterpart does [132]. In other words, one can expect that it is enough to take the planar approximation of an ordinary theory, find divergences within this approximation and write counterterms in the noncommutative theory as divergent parts of planar graphs multiplied on the phase factors. However, these arguments work only for superficial divergences. The situation is more subtle for divergent subgraphs. The reason is that a planar graph can contain nonplanar subgraphs (see simple example on Fig.1) and these divergences should be also removed. Therefore, renormalizability of the theory (3.1) is not obvious. This gave us [27] a reason to consider explicitly the two-loop renormalization of the theory (3.1) and the authors of [28] to study the renormalizability of (3.1) up to an arbitrary order.

In explicit calculations we will use single-line graphs and symmetric vertices. After the symmetrization of (3.2) we get

$$S_{int} = \frac{g}{3\cdot 4!}\frac{1}{(2\pi)^d}\int dp_1dp_2dp_3dp_4\,\varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\delta(p_1+p_2+p_3+p_4)$$
$$\times\left[\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)+\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)+\cos(p_1\wedge p_4)\cos(p_2\wedge p_3)\right]$$

and the vertex is a sum of three terms (see Fig.2).

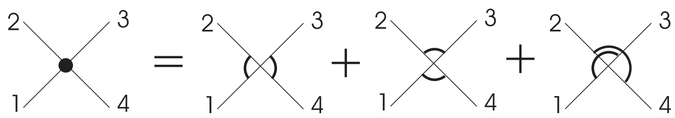


Figure 2: The symmetric vertex.

Here we would like to say a few words about a general technique of calculations in the presence of non-local phase factors in the vertex. In our calculations we always denote external momenta by p_i and loop momentum by k. In order to perform an integration over loop momenta the α-representation and Feynman parametrization are used. The useful formulae are the following [133]

$$\frac{1}{F_1F_2\dots F_n} = (n-1)!\int_0^1 \frac{\delta(x_1+x_2+\dots+x_n-1)dx_1dx_2\dots dx_n}{(x_1F_1+x_2F_2+\dots+x_nF_n)^n},$$

$$\frac{1}{F^s} = \frac{1}{\Gamma(s)}\int_0^\infty \alpha^{s-1}e^{-\alpha F}d\alpha.$$

These expressions are used to pass to the Gauss type integral over internal momentum

$$\int e^{-\omega\mathcal{A}\omega+l\omega}d^D\omega = \frac{\pi^{D/2}}{\sqrt{\det\mathcal{A}}}e^{\frac{1}{4}l\mathcal{A}^{-1}l}. \tag{3.3}$$

Namely, an expression for an arbitrary graph is a product of n propagators with exponential factor coming from trigonometric structure of the vertex. We put the parameter α_i in correspondence to the i-th propagator.

The most general expression for a one loop graph can be written schematically as

$$L_1 = \int d^dk\,\exp\left[-(ak^2+lk+M^2)+ib\theta k\right],$$

where a depends on α_i, l and M depend on α_i and p_j, b depends on p_j. Using (3.3) the integration over k is evident and the result is

$$L_1 = \frac{\pi^{d/2}}{a^{d/2}}e^{-M^2+\frac{1}{4a}(l^2-b^2+2il\theta b)}$$

The next step is an integration over parameters α_i. There are two formulae we work with

$$\int_0^\infty \alpha^{\nu-1}e^{-\mu\alpha}d\alpha = \frac{1}{\mu^\nu}\Gamma(\nu), \tag{3.4}$$

$$\int_0^\infty \alpha^{\nu-1}e^{-\gamma\alpha-\frac{\beta}{\alpha}}d\alpha = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{\beta\gamma}). \tag{3.5}$$

Note that function $\Gamma(\nu)$ has simple poles at points $\nu = 0,-1,-2,\dots$, which correspond to UV divergencies (the momentum space ultraviolet divergencies become a small α divergence). Meanwhile the RHS of (3.5) for $\beta\neq 0$ has no singularities at points $\nu = 0,-1,-2,\dots$, since the factor $e^{-\frac{\beta}{\alpha}}$ produces a regularization for small α. It is easy to show that the integral

$$\int L_1 d\alpha_1\dots d\alpha_n$$

also is well defined if $b\neq 0$.

Now we are going to compute explicitly one-loop counterterms using dimensional regularization $d = 4 - 2\epsilon$. We will also present the explicit form of finite part for two point and four point 1PI functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ in the one loop approximation. We use the standard notations for perturbation expansion of 1PI-functions

$$\Gamma^{(i)} = \sum_n g^n \Gamma^{(i)}_n,$$

$$\Gamma^{(2)} = \Gamma^{(2)}_{f.p.} + \Delta\Gamma^{(2)}$$

and

$$\Gamma^{(4)} = \Gamma^{(4)}_{f.p.} + \Delta\Gamma^{(4)}.$$

The only graph 3a contributes to $\Gamma^{(2)}_1$ and

Figure 3: $\Gamma^{(2)}_1 - \Delta\Gamma^{(2)}_1$.

$$\Gamma^{(2)}_1 = -\frac{g(\mu^2)^\epsilon}{6(2\pi)^d} \int dk \frac{2 + \cos 2p \wedge k}{k^2 + m^2}.$$

Here we have introduced a new parameter μ of dimension of mass to leave the action dimensionless in $4 - 2\epsilon$ dimensions.

We present this expression as a sum of planar and nonplanar parts

$$\Gamma^{(2)}_1 = \Gamma^{(2)}_{1,pl} + \Gamma^{(2)}_{1,npl}$$

where

$$\Gamma^{(2)}_{1,pl} = -\frac{g(\mu^2)^\epsilon}{6(2\pi)^d} \int dk \frac{2}{k^2 + m^2},$$

$$\Gamma^{(2)}_{1,npl} = -\frac{g(\mu^2)^\epsilon}{6(2\pi)^d} \int dk \frac{\cos 2p \wedge k}{k^2 + m^2}.$$

We see that only the planar part has a divergence

$$\Gamma^{(2)}_{1,pl} = \frac{g}{32\pi^2} \frac{2}{3} m^2 \left(\frac{1}{\epsilon} + \psi(2) - \ln \frac{m^2}{4\pi\mu^2} + O(\epsilon) \right) \tag{3.6}$$

and this divergent part is subtracted by the counterterm 3b.

