Triangulated and Derived Categories in Algebra and Geometry
Lecture 1

0. Monoids

Def A monoid is a pair $(M, \circ)$, where M is a set and $\circ : M \times M \longrightarrow M$ is a binary operation subject to two properties:

a) neutral element
$\exists\, e \in M : \forall m \in M \quad e \circ m = m \circ e = m,$

b) associativity
$\forall\, l, m, n \in M \quad (l \circ m) \circ n = l \circ (m \circ n).$

Examples

0) trivial monoid $M = \{*\}$,
1) natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ under addition,
2) positive integers $(\mathbb{Z}_{>0}, \times)$ under multiplication,
3) strings in an alphabet Σ = free monoid, ⟵ will discuss later

4) End (X) : maps from a set to itself.

<u>Lm</u> There is only one neutral element in a monoid.

<u>Pf</u> Assume $e_1, e_2 \in M$ are both neutral. Then

$$e_1 = e_1 \circ e_2 = e_2$$

since e_2 is neutral on the right

since e_1 is neutral on the left

$\square$

<u>Def</u> An element $m \in M$ is called right (left) invertible if there exists $r \in M$ ($l \in M$) such that

$$m \circ r = e \quad (l \circ m = e).$$

<u>Lm</u> If an element $m \in M$ is both right and left invertible, then any left inverse equals any right inverse. In particular, the inverse element is unique (denoted by m^{-1}).

<u>Pf</u>

$$l = l \circ e = l \circ (m \circ r) = (l \circ m) \circ r = e \circ r = r.$$

$\square$

Example Let X be a set, $M = \mathrm{End}(X)$, $f \in \mathrm{End}(X)$.

- f is right invertible $\iff$ f is surjective,
- f is left invertible $\iff$ f in injective.

Def A group is a monoid every element of which is invertible.

Rmk Every group is by definition a monoid. Later we will see how to pass from commutative monoids to groups.

Homomorphisms of monoids

Def Let $(M, e, \circ)$ and $(M', e', \circ')$ be monoids. A homomorphism is a map of sets $\varphi : M \to M'$ subject to the following conditions:

a) $\varphi(e) = e'$ — preservation of identity,

b) $\varphi(m \circ n) = \varphi(m) \circ' \varphi(n) \quad \forall m, n \in M$ — preservation of composition.

Examples

we drop the prefix homo-

0) For any monoid M $\exists$ unique morphisms $\{*\} \to M$ and $M \to \{*\}$.

1) For any monoid M and $m \in M$ $\exists$ a unique morphism $\varphi : \mathbb{N} \to M$ such that $\varphi(1) = m$. Indeed, put

$$n \cdot m = \underbrace{m \circ m \circ \ldots \circ m}_{n \text{ times}},$$

then $\varphi(n) = \varphi(1 + 1 + \ldots + 1) = \varphi(1) \circ \varphi(1) \circ \ldots \circ \varphi(1) = n \cdot m$.
We will later see that $\mathbb{N}$ is a free monoid.

2) The identity map $\mathrm{Id} : M \to M$ is a morphism.

Lm Morphisms of monoids preserve left (right) invertible elements.

Pf If $m \circ r = e$, then $e = \varphi(e) = \varphi(m \circ r) = \varphi(m) \circ \varphi(r)$.
Thus, $\varphi(r)$ is a right inverse. $\square$

Lm If $\varphi : M \to M'$, $\psi : M' \to M''$ are morphisms of monoids, so is $\psi \circ \varphi$.

Cancellation property

Def An element $m \in M$ has the right (left) cancellation property if $\forall\, a,b \in M$ $\quad a \circ m = b \circ m \Rightarrow a = b \quad (m \circ a = m \circ b \Rightarrow a = b)$.

Ex Right (left) invertible $\Rightarrow$ right (left) cancellation property.

Ex Let X be a set. Which elements of $\mathrm{End}(X)$ have the right (left) cancellation property?

Ex Give an example of a monoid and an element with no cancellation property.

Ex Is it true that if an element has the right cancellation property, then it is right invertible?

