Triangulated & Derived Categories in Algebra & Geometry
Lecture 10

1. Classical derived functors

<u>Motivation</u> Many objects in abelian cat's are described as extensions $0 \to M' \to M \to M'' \to 0$.
Want to study M via M' & M''.

Want to apply functors, but good / interesting functors are rarely exact.

Fortunately, most of them are left / right exact.

Recall: $F \dashv G \Rightarrow$ F is right exact (preserves colimits)
G is left exact (preserves limits)

Assume F is left exact, $0 \to M' \to M \to M'' \to 0$

$$\rightsquigarrow \quad 0 \to F(M') \to F(M) \to F(M'')$$

Classical derived functors extend to a LES.

<u>Ex</u> $F : \text{Mod-}A \to \text{Mod-}A$, $r \in A$
$F(M) = M_r = \{ m \in M \mid mr = 0 \}$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "r"] & M \arrow[r] \arrow[d, "r"] & M'' \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

Snake lemma $\rightsquigarrow$

$$\begin{array}{ccccccccccccc}
0 \to & M'_r & \to & M_r & \to & M''_r & \to & M'/M'r & \to & M/Mr & \to & M''/M''r & \to 0 \\
& \shortparallel & & \shortparallel & & \shortparallel & & \shortparallel & & & & \shortparallel & \\
& F(M') & & F(M) & & F(M'') & & R^1F(M') & & \cdots & & R^1F(M'') &
\end{array}$$

<u>Def</u> A δ-functor (right) is a collection of functors $T^i : \mathcal{A} \to \mathcal{B}$, $i \geq 0$ together with

natural transformations of functors

category of SES's in $\mathcal{A}$ $\rightarrow$ $SES(\mathcal{A}) \longrightarrow \mathcal{B}$

$$(0 \to M' \to M \to M'' \to 0) \longmapsto T^i(M'')$$
$$(0 \to M' \to M \to M'' \to 0) \longmapsto T^{i+1}(M') \quad \Big\downarrow \delta$$

s.t. $\forall\ 0 \to M' \to M \to M'' \to 0$ there is a LES

$$0 \to T^0(M') \to T^0(M) \to T^0(M'') \xrightarrow{\delta} T^1(M') \to T^1(M) \to T^1(M'') \xrightarrow{\delta} \cdots$$

<u>Comments</u> 1) T^0 is left exact (look at the LES)

2) δ being a morphism of functors from $SES(\mathcal{A})$ means that LES is functorial in SES's.

<u>Attempt</u> Look for a δ-functor s.t. $T^0 \simeq F$, where F is left exact.

<u>Problem</u> Might not be unique!

Why? Assume F is already exact.

Two extensions for a δ-functor s.t. $T^0 \simeq F$:

1) Put $T^0 \simeq F$, $T^i = 0$, $i > 0$.

$$0 \to M' \to M \to M'' \to 0$$

$$0 \to F(M') \to F(M) \to F(M'') \to 0 \to 0 \to 0 \to \ldots$$

2) Put $T^i = F$ for all $i \geq 0$, $\delta = 0$.

$$0 \to M' \to M \to M'' \to 0$$

$$0 \to \underbrace{F(M') \to F(M) \to F(M'')}_{\text{exact}} \xrightarrow{0} \underbrace{F(M') \to F(M) \to F(M'')}_{\text{exact}} \xrightarrow{0} \ldots$$

Solution to uniqueness: introduce a universal property.

Define morphisms of δ-functors:

$\varphi : T \to S$ is a collection $\varphi^i : T^i \to S^i$

s.t. $\forall\ 0 \to M' \to M \to M'' \to 0$, $i \geq 0$

$$\begin{tikzcd} T^i(M'') \arrow[r, "\delta"] \arrow[d, "\varphi^i"'] & T^{i+1}(M') \arrow[d, "\varphi^{i+1}"] \\ S^i(M'') \arrow[r, "\delta"] & S^{i+1}(M') \end{tikzcd}$$

(get morphisms of LES's).

<u>Def</u> T is a <u>universal δ-functor</u> if $\forall$ δ-functor S & $\psi : T^0 \to S^0$ $\exists!$ extension to $\varphi : T \to S$ s.t. $\varphi^0 = \psi$.

<u>Def</u> <u>Classical right derived functors</u> of a (left exact) $F : \mathcal{A} \to \mathcal{B}$ is a universal δ-functor $T : \mathcal{A} \to \mathcal{B}$ s.t. $T^0 \simeq F$.