The nonplanar part can be calculated explicitly using α-representation

$$\Gamma^{(2)}_{1,npl} = -\frac{g}{32\pi^2}\frac{2}{3}\sqrt{\frac{m^2}{\xi^2 p^2}}K_1(2m\xi|p|) \tag{3.7}$$

and as has been mentioned above has no divergencies. Note that this representation takes place only for $p \neq 0$.

$\Gamma^{(4)}_2$ is a sum of s-, t- and u-channel graphs, $\Gamma^{(4)}_2 = \Gamma^{(4)}_{2,s} + \Gamma^{(4)}_{2,t} + \Gamma^{(4)}_{2,u}$. The explicit form of $\Gamma^{(4)}_{2,s}$ is

$$\Gamma^{(4)}_{2,s} = \frac{g^2(\mu^2)^{2\epsilon}}{18(2\pi)^d}\int \frac{\mathcal{P}_{4a}(\{p\},k)}{(k^2+m^2)((P+k)^2+m^2)}dk$$

where $P = p_1 + p_2$,

$$\mathcal{P}_{4a}(\{p\},k) = [2\cos(p_1\wedge p_2)\cos(k\wedge P) + \cos(p_1\wedge p_2 + (p_1-p_2)\wedge k)]\times$$
$$[2\cos(p_3\wedge p_4)\cos(k\wedge P) + \cos(p_3\wedge p_4 + (p_4-p_3)\wedge k)].$$

The trigonometric polynomial $\mathcal{P}_{4a}$ can be rewritten in the form

$$\mathcal{P}_{4a} = 2\cos(p_1\wedge p_2)\cos(p_3\wedge p_4) + \sum_j{}' c_j e^{i\Phi^j(p)+ib^j(p)\wedge k} \tag{3.8}$$

where the sum $\sum'$ goes over all j for which linear functions $b^j(p)$ are nonzero for almost all $\{p\}$. This representation is useful to separate the terms which do not depend on internal momentum and, therefore, produce divergencies. Other terms have phase factors which contain k and provide a finite answer. Thus, the only first term in (3.8) contributes to a pole part (see Fig. 4a) and we have

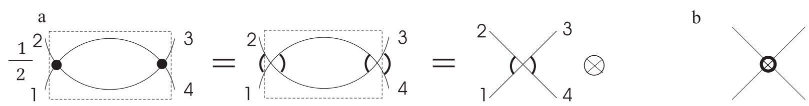


Figure 4: a) $\Delta\Gamma^{(4)}_{2,s}$, b) Cross denotes the 4-vertex counterterm.

$$\Gamma^{(4)}_{2,s} = \frac{g^2}{32\pi^2}\frac{2}{9}\left[\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)\left(\frac{1}{\epsilon}+\psi(1)-\int_0^1 dx\ln\frac{m^2+x(1-x)P^2}{4\pi\mu^2}\right)+\right.$$
$$\left.+\sum_j{}' c_j e^{i\Phi^j(p)}\int_0^1 dx K_0(\sqrt{\xi^2 {b^j}^2(m^2+P^2x(1-x))})e^{ixb^j\wedge P}\right].$$

This representation is well defined only if all b^j are nonzero, i.e. for non exceptional momenta. It is the matter of a simple algebra to sum up the divergent parts of $\Gamma^{(4)}_{2,s}$, $\Gamma^{(4)}_{2,t}$ and $\Gamma^{(4)}_{2,u}$ to obtain a graph with the symmetric vertex.

Therefore, at the one-loop we have

$$\Delta\Gamma^{(2)}_{1l} = \frac{g}{48\pi^2}\frac{m^2}{\epsilon},$$

$$\Delta\Gamma^{(4)}_{1l} = \frac{g^2}{16\pi^2}\frac{1}{9\epsilon}[\cos(p_1\wedge p_2)\cos(p_3\wedge p_4)+\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)+\cos(p_1\wedge p_4)\cos(p_2\wedge p_3)].$$

3.2 Noncommutative Complex φ^4-model.

Here we consider noncommutative quantum field theories of complex scalar field [21] whose commutative analogue $(\phi^*\phi)^2$ is renormalizable in four-dimensional case. There are only two noncommutative structures that generalize a commutative quartic interaction $(\phi^*\phi)^2$:

(a) $\quad \phi^*\star\phi\star\phi^*\star\phi,$

(b) $\quad \phi^*\star\phi^*\star\phi\star\phi.$

In the commutative case the quartic interaction $(\phi^*\phi)^2$ is invariant under local $U(1)$-transformations. In the noncommutative theory we can consider a "deformed" $U(1)$-symmetry ($U\star U^*=1$). One sees that only the structure (a) is invariant under these transformations. Using (a) and (b) we can construct an interaction

$$V(\phi^*,\phi) = A\phi^*\star\phi\star\phi^*\star\phi + B\phi^*\star\phi^*\star\phi\star\phi =$$

$$(A-B)\phi^*\star\phi\star\phi^*\star\phi + \frac{B}{2}(\{\phi^*,\phi\}\star\{\phi^*,\phi\}),$$

where $\{,\}$ is the Moyal anticommutator $\{f,g\}=f\star g+g\star f$. The action of the theory is

$$S=\int d^dx\left(\partial_\mu\phi^*\partial_\mu\phi + m^2\phi^*\phi + \lambda V(\phi^*,\phi)\right). \qquad (3.9)$$

Let us rewrite the interaction term in the Fourier components and symmetrize it, i.e.

$$V(\phi^*,\phi) = \frac{1}{(2\pi)^4}\int dp_1\dots dp_4\delta(\sum p_i)\times$$

$$\times[A\cos(p_1\wedge p_2+p_3\wedge p_4)+B\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)]\phi^*(p_1)\phi(p_2)\phi^*(p_3)\phi(p_4).$$

We would like to analyze counterterms to one loop Feynman graphs in the theory (3.9) and find conditions when this theory is renormalizable by renormalization of one coupling constant λ. We will show that this takes place only in two special cases: $B=0$ and $A=B$.