Commutation

Def A monoid M is commutative if $\forall\, m, m' \in M$ $\quad m \circ m' = m' \circ m$.

Def The opposite monoid M^{op} is the monoid $(M^{op}, \circ')$ such that $M^{op} = M$ and $m \circ' n = n \circ m$.

1. Categories

<u>Mantra</u> A category is a monoid with several objects.

<u>Warning</u> There will be obvious set-theoretic issues which we will not touch in this course. Possible solutions: classes, Grothendieck universes. Very rarely need to worry in practice.

<u>Def</u> A <u>category</u> $\mathcal{C}$ consists of

1) a set $\mathrm{Ob}\,\mathcal{C}$ whose elements are called <u>objects</u>,
2) $\forall X, Y \in \mathrm{Ob}\,\mathcal{C}$ a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of <u>morphisms</u> from X to Y,
3) $\forall X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$ a map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ \cup\!\!\!\!| & & \\ (g,f) & \longmapsto & g \circ f \end{array}$$

called <u>the composition map</u>,

which satisfy the following:

a) composition is associative,

b) $\forall\ X \in \mathrm{Ob}\,\mathcal{C}\ \ \exists\ id_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ s.t. $\forall\ f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\forall\ g \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$

$$f \circ id_X = f \qquad \text{and} \qquad id_X \circ g = g.$$

Common notation

- Instead of $X \in \mathrm{Ob}\,\mathcal{C}$ write $X \in \mathcal{C}$.
- Instead of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ write $\mathcal{C}(X,Y)$.
- For $f \in \mathcal{C}(X,Y)$ write $f: X \longrightarrow Y$. target of f source
- Draw pictures. Associativity:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rrr, bend left=50, "h\circ(g\circ f)"] \arrow[rrr, bend right=50, "(h\circ g)\circ f"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & Z \arrow[r, "h"] & W
\end{tikzcd}$$

these two should be the same

Examples of categories

- For any set X there is a discrete category.

  $\mathrm{Ob}\, X = X$

  $$X(x,y) = \begin{cases} \mathrm{id} & , x = y, \\ \emptyset & , x \neq y. \end{cases}$$

- Sets

  Objects : sets.

  Morphisms : maps of sets.

  Variation: finite sets.

- Abelian groups $\mathcal{A}b$
- Groups $\mathcal{G}rp$

  $\Big\}$ morphisms = homomorphisms

- Let k be a field

  Vect-k : vector spaces$/k$, morphisms – linear maps.

  Variation: finite-dimensional.

- $\mathcal{T}op$: topological spaces + continuous maps.

- Let M be a monoid. There is a category $\mathcal{M}$:
  $\mathrm{Ob}\,\mathcal{M} = \{*\}$, $\mathcal{M}(\{*\},\{*\}) = M$.

Exc Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$. Then $\mathcal{C}(X,X)$ is a monoid.

- Category of monoids.

Types of morphisms

Def A morphism $f \in \mathcal{C}(X,Y)$ is left (right) invertible if $\exists\, g \in \mathcal{C}(Y,X)$ $(h \in \mathcal{C}(Y,X))$ s.t. $g \circ f = id_X$ $(f \circ h = id_Y)$.
Invertible = left + right invertible.

Lm 1) Identity morphisms are unique.
2) If $f: X \to Y$ is invertible, then any left inverse equals any right inverse $\Rightarrow$ unique inverse $f^{-1}: Y \to X$.

Pf 1) $id_X = id_X \circ id'_X = id'_X$.

2) $g = g \circ id_Y = g \circ (f \circ h) = (g \circ f) \circ h = id_X \circ h = h$. $\square$

Invertible morphisms are called isomorphisms.

Exc The composition of two left (right) invertible morphisms is left (right) invertible. The same for monoids.

Exc Find all left (right) invertible morphisms in the categories Sets, Ab, Vect-k.

Def A groupoid is a category with only invertible morphisms.

Def A morphism $f: X \to Y$ is a (categorical) monomorphism if $\forall\, g, h: Z \to X \quad f \circ g = f \circ h \Rightarrow g = h$.

