

Triangulated and Derived Categories in Algebra and Geometry

Lecture 1

0. Monoids

Def A monoid is a pair (M, o) , where M is a set and $o: M \times M \rightarrow M$ is a binary operation subject to two properties:

a) neutral element

$$\exists e \in M : \forall m \in M \quad eom = moe = m,$$

b) associativity

$$\forall l, m, n \in M \quad (lom)on = lo(mon).$$

Examples

- 0) trivial monoid $M = \{*\}$,
- 1) natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ under addition,
- 2) positive integers $(\mathbb{Z}_{>0}, \times)$ under multiplication,
- 3) strings in an alphabet $\Sigma =$ free monoid, ← will discuss later

4) End (x) : maps from a set to itself.

Lm There is only one neutral element in a monoid.

Pf Assume $e_1, e_2 \in M$ are both neutral. Then

$$e_1 = e_1 o e_2 = e_2$$

since e_2 is neutral on the right \rightarrow \leftarrow since e_1 is neutral on the left \square

Def An element $m \in M$ is called right (left) invertible if there exists $r \in M$ ($l \in M$) such that $m o r = e$ ($l o m = e$).

Lm If an element $m \in M$ is both right and left invertible, then any left inverse equals any right inverse. In particular, the inverse element is unique (denoted by m^{-1}).

Pf

$$l = l o e = l o (m o r) = (l o m) o r = e o r = r. \quad \square$$

Example Let X be a set, $M = \text{Eud}(X)$, $f \in \text{Eud}(X)$.

- f is right invertible $\Leftrightarrow f$ is surjective,
- f is left invertible $\Leftrightarrow f$ is injective.

Def A group is a monoid every element of which is invertible.

Prk Every group is by definition a monoid. Later we will see how to pass from commutative monoids to groups.

Homomorphisms of monoids

Def Let (M, e, \circ) and (M', e', \circ') be monoids. A homomorphism is a map of sets $\varphi: M \rightarrow M'$ subject to the following conditions:

a) $\varphi(e) = e'$

b) $\varphi(m \circ n) = \varphi(m) \circ' \varphi(n) \quad \forall m, n \in M$

preservation of identity,
preservation of composition.

Examples

- 0) For any monoid $M \exists$ unique morphisms $\{x\} \rightarrow M$ and $M \rightarrow \{x\}$.
1) For any monoid M and $m \in M \exists$ a unique morphism $\varphi: M \rightarrow M$ such that $\varphi(1) = m$. Indeed, put

$$n \cdot m = \underbrace{m \circ m \circ \dots \circ m}_n,$$

then $\varphi(n) = \varphi(1 + 1 + \dots + 1) = \varphi(1) \circ \varphi(1) \circ \dots \circ \varphi(1) = n \cdot m$.
We will later see that M is a free monoid.

- 2) The identity map $\text{Id}: M \rightarrow M$ is a morphism.

Lm Morphisms of monoids preserve left (right) invertible elements.

Pf If $m\sigma = e$, then $e = \varphi(e) = \varphi(m\sigma) = \varphi(m) \circ \varphi(\sigma)$.

Thus, $\varphi(\sigma)$ is a right inverse. \square

Lm If $\varphi: M \rightarrow M'$, $\psi: M' \rightarrow M''$ are morphisms of monoids, so is $\psi \circ \varphi$.

Cancellation property

Def An element $m \in M$ has the right (left) cancellation property if $\forall a, b \in M \quad am = bm \Rightarrow a = b$ ($ma = mb \Rightarrow a = b$).

Ex Right (left) invertible \Rightarrow right (left) cancellation property.

Ex Let X be a set. Which elements of $\text{End}(X)$ have the right (left) cancellation property?

Ex Give an example of a monoid and an element with no cancellation property.

Ex Is it true that if an element has the right cancellation property, then it is right invertible?

Commutation

Def A monoid M is commutative if $\forall m, m' \in M \quad mm' = m'm$.

Def The opposite monoid M^{op} is the monoid (M^{op}, o') such that $M^{\text{op}} = M$ and $m'o'n = nom$.

1. Categories

Mantra A category is a monoid with several objects.

Warning There will be obvious set-theoretic issues which we will not touch in this course. Possible solutions: classes, Grothendieck universes. Very rarely need to worry in practice.