<u>Exc</u> If exists, unique up to iso of δ-functors.

<u>Notation</u> If exists, $\{R^iF : \mathcal{A} \to \mathcal{B}\}$. $R^0F = F$.

<u>Exc</u> F - exact. Check that $R^iF = 0$, $i > 0$ give a universal δ-functor.

One can repeat everything for left δ-functors:

$$\{T_i : \mathcal{A} \to \mathcal{B}\} \quad \text{s.t.}$$

$$0 \to M' \to M \to M'' \to 0$$

$$\rightsquigarrow$$

$$\cdots \to T_1(M'') \xrightarrow{\delta} T_0(M') \to T_0(M) \to T_0(M'') \to 0$$

As usual, reverse the arrows.
Universal left δ-functors satisfy the lifting property for morphisms into.

Define left derived functors for right exact functors.

<u>Notation</u> $\{L_i F : \mathcal{A} \to \mathcal{B}\}$, $i \geq 0$.

2. Projective resolutions

Recall P - projective $\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.

Warning / exc Projective objects in $C(\mathcal{A})$.

Def A complex $X^\bullet \in C(\mathcal{A})$ is split exact if
$X^\bullet \xrightarrow{id} X^\bullet$ is homotopic to 0
(in particular, $H^i(X^\bullet) = 0$ since $id \sim 0$
and must induce the same morphisms
on cohomology)

Show that $X^\bullet \in C(\mathcal{A})$ is a projective object
if and only if 1) X^i are projective, 2) $X^\bullet$ is split exact.

Hint Look at $0 \to P[-1] \to C(id_P) \to P \to 0$
surjective!

Last time we discussed projective resolutions.

Lm Every $M \in \mathcal{A}$ has a projective resolution $\Longleftrightarrow$ $\mathcal{A}$ has enough projectives.

Pf $\Rightarrow$ $\to P_2 \to P_1 \to P_0 \to M \to 0 \quad \Rightarrow \quad P_0 \twoheadrightarrow M$!

$\Leftarrow$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 0 \arrow[dr] & & 0 & & & & \\
 & & & K_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & P_1 \arrow[ur] \arrow[rr] & & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
\arrow[ur] & & & & & K_0 \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$\square$

Lm Let $M, N \in \mathcal{A}$. Assume we are given complexes

$$\begin{array}{ccccccccccc}
\cdots \to & P_2 & \to & P_1 & \to & P_0 & \to & M & \to & 0 \\
 & & & & & & & \downarrow f & & \\
\cdots \to & Q_2 & \to & Q_1 & \to & Q_0 & \to & N & \to & 0
\end{array}$$

P_i – projective just a complex!

Q_i – arbitrary but acyclic.

$\exists!$ lift up to homotopy of $f: M \to N$ to a morphisms of complexes.

Cor In $K^-(\mathcal{A})$ ← homotopy category of complexes bounded from above $\forall$ $P^\bullet$ with projective terms & any $Q^\bullet$ acyclic

$$\mathrm{Hom}_{K^-(\mathcal{A})}(P^\bullet, Q^\bullet) = 0.$$

Pf Inductive proof.

base

$$\begin{tikzcd}
P_1 \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\exists"'] \arrow[dr] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

surjective

inductive step

$$\begin{tikzcd}
P_{i+1} \arrow[r] \arrow[d, dashed, "\exists"'] \arrow[dr, "g"] & P_i \arrow[r] \arrow[d, "f_i"] & P_{i-1} \arrow[d, "f_{i-1}"] \\
Q_{i+1} \arrow[r] & Q_i \arrow[r, "d_i"] & Q_{i-1}
\end{tikzcd}$$

g factors through $\ker d_i = \mathrm{Im}\, d_{i+1}$

So far we only used that P_i are projective, $Q_\bullet$ is exact.

In order to show uniqueness, enough to show that any lift of the zero $M \xrightarrow{0} N$ is null-homotopic.