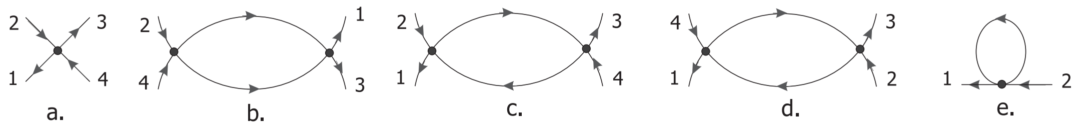


Figure 5: The vertex and one-loop graphs.

Moreover, in the case $B = 0$ the model does not suffer from IR divergencies at least at one-loop insertions level. All one-loop corrections to the interaction are presented on Fig.5:b,c,d. "In" arrows are the fields "ϕ" and "out" arrows are the fields "ϕ^*".

The following analytic expression corresponds to the graph on Fig.5:b

$$\Gamma_{5b} = \frac{N_b(\mu^2)^{2\epsilon}}{(2\pi)^d}\int d^dk \frac{\mathcal{P}_{5b}(p,k)}{(k^2+m^2)((k+P)^2+m^2)}, \tag{3.10}$$

where N_b is a number of graphs (N_b=8), $P = p_2 + p_4 = -p_1 - p_3$ and $\mathcal{P}_{5b}(p,k)$ is the trigonometric polynomial

$$\mathcal{P}_{5b}(p,k) = [A\cos(k\wedge p_2 + (-k-p_2)\wedge p_4) + B\cos(p_2\wedge p_4)\cos(k\wedge P)]$$

$$\times[A\cos(p_1\wedge(-k) + p_3\wedge(k-p_1)) + B\cos(p_1\wedge p_3)\cos(k\wedge P)].$$

The terms containing $\exp[(\dots)\wedge k]$ give a finite contribution to (3.10). Divergencies come from the terms $\Delta\mathcal{P}_{5b}$ of the polynomial $\mathcal{P}_{5b}$

$$\Delta\mathcal{P}_{5b} = \frac{B^2}{2}\cos(p_1\wedge p_3)\cos(p_2\wedge p_4).$$

The graphs Fig.5:c and 5:d mutually differ by permutation of momenta $1\leftrightarrow 3$ only and the analytic expressions for these graphs coincide. For the graph Fig.5:c we have

$$\Gamma_{5c} = \frac{N_c(\mu^2)^{2\epsilon}}{(2\pi)^d}\int d^dk \frac{\mathcal{P}_{5c}(p,k)}{(k^2+m^2)((k+P)^2+m^2)},$$

where N_c is a number of graphs ($N_c = 16$), $P = p_1 + p_2 = -p_3 - p_4$ and $\mathcal{P}_{5c}(p,k)$ is the trigonometric polynomial

$$\mathcal{P}_{5c}(p,k) = [A\cos(p_1\wedge p_2 + (-k-P)\wedge k) + B\cos(p_1\wedge(k+P))\cos(p_2\wedge k)]$$

$$\times[A\cos(p_3\wedge p_4 + (-k)\wedge(k+P)) + B\cos(p_3\wedge k)\cos(4\wedge(k+P))].$$

The polynomial $\Delta\mathcal{P}_{5c}$ that gives contribution to a divergent part of this graph is equal (after symmetrization $p_2\leftrightarrow p_4$) to

$$\Delta\mathcal{P}_{5c} = \cos(p_1\wedge p_2 + p_3\wedge p_4)\left[\frac{A^2}{2} + \frac{B^2}{8}\right] + \frac{AB}{2}\cos(p_1\wedge p_3)\cos(p_2\wedge p_4).$$

We obtain the same answer for the graph on Fig.5:d, i.e.

$$N_d = N_c, \qquad \Delta\mathcal{P}_{5c} = \Delta\mathcal{P}_{5d}.$$

The following condition is equal to one coupling constant renormalizability of the theory (3.9) at one-loop level

$$N_b\Delta\mathcal{P}_{5b} + 2N_c\Delta\mathcal{P}_{5c} = C[A\cos(p_1\wedge p_2 + p_3\wedge p_4) + B\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)],$$

where C is a constant. This condition yields two algebraic equations:

$$N_c\left[A^2 + \frac{B^2}{4}\right] = AC,$$
$$N_b\frac{B^2}{2} + N_cAB = BC.$$

This system is self consistent if

$$B(BN_c - 2AN_b) = 0.$$

The latter equation has two solutions: $B = 0$ and $A = B$. Therefore, one coupling constant renormalizability takes place at one-loop only in two cases

$$B = 0 \quad \text{and} \quad V(\phi^*,\phi) = A(\phi^*\star\phi)^2,$$
$$A = B \quad \text{and} \quad V(\phi^*,\phi) = \frac{B}{2}(\{\phi^*,\phi\})^2.$$

It is straightforward to check the renormalizability of the propagator. We do not perform here explicit calculations of the one-loop correction to the propagator.

Here and in three next subsections we consider the corrections to the interaction only. Corrections to the propagator are more simple and we do not write down explicit calculations. The result is that these corrections do not impose any new restrictions on the coupling constants in the case of the complex scalar field, noncommutative $U(1)$ Yang-Mills theory, noncommutative scalar electrodynamics or $\mathcal{N} = 2$ noncommutative super Yang-Mills theory. Nevertheless, we consider one-loop corrections to the propagator describing the UV/IR mixing in the last subsection of this section.