Epimorphism: $\forall\, g, h: Y \to Z \quad g \circ f = h \circ f \Rightarrow g = h$.

Exc Every left (right) invertible morphism is a mono(epi)-morphism.

Exc Find all mono(epi)-morphisms in Sets, Ab, Vect-k.

Exc Show that in the category of monoids
$\iota : \mathbb{N} \hookrightarrow \mathbb{Z}$
is a (non-surjective !!!) epimorphism.

Two fundamental constructions

Def Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{op}$:
$\mathrm{Ob}\,\mathcal{C}^{op} = \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$.
"All the arrows are reversed."

Def Let $\mathcal{C}, \mathcal{C}'$ be categories. Their product $\mathcal{C} \times \mathcal{C}'$:

$\mathrm{Ob}(\mathcal{C} \times \mathcal{C}') = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}'$, Cartesian product

$\mathcal{C} \times \mathcal{C}'\big((X,X'),(Y,Y')\big) = \mathcal{C}(X,Y) \times \mathcal{C}'(X',Y')$.

Def $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if $\mathrm{Ob}\,\mathcal{C}' \subset \mathrm{Ob}\,\mathcal{C}$ and
$\forall\, X,Y \in \mathcal{C}'$ $\mathcal{C}'(X,Y) \subset \mathcal{C}(X,Y)$. If $\mathcal{C}'(X,Y) = \mathcal{C}(X,Y)$,
the subcategory is full.

the composition is induced

Examples $Ab \subset Grp$, $Sets^f \subset Sets$, $Vect^f\text{-}k \subset Vect\text{-}k$ are full subcategories.

Special objects

Def An object $X \in \mathcal{C}$ is initial (final) if $\forall\, Y \in \mathcal{C}$ $\mathcal{C}(X, Y) = \{pt\}$ $(\mathcal{C}(Y, X) = \{pt\})$.

Exc Initial (final) objects, if exist, are unique up to a unique isomorphism.

Exc $\mathcal{C}$ has an initial (final) object $\Leftrightarrow$ $\mathcal{C}^{op}$ has a final (initial) object.

Example In Sets the only initial object is $\emptyset$, final objects — singletons.

Final objects are also called terminal.

Universal constructions

Def Let $X, Y \in \mathcal{C}$. A product of X and Y is a triple $(X \times Y, p_X, p_Y)$, $X \times Y \in \mathcal{C}$, $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ such that $\forall\ Z \in \mathcal{C}$, $f : Z \to X$, $g : Z \to Y$ $\exists!\ h : Z \to X \times Y$ such that $f = p_X \circ h$, $g = p_Y \circ h$.

In pictures

product in the opposite category

Product

$$
\begin{tikzcd}
 & Z \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, bend left, "g"] & \\
X & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y
\end{tikzcd}
$$

Coproduct

$$
\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[dr, bend right, "f"'] & X \sqcup Y \arrow[d, dashed, "\exists! h"] & Y \arrow[l, "\iota_Y"'] \arrow[dl, bend left, "g"] \\
 & Z &
\end{tikzcd}
$$

A very common argument goes as follows.

Lm If a product exists, it is unique up to natural isomorphism.

Pf Picture

$$
\begin{tikzcd}
& (X\times Y) \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] \arrow[dd, dashed, bend right, "\exists! h'"'] & \\
X & & Y \\
& (X\times Y)' \arrow[ul, "p'_X"] \arrow[ur, "p'_Y"'] \arrow[uu, dashed, bend right, "\exists! h"'] &
\end{tikzcd}
$$

Since $h\circ h'$ & $\mathrm{id}_{X\times Y}$ both satisfy

$$p_X = p_X \circ \mathrm{id}_{X\times Y} = p_X\circ(h\circ h')$$

$$\parallel$$

$$(p_X\circ h)\circ h' = p'_X\circ h' = p_X$$

and such a morphism is unique, $h\circ h' = \mathrm{id}_{X\times Y}$. By symmetry, $h'\circ h = \mathrm{id}_{(X\times Y)'}$.

$\square$

Rmk Since coproduct = product in $\mathcal{C}^{op}$, we get a similar statement for free.