Def A category \mathcal{C} consists of

- 1) a set $\text{Ob } \mathcal{C}$ whose elements are called objects,
- 2) $\forall x, y \in \text{Ob } \mathcal{C}$ a set $\text{Hom}_{\mathcal{C}}(x, y)$ of morphisms from x to y ,
- 3) $\forall x, y, z \in \text{Ob } \mathcal{C}$ a map

$$\text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \longrightarrow \text{Hom}_{\mathcal{C}}(x, z)$$

\cup
 $(g, f) \longmapsto g \circ f$

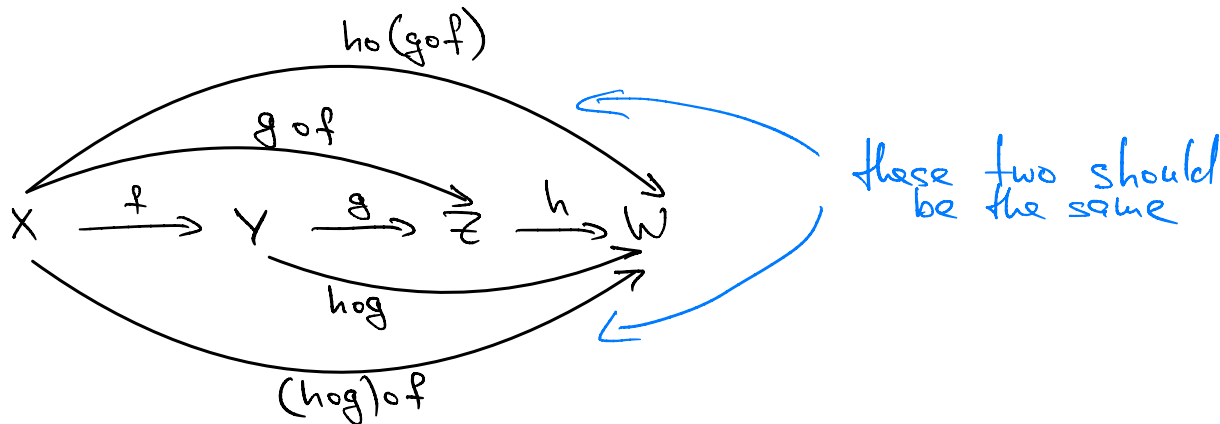
called the composition map,

which satisfy the following:

- a) composition is associative,
 b) $\forall X \in \text{Ob } \mathcal{C} \exists \text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ s.t. $\forall f \in \text{Hom}_{\mathcal{C}}(X, Y)$
 and $\forall g \in \text{Hom}_{\mathcal{C}}(Z, X)$
 $f \circ \text{id}_X = f$ and $\text{id}_X \circ g = g$.

Common notation

- Instead of $X \in \text{Ob } \mathcal{C}$ write $X \in \mathcal{C}$.
- Instead of $\text{Hom}_{\mathcal{C}}(X, Y)$ write $\mathcal{C}(X, Y)$.
- For $f \in \mathcal{C}(X, Y)$ write $f: X \rightarrow Y$. ← target of f
← source of f
- Draw pictures. Associativity:



Examples of categories

- For any set X there is a discrete category.

$$\text{Ob } X = X$$

$$X(x, y) = \begin{cases} \text{id} & , x=y, \\ \emptyset & , x \neq y. \end{cases}$$

- Sets

Objects: sets.

Morphisms: maps of sets.

Variation: finite sets.

- Abelian groups Ab
 - Groups Gp
- } morphisms = homomorphisms

- Let k be a field

$\text{Vect-}k$: vector spaces / k , morphisms - linear maps.

Variation: finite-dimensional.

- Top : topological spaces + continuous maps.

- Let M be a monoid. There is a category \mathcal{M} :
 $Ob \mathcal{M} = \{x, y\}$, $\mathcal{M}(\{x, y\}, \{x, y\}) = M$.

Exc Let \mathcal{C} be a category, $X \in \mathcal{C}$. Then $\mathcal{C}(X, X)$ is a monoid.

- Category of monoids.

Types of morphisms

Def A morphism $f \in \mathcal{C}(X, Y)$ is left (right) invertible if
 $\exists g \in \mathcal{C}(Y, X)$ ($h \in \mathcal{C}(Y, X)$) s.t. $g \circ f = id_X$ ($f \circ h = id_Y$).
Invertible = left + right invertible.

Lem 1) Identity morphisms are unique.

2) If $f: X \rightarrow Y$ is invertible, then any left inverse equals
any right inverse \Rightarrow unique inverse $f^{-1}: Y \rightarrow X$.

Pf 1) $id_X = id_X \circ id'_X = id'_X$.

2) $g = g \circ id_Y = g \circ (f \circ h) = (g \circ f) \circ h = id_X \circ h = h$. \square

Invertible morphisms are called isomorphisms.

Exc The composition of two left (right) invertible morphisms is left (right) invertible. *The same for monoids.*

Exc Find all left (right) invertible morphisms in the categories Sets, Ab, Vect- k .

Def A groupoid is a category with only invertible morphisms.