3.3 Noncommutative Yang-Mills Theory.

In this subsection we explain the very important example of noncommutative theories: the noncommutative Yang-Mills Theory. But in contrast to the ordinary commutative theories we have a nontrivial interaction already with $U(1)$ group. Roughly speaking, this is noncommutative electrodynamics but it exhibits properties of the Yang-Mills theory with non-abelian gauge group. This is because of the deformation of the abelian gauge group. This deformation was described in [34]. We have to replace the ordinary product by the Moyal one. Such a replacement yields the following form of the deformed group element:

$$U = e^{\star i\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\lambda)^{\star n}$$

where the $\star$ denotes the Moyal multiplication.

The $U(1)$ noncommutative connection $\mathcal{D}_\mu$ is the following:

$$\mathcal{D}_\mu = \partial_\mu - ig[A_\mu, \cdot]_\star,$$

where g is a coupling constant and

$$[f,g]_\star(x) = (f \star g)(x) - (g \star f)(x) = 2i \sin\left(\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu\right) f(x) \otimes g(x).$$

The expressions for the curvature is the following

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]_\star.$$

Indeed, the commutator does not vanish and it produces three- and four-point vertices for the gauge field and a nontrivial interaction with the Faddeev-Popov ghosts to be introduced below.

The noncommutative $U(1)$ Yang-Mills (NCYM) action is given by

$$S = -\frac{1}{4}\int d^d x F_{\mu\nu}F^{\mu\nu}, \tag{3.11}$$

where we have dropped the $\star$ because of the property of the integral. This action is invariant under the following transformation:

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger$$

A gauge fixed generating functional is of the form

$$\mathcal{Z}[J,\eta,\bar{\eta}] = \int [dA][dX][d\Theta][d\bar{C}][dC] e^{-S[A,X,\Theta]+S_{GF}[A,X,\Theta]+S_{FP}[A,C,\bar{C}]+\text{sources}},$$

where C and $\bar{C}$ are Faddeev-Popov ghosts, S_{GF} is the gauge fixing term

$$S_{GF}[A_\mu] = -\frac{1}{2\alpha}\int (\partial_\mu A_\mu)^2$$

and S_{FP} is Faddeev-Popov term

$$S_{FP}[A_\mu, C, \bar{C}] = \int \partial_\mu \bar{C}(\partial_\mu C - ig[A_\mu, C]).$$

The generating functional can be computed perturbatively using Feynman graphs [24]-[26]. The quadratic terms (propogators) are identical to the ones appearing in non abelian gauge theories. However, interaction vertices involve non polynomial functions of the momenta (see explicit definition of the Feynman rules in the next subsections).

The generalization of this action to the other groups $U(N)$ is possible. However, Lie algebras of $SU(N)$, $SO(N)$ or $Sp(N)$ are not closed under the Moyal commutator and, therefore, cannot be considered. This fact was shown in [33]. Also this action can be made supersymmetric by adding the correct fermionic and scalar degrees of freedom [34, 25].

Explicit calculations [24]-[26] show that noncommutative U(1) gauge theory is renormalizable at one-loop.

3.4 Noncommutative Scalar Electrodynamics.

Next, we proceed with a consideration of noncommutative scalar electrodynamics in Euclidean space $\mathbb{R}^4$. The classical action is given by

$$\begin{aligned} S &= \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}\star F^{\mu\nu} + (\mathcal{D}_\mu\phi^*)\star(\mathcal{D}^\mu\phi) + V[\phi^*,\phi]\right), \\ V[\phi^*,\phi] &= \lambda^2(a\phi^*\star\phi\star\phi^*\star\phi + b\phi^*\star\phi^*\star\phi\star\phi), \end{aligned} \tag{3.12}$$

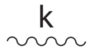
where the covariant derivative is defined by $\mathcal{D}_\mu\phi = \partial_\mu\phi - ig[A_\mu,\phi]_\star$. g and λ are coupling constants and a and b are fixed real numbers. It has been shown in [32], that the pure complex scalar field theory is one-loop renormalizable only if $a = b$ or $b = 0$. The purpose of the present analysis is to find the analogous restrictions on a and b in the case of scalar electrodynamics.

The action (3.12) is invariant under the following gauge transformations

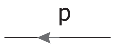
$$\phi \mapsto U\star\phi\star U^\dagger, \quad \phi^* \mapsto U\star\phi^*\star U^\dagger, \quad A_\mu \mapsto U\star A_\mu\star U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger,$$

where U is an element of the noncommutative $U(1)$ group [34]. Note that since our fields are in adjoint representation we could consider the theory with two real scalar fields instead of one complex.

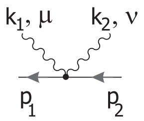
The Feynman rules for the theory (3.12) are presented in Table 1. Solid lines denote the scalar fields, "in" arrows stand for the field ϕ and "out" arrows stand for the field ϕ^*.



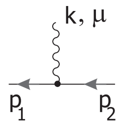
$$D_{\mu\nu}(k) = \frac{1}{k^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right]$$



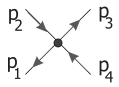
$$D(p) = \frac{1}{p^2}$$



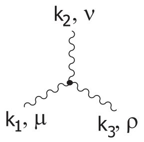
$$4g^2\delta^{\mu\nu}[\cos(k_1\wedge p_1 + k_2\wedge p_2) - \cos(p_1\wedge p_2)\cos(k_1\wedge k_2)]$$



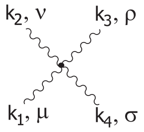
$$2ig(p_1-p_2)_\mu \sin(p_1\wedge p_2)$$



$$-4\lambda^2[a\cos(p_1\wedge p_2 + p_3\wedge p_4) + b\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)]$$



$$-2ig\sin(k_1\wedge k_2)[(k_1-k_2)_\rho\delta_{\mu\nu} + (k_2-k_3)_\mu\delta_{\nu\rho} + (k_3-k_1)_\nu\delta_{\mu\rho}]$$



$$\begin{aligned}4g^2[&(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\sin(k_1\wedge k_2)\sin(k_3\wedge k_4)\\ &+(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\sin(k_1\wedge k_3)\sin(k_2\wedge k_4)\\ &+(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})\sin(k_1\wedge k_4)\sin(k_2\wedge k_3)]\end{aligned}$$

Table 1: Feynman rules for scalar electrodynamics.