Def Let $f, g : X \to Y$. Their equalizer is a pair $Eq(f,g)$, $\iota : Eq(f,g) \longrightarrow X$ such that $f \circ \iota = g \circ \iota$ and $\forall\ p : Z \longrightarrow X$ s.t. $f \circ p = g \circ p$ $\exists!\ h : Z \to Eq(f,g) : p = \iota \circ h$.

Picture:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! h"'] \arrow[drr, bend left, "p"] & & \\
& Eq(f,g) \arrow[r, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

Exc Define and draw the picture for the coequalizer.

Exc Show that if an equalizer of f & g exists, it is unique up to iso.

In Anton's & = "and".

Examples

Products exist in Sets: $X \times Y$ – Cartesian product.
Coproducts — " — : $X \sqcup Y$ – disjoint union.
Equalizers in Sets:

$$Eq(f,g) \subset X, \quad Eq(f,g) = \{x \in X \mid f(x) = g(x)\}. \qquad Eq(f,g) \subset X$$

Coequalizers in Sets:

$Coeq(f,g) = Y/\sim$, where $\sim$ – smallest equivalence relation s.t. $\forall x \in X \quad f(x) \sim g(x)$.

Exc Describe (co)products and (co)equalizers in Ab.

Problem Very similar definitions, repetitive proofs "if exists, then unique up to iso."

There are various solutions, here is one of them.

Let $\mathcal{C}$ be a category, $X, Y \in \mathcal{C}$.
Define a new category.
Objects: triples (Z, f, g), where $X \xleftarrow{f} Z \xrightarrow{g} Y$.

Morphisms:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ X & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y \end{tikzcd}$$

$g' \circ h = g$
$f' \circ h = f$

<u>Exc</u> A final object in this category is what we call a product of X & Y.

Let $f, g : X \to Y$. Define a new category:

objects $\quad Z \xrightarrow{h} X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $\quad f \circ h = g \circ h$,

morphisms

$$\begin{tikzcd} Z \arrow[d, "p"'] \arrow[dr, "h"] & & \\ Z' \arrow[r, "h'"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

$h' \circ p = h$.

Exc A final object in this category is $Eq(f,g)$.

Functors

Def A functor F from $\mathcal{C}$ to $\mathcal{C}'$ is a map $F: \operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{C}'$ and for any $X, Y \in \mathcal{C}$
$\mathcal{C}(X,Y) \longrightarrow \mathcal{C}'(F(X), F(Y))$ such that
a) $\forall X \in \mathcal{C}$ $\quad F(id_X) = id_{F(X)}$,
b) $F(g \circ f) = F(g) \circ F(f) \quad \forall f: X \to Y,\ g: Y \to Z$.

Exc The identity functor is a functor, the composition of functors is a functor.

Exc Categories + functors as morphisms form a category (up to set-theoretical issues).

The most fundamental example.

Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$.

Define the functor $h^X : \mathcal{C} \to \text{Sets}$ by the rule:

$$\mathcal{C} \ni Y \longmapsto h^X(Y) = \mathcal{C}(X,Y).$$

Let $f : Y \to Z$. Need to construct a map

$$\begin{array}{ccc}
\mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}(X,Z) \\
\shortparallel & & \shortparallel \\
h^X(Y) & & h^X(Z) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
g & \longmapsto & f \circ g
\end{array}$$

$$\begin{tikzcd}
& Y \arrow[r, "f"] & Z \\
X \arrow[ur, "g"] \arrow[urr, bend right, "f \circ g"'] & &
\end{tikzcd}$$

To check that h^X is a functor:

$$\begin{tikzcd}
& Y \arrow[r, "id_Y"] & Y \\
X \arrow[ur, "g"] \arrow[urr, bend right, "id_Y \circ g = g"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[r, "u"] \arrow[rr, bend left, "v \circ u"] & Z \arrow[r, "v"] & W \\
X \arrow[ur, "g"] \arrow[urr, bend right, "u \circ g"'] \arrow[urrr, bend right, "v \circ (u \circ g)"'] & & &
\end{tikzcd}$$