Def A morphism $f: X \rightarrow Y$ is a (categorical) monomorphism if
 $\forall g, h: Z \rightarrow X \quad f \circ g = f \circ h \Rightarrow g = h.$

Epimorphism:

$\forall g, h: Y \rightarrow Z \quad g \circ f = h \circ f \Rightarrow g = h.$

Exc Every left (right) invertible morphism is a mono(epi)-morphism.

Exc Find all mono(epi)-morphisms in Sets, Ab, Vect- k .

Exc Show that in the category of monoids

$$L: \mathbb{N} \hookrightarrow \mathbb{Z}$$

is a (non-surjective!!!) epimorphism.

Two fundamental constructions

Def Let \mathcal{C} be a category. The opposite category \mathcal{C}^{op} :
 $\text{Ob } \mathcal{C}^{\text{op}} = \text{Ob } \mathcal{C}$, $\mathcal{C}^{\text{op}}(X, Y) = \mathcal{C}(Y, X)$.

"All the arrows are reversed."

Def Let $\mathcal{C}, \mathcal{C}'$ be categories. Their product $\mathcal{C} \times \mathcal{C}'$:

$$\text{Ob}(\mathcal{C} \times \mathcal{C}') = \text{Ob } \mathcal{C} \times \text{Ob } \mathcal{C}', \quad \text{Cartesian product}$$

$$\mathcal{C} \times \mathcal{C}'((x, x'), (y, y')) = \mathcal{C}(x, y) \times \mathcal{C}'(x', y').$$

Def \mathcal{C}' is a subcategory of \mathcal{C} if $\text{Ob } \mathcal{C}' \subset \text{Ob } \mathcal{C}$ and
 $\forall x, y \in \text{Ob } \mathcal{C}' \quad \mathcal{C}'(x, y) \subset \mathcal{C}(x, y)$. If $\mathcal{C}'(x, y) = \mathcal{C}(x, y)$,
the subcategory is full. \leftarrow the composition is induced

Examples $Ab \subset Grp$, $Sets^I \subset Sets$, $Vect^I-k \subset Vect-k$
are full subcategories.

Special objects

Def An object $X \in \mathcal{C}$ is initial (final) if $\forall Y \in \mathcal{C}$
 $\mathcal{C}(X, Y) = \{pt\}$ ($\mathcal{C}(Y, X) = \{pt\}$).

Exc Initial (final) objects, if exist, are unique up
to a unique isomorphism.

Exc \mathcal{C} has an initial (final) object $\Leftrightarrow \mathcal{C}^{op}$ has a final (initial)
object.

Example In $Sets$ the only initial object is \emptyset , final objects —
singletons.

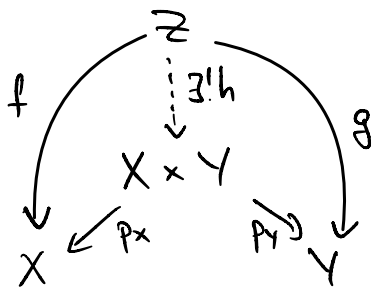
Final objects are also called terminal.

Universal constructions

Def Let $X, Y \in \mathcal{C}$. A product of X and Y is a triple $(X \times Y, p_x, p_y)$, $X \times Y \in \mathcal{C}$, $p_x: X \times Y \rightarrow X$, $p_y: X \times Y \rightarrow Y$ such that $\forall Z \in \mathcal{C}$, $f: Z \rightarrow X$, $g: Z \rightarrow Y$ $\exists! h: Z \rightarrow X \times Y$ such that $f = p_x \circ h$, $g = p_y \circ h$.

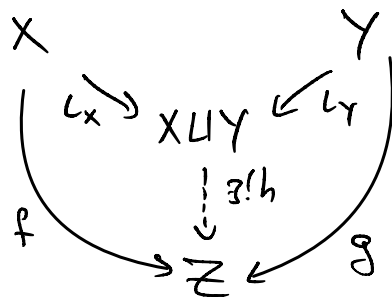
In pictures

Product



product in the opposite category

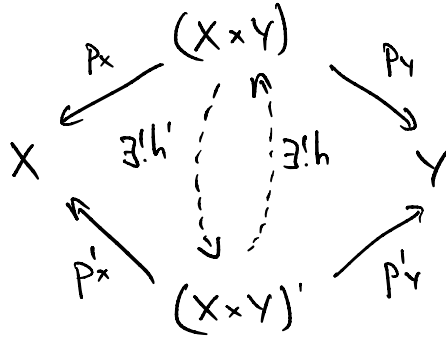
→ Coproduct



A very common argument goes as follows.

Ln If a product exists, it is unique up to ^{natural} isomorphism.