We do not specify the Feynman rules for ghosts, since they do not contribute to one-loop graphs with the external matter lines.

All calculations are performed in Landau gauge $\alpha = 0$. This is a convenient choice, since in this gauge a great number of graphs do not have divergent parts. One can prove that the theory is gauge invariant on quantum level (cf. [24, 25, 26, 33]), so our results and conclusions are valid for an arbitrary value of α.

The above mentioned restriction on a and b can come from one-loop corrections to the 4-scalar and 2-scalar-2-gluon vertices. First we consider one-loop corrections to the 4-point scalar vertex. The graphs that have non zero divergent parts in Landau gauge are presented in Figure 6. Using the dimensional regularization ($d = 4-2\epsilon$) we find that the sum of divergent parts of these graphs is equal to

$$\frac{4}{(4\pi)^2\epsilon}[(3g^4+4\lambda^4a^2+\lambda^4b^2)\cos(p_1\wedge p_2+p_3\wedge p_4)+(3g^4+4\lambda^4ab+\lambda^4b^2)\cos(p_1\wedge p_3)\cos(p_2\wedge p_4)]. \tag{3.13}$$

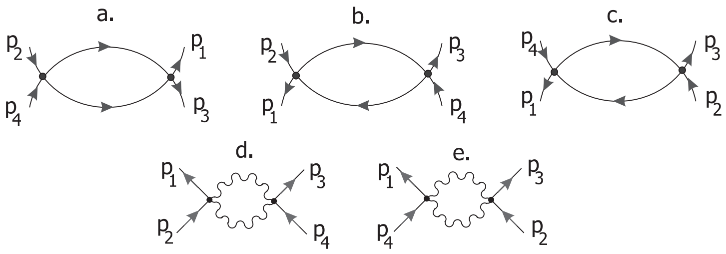


Figure 6: One-loop corrections to the 4-point scalar vertex.

The condition of one-loop renormalizability yields a system of two algebraic equations on a and b

$$\begin{aligned} 3g^4 + 4\lambda^4 a^2 + \lambda^4 b^2 &= ca, \\ 3g^4 + 4\lambda^4 ab + \lambda^4 b^2 &= cb, \end{aligned}$$

where c is a constant. These equations are self-consistent only in the case $a = b$. Therefore the renormalizable potential for scalar electrodynamics has the form

$$V[\phi^*, \phi] = a\frac{\lambda^2}{2}(\{\phi^*, \phi\}_\star)^2, \tag{3.14}$$

where $\{f, g\}_\star = f \star g + g \star f$. Note that in contrast to the pure noncommutative complex scalar field theory [32] we do not have the solution $b = 0$.

Let us turn to an analysis of one-loop corrections to the 2-scalar-2-gluon vertex. The graphs that have non zero divergent parts in the $\alpha = 0$ gauge are presented in Figure 7.

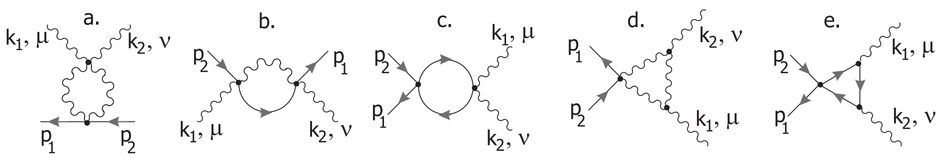


Figure 7: One-loop corrections to the 2-scalar-2-gluon vertex.

The sum of the divergent parts of these graphs is

$$\frac{12}{(4\pi)^2\epsilon} g^4 \delta_{\mu\nu}[\cos(p_1 \wedge k_1 + p_2 \wedge k_2) - \cos(p_1 \wedge p_2)\cos(k_1 \wedge k_2)]. \tag{3.15}$$

Note that the graphs 7c and 7e do contain 4-point scalar vertex and there are terms depending on a and b. However the contributions of these graphs mutually cancel and the sum does not depend on a and b. Therefore, there are no new restrictions on these constants. We see that a counterterm requiring for a cancellation of (3.15) has just the same trigonometric structure as the initial 2-scalar-2-gluon vertex in the action, i.e. this vertex is one-loop renormalizable.

Thus the above analysis leads to the conclusion that noncommutative scalar electrodynamics (3.12) is one-loop renormalizable only if the scalar potential has the anticommutator form (3.14).

3.5 Noncommutative $\mathcal{N}=2$ Super Yang-Mills Theory.

The action for the Euclidean noncommutative $\mathcal{N}=2$ SUSY Yang-Mills theory reads [74, 75]

$$S = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + (\mathcal{D}_\mu \phi_-) \star (\mathcal{D}_\mu \phi_+) - i\chi^* \star \widehat{\mathcal{D}}\chi \right.$$
$$\left. - g\sqrt{2}\chi^* \star (R[\chi,\phi_+]_\star + L[\chi,\phi_-]_\star) - \tfrac{g^2}{2}([\phi_-,\phi_+]_\star)^2 \right), \tag{3.16}$$

where $\mathcal{D}_\mu = \partial_\mu - ig[A_\mu,\cdot]_\star$, $L,R = \frac{1}{2}(1 \pm \Gamma^5)$, $\phi_\pm$ are real scalar fields, χ is a complex four-component spinor[6]. The action (3.16) is a noncommutative generalization of Euclidean $\mathcal{N}=2$ SYM theory. A formulation of noncommutative $\mathcal{N}=2$ supersymmetric theories in terms of superfields was given in [34].