<u>base</u>

$$\begin{tikzcd}
P_1 \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "0"] \arrow[dl, "0"] & 0 \arrow[dl, "0"] \\
P_2 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Put $\varphi_{-1} : M \to Q_0 = 0$

<u>inductive step</u>

$$\begin{tikzcd}
 & P_i \arrow[r] \arrow[d, "f_i"] & P_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] \arrow[dl, "\varphi_{i-1}"] & P_{i-2} \arrow[d, "f_{i-2}"] \arrow[dl, "\varphi_{i-2}"] \\
Q_{i+1} \arrow[r] & Q_i \arrow[r] & Q_{i-1} \arrow[r] & Q_{i-2}
\end{tikzcd}$$

$f_{i-1} = \varphi_{i-2} d + \underbrace{d \varphi_{i-1}}_{}$, want $f_i = \varphi_{i-1} d + d \varphi_i$

Look at $f_i - \varphi_{i-1} d$, apply d:

$$d f_i - d\underbrace{\varphi_{i-1} d} = f_{i-1} d - (f_{i-1} - \varphi_{i-2} d) d = \varphi_{i-2} d^2 = 0$$

Thus, $f_i - \varphi_{i-2} d$ factors through the kernel of $d : Q_i \to Q_{i-1}$.

Again use P_i - projective to lift it to $Q_{i+1} \twoheadrightarrow \mathrm{Im}\, d_{i+1} = \ker d_i$. $\square$

__Cor__ Any two projective $P'_\bullet \to M$ & $P''_\bullet \to M$ are isomorphic in $K(\mathcal{A})$.

__Pf__

$$\begin{tikzcd}
M \arrow[r, "id"] & M \arrow[r, "id"] & M \\
P'_\bullet \arrow[u] \arrow[r, "\varphi"] & P''_\bullet \arrow[u] \arrow[r, "\psi"] & P'_\bullet \arrow[u]
\end{tikzcd}$$

$\psi \circ \varphi$ lifts id $\Rightarrow$

$\Rightarrow$ $\psi \circ \varphi \sim id_{P'_\bullet}$ since the latter is an (obvious) lift of $M \xrightarrow{id} M$. $\square$

In other words,

$\mathcal{A}$ is equivalent to the full subcategory in $K(\mathcal{A})$ consisting of objects $P^\bullet$ s.t. $P^i = 0$, $i > 0$, P^i - proj for all $i \in \mathbb{Z}$, $H^i(P^\bullet) = 0$, $i \neq 0$.

<u>Lm</u> Projective resolutions can be chosen in accordance with SES's: given $0 \to M' \to M \to M'' \to 0$ and $P'_\bullet \to M'$ & $P''_\bullet \to M''$ – proj. res's $\exists$ a proj. resol of M & a SES

$$\begin{tikzcd}
\boxed{0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0} \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

SES in $C(\mathcal{A})$

<u>Pf</u> If P exists, then $P_n \simeq P'_n \oplus P''_n$.

Indeed, $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ is a SES $\Longleftrightarrow$

$\Longleftrightarrow$ $0 \to P'_n \to P_n \to P''_n \to 0$ for all n.

But P''_n – projective $\Rightarrow$ splits.

Now we just need to construct a proj. resol of M with the terms $P'_n \oplus P''_n$.

Inductive construction.

From snake lemma we conclude that $P_0' \oplus P_0'' \to M$ is surjective. Pass to kernels & proceed.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
P_0' \arrow[r] \arrow[d] \arrow[dr] & M' \arrow[r] \arrow[d] & 0 \\
P_0' \oplus P_0'' \arrow[r] \arrow[d] & M \arrow[d] & \\
P_0'' \arrow[r] \arrow[d] \arrow[ur, dashed] & M'' \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

UP of P_0'' w/r to $M \twoheadrightarrow M''$

∎

3. Construction of classical derived functors

Thm Define $L_iF(M)$ by the formula

$$L_iF(M) = H_i(F(P_\bullet)),$$

complex obtained by term-wise application of F.

where $P_\bullet \to M$ is a projective resolution.

Then these form universal (left) δ-functors.

Correctness

↳ on objects:

since any two proj. resolutions are isomorphic in $K(\mathcal{A})$, $L_iF(M)$ is well-def up to isomorphism.

must fix a proj. resol for every object.

↳ on morphisms:

the lemma about the lift (again)

homotopic morphisms give the same morphisms on homology

↳ connecting homomorphisms:

the previous lemma gave us

$$0 \to M' \to M \to M'' \to 0$$

$$\wr$$

$$0 \to P_\bullet' \to P_\bullet \to P_\bullet'' \to 0$$

Since $0 \to P_n' \to P_n \to P_n'' \to 0$ all split,

$0 \to F(P_n') \to F(P_n) \to F(P_n'') \to 0$ are also exact (actually, split)

$0 \to F(P_\bullet') \to F(P_\bullet) \to F(P_\bullet'') \to 0$ is a SES

The LES of homology $\rightsquigarrow$ LES

$$\cdots \to L_1F(M) \to L_1F(M'') \to L_0F(M') \to L_0F(M) \to L_0F(M'') \to 0$$

So far: $\{L_iF\}$ give us a δ-functor.