Pf Picture



Since $h \circ h' \neq \text{id}_{(X \times Y)}$ both satisfy

$$p_x = p_x \circ \text{id}_{(X \times Y)} = p_x \circ (h \circ h')$$

$$(p_x \circ h) \circ h' = p'_x \circ h' = p_x$$

and such a morphism is unique,

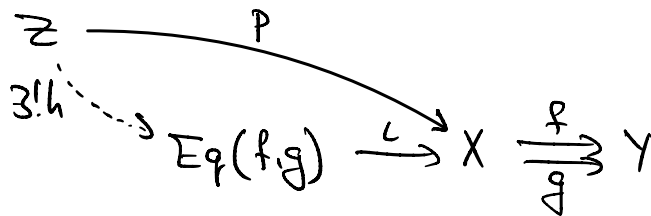
$h \circ h' = \text{id}_{(X \times Y)}$. By symmetry, $h' \circ h = \text{id}_{(X \times Y)'}$.

□

Remk Since coproduct = product in \mathcal{C}^{op} , we get a similar statement for free.

Def Let $f, g: X \rightarrow Y$. Their equalizer is a pair $Eq(f, g)$, $\iota: Eq(f, g) \rightarrow X$ such that $f \circ \iota = g \circ \iota$ and $\forall p: Z \rightarrow X$ s.t. $f \circ p = g \circ p \quad \exists! h: Z \rightarrow Eq(f, g): p = \iota \circ h$.

Picture:



Exc Define and draw the picture for the coequalizer.

Exc Show that if an equalizer of f, g exists, it is unique up to iso.

In Anton's
& = "and".

Examples

Products exist in Sets: $X \times Y$ - Cartesian product.

Coproducts — \cup — : $X \cup Y$ - disjoint union.

Equalizers in Sets:

$$\text{Eq}(f, g) \subset X, \quad \text{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\}. \quad \text{Eq}(f, g) \subset X$$

Coequalizers in Sets:

$\text{Coeq}(f, g) = Y/\sim$, where \sim - smallest equivalence relation s.t. $\forall x \in X \quad f(x) \sim g(x)$.

Exe Describe (co)products and (co)equalizers in Ab .

Problem Very similar definitions, repetitive proofs
"if exists, then unique up to iso."

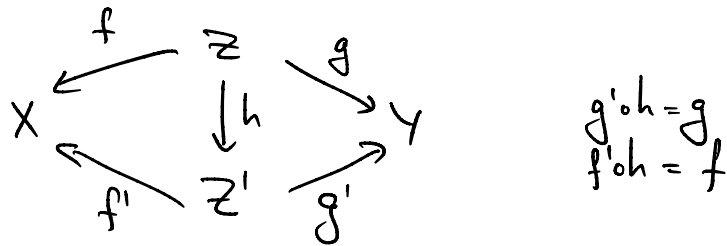
There are various solutions, here is one of them.

Let \mathcal{C} be a category, $X, Y \in \mathcal{C}$.

Define a new category.

Objects: triples (Z, f, g) , where $X \xleftarrow{f} Z \xrightarrow{g} Y$.

Morphisms:

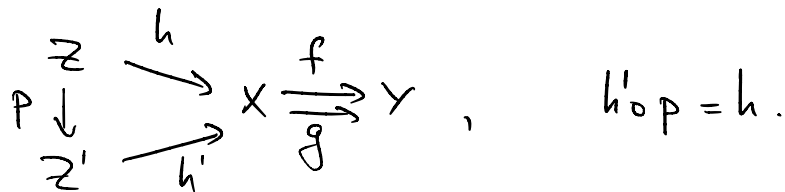


Exc A final object in this category is what we call a product of X & Y .

Let $f, g: X \rightarrow Y$. Define a new category:

objects $Z \xrightarrow{h} X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Y$, $f \circ h = g \circ h$,

morphisms



Exc A final object in this category is $E_g(f, g)$.

Functors

Def A functor F from \mathcal{C} to \mathcal{C}' is a map
 $F: \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{C}'$ and for any $X, Y \in \mathcal{C}$
 $\mathcal{C}(X, Y) \rightarrow \mathcal{C}'(F(X), F(Y))$ such that

a) $\forall X \in \mathcal{C} \quad F(\text{id}_X) = \text{id}_{F(X)},$

b) $F(g \circ f) = F(g) \circ F(f) \quad \forall f: X \rightarrow Y, g: Y \rightarrow Z.$

Exc The identity functor is a functor, the composition of functors is a functor.

Exc Categories + functors as morphisms form a category (up to set-theoretical issues).

The most fundamental example.

Let \mathcal{C} be a category, $X \in \mathcal{C}$.