Note that the scalar electrodynamics examined in the previous section can be considered as a bosonic part of $\mathcal{N}=2$ NCSYM. The identification is evident: ϕ and ϕ^* corresponds to ϕ_+ and ϕ_-, respectively. Also we should replace λ by g and take $a=-b=-1$. The fact that fields $\phi_\pm$ are not complex conjugate does not affect the performed calculations. The Feynman rules for the bosonic part of the action (3.16) can be easily obtained from the Feynman rules for scalar electrodynamics (see Table 1) using the above mentioned identification. The Feynman rules for fermion fields are presented in Table 2.

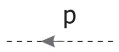
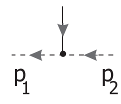
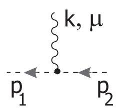
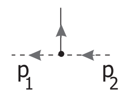
	$S(p) = \frac{\Gamma^\mu p_\mu}{p^2}$		$2ig\sqrt{2}\sin(p_1 \wedge p_2)R$
	$-2ig\Gamma^\mu \sin(p_1 \wedge p_2)$		$2ig\sqrt{2}\sin(p_1 \wedge p_2)L$

Table 2: Feynman rules for fermion fields.

As in the case of scalar electrodynamics we start with the examination of the one-loop corrections to the 4-point scalar vertex. The graphs with fermion loops are presented in Figure 8. The divergencies coming from these graphs are

$$-\frac{32}{(4\pi)^2\epsilon} g^4 \cos(p_1 \wedge p_2 + p_3 \wedge p_4). \tag{3.17}$$

[6]Note that $\mathcal{N}=2$ SYM in Minkowski space contains a complex scalar field.

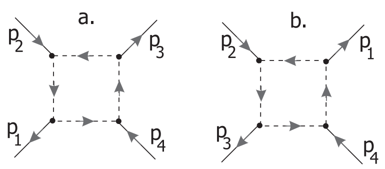


Figure 8: One-loop fermion corrections to the 4-point scalar vertex.

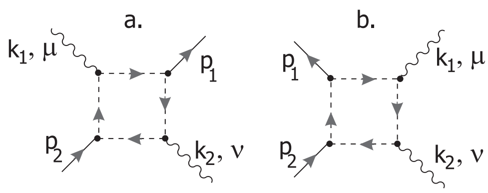


Figure 9: One-loop fermion corrections to the 2-scalar-2-gluon vertex.

Thus, taking into account the contribution (3.13) of the boson graphs we find that the boson and fermion divergencies mutually cancel. The similar result is valid for ordinary $\mathcal{N} = 2$ SUSY Yang-Mills theory where the 4-point scalar vertex is finite at one-loop [134].

Next we calculate one-loop corrections to the 2-scalar-2-gluon vertex. The graphs with fermion loops are presented in Figure 9. The sum of divergent parts of these graphs is

$$-\frac{16}{(4\pi)^2\epsilon}g^4\delta_{\mu\nu}[\cos(p_1\wedge \tag{3.18}$$

Summing the contributions of the boson (3.15) and fermion (3.18) graphs we get

$$-\frac{4}{(4\pi)^2\epsilon}g^4\delta_{\mu\nu}[\cos(p_1\wedge k_1 + p_2\wedge k_2) - \cos(p_1\wedge p_2)\cos(k_1\wedge k_2)].$$

Note that one-loop fermion corrections as well as boson ones restore the trigonometric structure of the initial vertex. So, we conclude that the noncommutative $\mathcal{N} = 2$ $D = 4$ SYM with the action (3.16) is one loop renormalizable.

3.6 UV/IR Problem.

It is evident that our renormalized 1PI functions do not have the limit $\xi \to 0$. Due to this there is a nontrivial mixing of UV and IR divergencies (see [29, 27] for details). It turns out that a type of the UV divergency of the integral corresponding to the Feynman graph in a local model

$$J_{UV}(p) = \int f(k,p)dk$$

is the same as a type of the IR behavior of the integral corresponding to the Feynman graph in a noncommutative model

$$J_{IR}(\xi p) = \int e^{i\xi k\theta p} f(k,p)dk.$$

For example, the logarithmically divergent integral $J_{UV} \sim \log\Lambda$ corresponds to a logarithmic singularity $\log(\xi|\theta p|)$ in J_{IR}, the quadratically divergent integral J_{UV} corresponds to a quadratic singularity $(\xi|\theta p|)^{-2}$ in J_{IR}, and so on.

Indeed, the UV behavior of the Feynman graph is defined by its index of divergence ω. For $\omega > 0$ one has the asymptotic UV behaviour as $\Lambda \to \infty$

$$J_{UV}^{as} = \int_0^\Lambda k^{\omega-1} dk = \frac{1}{\omega}\Lambda^\omega. \tag{3.19}$$

The IR behavior is described by integral

$$J_{IR}^{as} = \int_0^\infty k^{\omega-1} e^{i\xi k} dk \tag{3.20}$$

as $\xi \to 0$. This integral is the Fourier transform of the distribution $k_+^{\omega-1}$

$$J_{IR}^{as} = ie^{i\frac{\pi}{2}(\omega-1)} \frac{\Gamma(\omega)}{(\xi + i0)^\omega}, \tag{3.21}$$

i.e. for $\omega > 0$ one has the correspondence with UV behaviour (3.19).