<u>Exc</u> Check naturality. Will need to ensure

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0 \end{tikzcd}$$

$$\left.\begin{tikzcd} 0 \arrow[r] & P_\bullet' \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d, dashed] & P_\bullet'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Q_\bullet' \arrow[r] & Q_\bullet \arrow[r] & Q_\bullet \arrow[r] & 0 \end{tikzcd}\right\}$$ proj. resol'ns.

<u>Comment</u> By reversing arrows, you get functors $R^iF : \mathcal{A} \to \mathcal{B}$ for any

left exact $F: \mathcal{A} \to \mathcal{B}$ by putting

$R^iF(M) = H^i(F(I^\bullet))$, where $M \to I^\bullet$ is an injective resolution.

$$(0 \to M \to I^0 \to I^1 \to \dots)$$

<u>Lm</u> $L_0F(M) = F(M)$!

$$F(P_1) \to F(P_0) \to 0 \qquad \text{cohom} \simeq F(M)$$

since F is right exact

$$P_1 \to P_0 \to M \to 0$$

$$\leadsto$$

$$F(P_1) \to F(P_0) \to F(M) \to 0.$$

4. Universality

Dimension shifting

Consider a SES $0 \to K \to P \to M \to 0$, where P-proj.

P-proj $\Rightarrow$ $0 \to P \to 0$ is a proj. resol of P $\Rightarrow$
$\Rightarrow$ $\forall$ F-right exact $L_iF(P) = 0$, $i > 0$.

The associated LES of $L_i F$:

$$\begin{array}{l} \hookrightarrow F(K) \to F(P) \to F(M) \to 0 \\ \quad\quad \delta \\ \hookrightarrow L_1F(K) \to 0 \to L_1F(M) \to 0 \\ \quad\quad \sim \\ \cdots \to 0 \to L_2F(M) \end{array}$$

In particular, $L_iF(M) \simeq L_{i-1}F(K)$ for $i>1$,

$$L_1F(M) = \ker\big(F(K) \to F(P)\big).$$

Exc An object Q is F-acyclic if $L_iF(Q)=0$, $i>0$.
An F-acyclic resolution $Q_\bullet \to M$ is a resol with F-acyclic terms.

Show that if $Q_\bullet \to M$ is an F-acyclic resol'n, then $L_iF(M) = H_i(F(Q_\bullet))$.

Exc Show that if $G : \mathcal{B} \to \mathcal{C}$ is exact, then $G \circ L_i F \simeq L_i(G \circ F)$.

Thm $L_i F$ are a universal δ-functor (left). Thus, they form the left derived functors of F.

Pf Let $\{T_i : \mathcal{A} \to \mathcal{B}\}$ be a left δ-functor, $\varphi_0 : T_0 \to L_0 F$. Want to find a lift.

$$T_i(M) \to L_i F(M) \quad \text{for all } i.$$

Inductively: $0 \to K \to P \to M \to 0$

Recall that $L_i F(P) = 0$ for all $i > 0$

unique lift!

$$\begin{tikzcd}
T_{i+1}(P) \arrow[r] & T_{i+1}(M) \arrow[r] \arrow[dr] \arrow[d, bend left] & T_i(K) \arrow[r] \arrow[d, "\varphi_i"] & T_i(P) \arrow[r] \arrow[d] & T_i(M) \arrow[r] \arrow[d, "\varphi_i"] & {} \\
0 \arrow[r] & L_{i+1}(M) \arrow[r] & L_i F(K) \arrow[r] & L_i F(P) \arrow[r] & L_i F(M) \arrow[r] & {}
\end{tikzcd}$$

Remains to check that the constructed morphisms commute with δ.

Exc Use the same trickery to show that it's true. $\square$

Rmk The only thing we used is the following: $\forall M \in \mathcal{A}$ $\forall i > 0$ $\exists$ a surjection $Q \to M \to 0$ s.t. $L_i F(Q) = 0$. Such δ-functors are called ~~erasable~~ (look up the French word). effacable
The proof shows that any ~~erasable~~ is universal.

Next Many examples (sheaves especially).
Spectral sequences.