To see logarithmic singularities on ξ let us note that

$$\int_m^\Lambda k^{-1} dk \sim C \int_{-\Lambda}^\Lambda \frac{d^2k}{k^2+m^2} \sim \log\Lambda \tag{3.22}$$

and the corresponding non local integral is

$$J_{IR}^{as,0}(\xi p) = \int \frac{e^{i\xi p\wedge k}}{k^2+m^2} d^2k. \tag{3.23}$$

The latter integral can be calculated explicitly and the result is

$$J_{IR}^{as,0} = 2\pi K_0(m\xi|p|). \tag{3.24}$$

The modified Bessel function $K_0(z)$ has an expansion

$$K_0(z)(\xi p) = -\ln(z) + \ln(2) - C + O(z^2)$$

where C is Euler constant, i.e. $J_{IR}^{as,0}(\xi p)$ has the logarithmic dependence on ξ.

IR poles appear in the corrections to propagators and can produce IR divergencies in multi-loop graphs even in the massive theories. One has not to worry about logarithms, since in the origin the logarithm is an integrable function. Let us consider several examples.

Example 1. One can calculate explicitly the finite part of the tadpole graph in the noncommutative φ^4 theory. The answer is given by the sum of (3.7) and (3.6). The behavior of (3.7) in the limit $p^2 \to 0$ (the same as the limit $\xi \to 0$) is the following

$$\Gamma^{(2)}_{1,f.p.} \underset{p^2 \to 0}{\sim} \frac{c}{(\xi|p|)^2}.$$

This result can be easily obtained from the series expansion of the modified Bessel function $K_1(z)$

$$K_1(z) = 1/z + \frac{1}{2} z \ln(z) + (\frac{1}{2}C - \frac{1}{4} - \frac{1}{2}\ln(2))z + O(z^3)$$

where C is Euler constant. Caused by this asymptotic there are problems with an IR behavior of graphs with tadpoles. They produce divergence in the IR region if the number of insertions is three or more (compare with an example of ref. [21] in [28]). Thus, theories with a real scalar field have problems with infrared behavior [29, 27] originated in multi one-loop insertions.

Example 2. Another example is a tadpole Fig.5:e in the case of complex scalar field. The analytic expression for this graph is the following

$$\Gamma(p) = \frac{(\mu^2)^\epsilon}{(2\pi)^d} \int d^d k \frac{A + B\cos^2(k \wedge p)}{k^2 + m^2}$$

$$= \frac{(A + \frac{B}{2})(\mu^2)^\epsilon}{(2\pi)^d} \int d^d k \frac{1}{k^2 + m^2} + \frac{B(\mu^2)^\epsilon}{2(2\pi)^d} \int d^d k \frac{e^{i2k\wedge p}}{k^2 + m^2}. \tag{3.25}$$

Integrating this expression over momentum k we obtain

$$\Gamma(p) = \frac{m^{d-2}(\mu^2)^\epsilon}{(4\pi)^{d/2}}(A + \frac{B}{2})\Gamma(1 - d/2) + \frac{B}{(4\pi)^{d/2}} \left[\frac{m}{\xi|\theta p|}\right]^{d/2-1} K_{d/2-1}(2m\xi|\theta p|).$$

For $d = 4$ the second term is singular when $p \to 0$. But in the case $B = 0$ (one of the possible solution of the UV renormalizability) this term disappears and hence there is no IR problem at least at one-loop level.

Example 3. In the case of scalar electrodynamics all corrections to the scalar field propagator are presented in Figure 10a,b,c. For $a = b$ (see notations in the subsection about scalar electrodynamics) the sum of the divergent parts is:

$$\frac{6g^2(\mu^2)^{2\epsilon}}{(2\pi)^4} \int \frac{d^4k}{k^2} \left[1 - \frac{a\lambda^2}{g^2} - \left(1 + \frac{a\lambda^2}{3g^2}\right)\cos(2k \wedge p)\right]$$

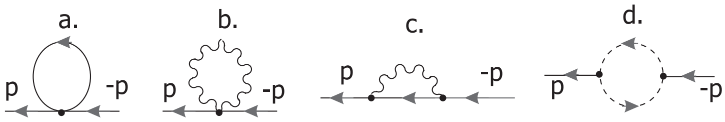


Figure 10: One-loop corrections to the scalar field propagator.

$$-\frac{8g^2(\mu^2)^{2\epsilon}}{(2\pi)^4}\int d^4k\frac{1-\cos(2k\wedge p)}{k^2(p+k)^2}\left[p^2-\frac{(pk)^2}{k^2}\right]$$

where p is an external momentum. A quadratic UV divergence here is removed by a mass renormalization. To remove IR poles one has to impose the condition $\lambda^2 a = -3g^2$.

Computing a one-loop correction to the scalar field propagator in the $\mathcal{N}=2$ $d=4$ NC SYM (3.16) we have one more graph presented in Figure 10d. Summing all contributions we have

$$-4g^2(\mu^2)^{2\epsilon}\int\frac{d^4k}{(2\pi)^4}\frac{1-\cos(2k\wedge p)}{k^2(p+k)^2}\left[p^2+2\frac{(pk)^2}{k^2}\right].$$

Since all UV divergencies are logarithmic there is no IR poles in the case of $\mathcal{N}=2$ $D=4$ NC SYM theory.

All calculations of the one-loop corrections was performed using a dimensional regularization. Now we would like to show that appearance of the IR singularities does not depend on a regularization. Namely, we compute an answer for the tadpole graph in the noncommutative complex scalar field theory using another methods. The expression for this graph is given by (3.25).

First, we perform calculations using Pauli-Villars regularization. To do this we rewrite (3.25) in the following form

$$\Gamma_{reg,PV}(p)=\frac{1}{(2\pi)^d}\int d^4k\,\left[A+B\cos^2(k\wedge p)\right]\left[\frac{1}{k^2+m^2}-\frac{C_1}{k^2+M_1^2}-\frac{C_2}{k^2+M_2^2}\right]$$

with $C_1+C_2=1$ and $C_1M_1^2+C_2M_2^2=m^2$. All integrals are finite and we get the following answer

$$\Gamma_{reg,PV}(p)=\frac{1}{8\pi^2}(A+\frac{B}{2})(M_1^2M_2^2-m^2(M_1^2+M_2^2-m^2))\times$$

$$\times\frac{1}{M_2^2-m^2}\left[\log\frac{M_1^2+M_2^2-m^2}{M_1^2}+\frac{M_2^2}{M_1^2-m^2}\log\frac{M_1^2+M_2^2-m^2}{M_2^2}+\frac{m^2}{M_1^2-m^2}\log\frac{m^2}{M_1^2}\right]$$

$$+\frac{B}{(4\pi)^2}\left(\frac{m}{\xi|\theta p|}K_1(2m\xi|\theta p|)-C_1\frac{M_1}{\xi|\theta p|}K_1(2M_1\xi|\theta p|)-C_2\frac{M_2}{\xi|\theta p|}K_1(2M_2\xi|\theta p|)\right).\qquad(3.26)$$

Note, that for fixed values of M_i there is a cancellation of an IR pole $1/|p\theta|^2$, i.e. the sum of the last three terms has the following asymptotic behavior for small $|p\theta|^2$ up to a numeric factor

$$m^2 \log(2m\xi|\theta p|) - C_1 M_1^2 \log(2M_1\xi|\theta p|) - C_2 M_2^2 \log(2M_2\xi|\theta p|).$$

It is easy to see, that one cannot do this calculation for large M_i.

To remove a regularization (to take limits $M_1 \to \infty$ and $M_2 \to \infty$) one has to do the standard UV renormalization which corresponds to the UV renormalization of the planar part of this graph. The first two lines in the (3.26) contain these UV divergencies of the tadpole.

The non-planar part, i.e. the third line of (3.26), does not require a renormalization. If one takes limit $M_1 \to \infty$ and $M_2 \to \infty$ than two last terms in the third line of (3.26) go to zero since

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad \text{for} \quad z \to \infty.$$

Therefore, after removing a regularization we are left with the following term

$$\frac{B}{(4\pi)^2} \frac{m}{\xi|\theta p|} K_1(2m\xi|\theta p|)$$

that contains IR pole.

The same result is true for the "cut-off" regularization. We have (up to numeric factors)

$$\Gamma_{reg,\Lambda}(p) = \int_0^\infty d\alpha \int d^4k \, \left[(A + \frac{B}{2})\, e^{-\alpha(k^2+m^2)} + B\, e^{ik\wedge p - \alpha(k^2+m^2)} \right] \cdot e^{-\frac{1}{4\alpha\Lambda^2}}$$

$$= \frac{1}{(2\pi)^4} \int_0^\infty d\alpha \left(\frac{\pi}{\alpha}\right)^2 \left[(A + \frac{B}{2})\, e^{-\alpha m^2} + B\, e^{-\alpha m^2 - \frac{\xi^2|\theta p|^2}{4\alpha}} \right] \cdot e^{-\frac{1}{4\alpha\Lambda^2}}$$

where Λ plays the role of a cut-off. For non-planar part we have

$$\frac{B}{(2\pi)^4} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{1}{4\alpha\Lambda_{eff}^2}} \sim m^2 \sqrt{\frac{\Lambda_{eff}^2}{m^2}}\, K_1\left(\frac{m}{\Lambda_{eff}}\right)$$

where

$$\frac{1}{\Lambda_{eff}^2} = \xi^2|p\theta|^2 + \frac{1}{\Lambda^2}.$$

We see that in the limit $\Lambda \to \infty$ the integral remains finite and it has infrared pole. This is in an agreement with an absence of the pole in (3.26) for fixed regularization and a presence of the pole after the regularization is removed.

The mixing of UV and IR divergencies was noted in [29, 27] and as well as consistency of noncommutative quantum field theories were discussed in many papers [27]-[78] in last two years.

As it is seen from examples, theories without fermions have IR singularities (usually poles) at one loop level. There is a possibility to avoid appearance of IR singularities for complex scalar field in one special case: $B = 0$ in our notations. However, it should be checked that this one loop condition is not violated at higher loop levels. In the case of scalar electrodynamics it is possible to avoid an appearance of IR poles imposing additional condition on coupling constant: $\lambda^2 a = -3g^2$.

In [33] the $U(1)$ theory with additional fermion degrees of freedom was analyzed. It was shown that the IR pole modifies the dispersion relation of the photon in the following way

$$E^2 = p^2 - (N_B - N_F)\frac{g^2}{\pi^2}\frac{1}{(\theta p)^2}$$

where N_B and N_F are numbers of bosons and fermions in the adjoint representation. In particular, the energy becomes imaginary for $N_B > N_F$, indicating that such a theory may suffer from instability.

It is surprising that for the pure gauge theory without fermions we have IR poles which are not expected from the UV/IR correspondence found for the scalar models (See [30] for detailed calculations). The most intriguing point is that gauge invariance does not protect from appearence of the IR singularities. However, such singularities have pure noncommutative nature becuase of trigonometric factors in the vertices. This phenomenon is clear technically but not physically.

On the other hand supersymmetric theories do not have pole singularities in the IR region and this was shown explicitly in the third example. This became possible thanks to presence of supersymmetry because there is only one coupling constant which can be tuned and one is unable to remove singularities using this constant. Although logarithms are still present in the $\mathcal{N} = 2$ NC SYM theory, we emphasize that the logarithm function is integrable in the origin and cannot produce non-integrable singularities in multi-loop graphs.

Moreover, it seems likely that the theories with vanishing β-function on the ordinary commutative space are to be free of nonanalytic dependence on θ in noncommutative case. In particular, one could expect that $\mathcal{N} = 4$ NC SYM theory should be finite in both UV and IR regions [33].

Therefore, we see that all theories except $\mathcal{N} = 4$ NC SYM theory have no a smooth limit $\theta \to 0$ in one-loop approximation and some of them have IR poles. This makes the issue of self-consistency of such theories rather questionable.

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