Triangulated & Derived Categories in Algebra & Geometry
Lecture 11

0. Recall about derived functors

Problem: many nice functors are not exact.

Typical example

A - ring (say, commutative)

$A \longrightarrow B$ - A-algebra

$$-\otimes_A B : \begin{array}{ccc} A\text{-Mod} & \longrightarrow & B\text{-Mod} \\ \cup & & \cup \\ M & \longmapsto & M \otimes_A B \end{array}$$

In general, $-\otimes_A B$ is only right exact.

How to apply to SES's?

δ-functor: SES $\rightsquigarrow$ LES's

Left derived functors of a right exact F is a universal δ-functor (left) $\{L_i F : \mathcal{A} \to \mathcal{B}\}$ s.t. $L_0 F \simeq F$.

$$0 \to M' \to M \to M'' \to 0 \quad \rightsquigarrow$$

$$\rightsquigarrow \quad \ldots \to L_2 F(M'') \xrightarrow{\delta} L_1 F(M') \to L_1 F(M) \to \ldots$$

Similar story for left exact functors.

<u>Thm</u> If $\mathcal{A}$ has enough injectives, then $R^i F$ exist for any left exact $F : \mathcal{A} \to \mathcal{B}$.

Construction: $M \to I^\bullet$ ← choose an injective resolution

Put $R^i F(M) = H^i(F(I^\bullet))$, where

$$F(I^\bullet) = 0 \to F(I^0) \to F(I^1) \to \ldots$$

Main examples to keep in mind: $\otimes$, Hom.

For simplicity - comm. rings.

Given $M \in A\text{-Mod}$, consider

$$M \otimes_A - \ , \quad - \otimes_A M \quad - \text{ right exact}$$

$$\mathrm{Hom}_A(M,-) \ , \quad \mathrm{Hom}_A(-,M) \quad - \text{ left exact}$$

The corresponding (classical) derived are

$$\mathrm{Tor}_i(M,N) \quad \& \quad \mathrm{Ext}^i(M,N).$$

<u>Q:</u> Both are functors in two arguments!

$\mathrm{Tor}_i(M,N)$ - two choices

$P_\bullet \to M$ - proj. put $\mathrm{Tor}_i(M,N) = H_i(P_\bullet \otimes_A N)$

$Q_\bullet \to N$ - proj put $\mathrm{Tor}_i(M,N) = H_i(M \otimes_A Q_\bullet)$

Which one to choose?

For $\mathrm{Ext}^i(M,N)$:

$$P_\bullet \to M \qquad\qquad H^i(\mathrm{Hom}(P_\bullet, N))$$

$$N \to I^\bullet \qquad\qquad K^i(\mathrm{Hom}(M, I^\bullet))$$

What about compositions?

$$A \to B \to C$$

<u>Want</u>: $- \otimes_A B$, then $- \otimes_B C$

Can we get any information about the derived functors of the comp out of the derived functors of those being composed.

Answer to both questions – spectral sequences!

1. Double complexes

Def A double complex is a collection $\{E^{p,q}\}$, $p\in\mathbb{Z}, q\in\mathbb{Z}$ of objects in $\mathcal{A}$ together with $d_h : E^{p,q} \to E^{p+1,q}$ & $d_v : E^{p,q} \to E^{p,q+1}$ s.t.

1) $d_h \circ d_h = 0$
2) $d_v \circ d_v = 0$
3) every square (anti) commutes:

$$\begin{tikzcd} E^{p,q+1} \arrow[r, "d_h"] & E^{p+1,q+1} \\ E^{pq} \arrow[u, "d_v"] \arrow[r, "d_h"] & E^{p+1,q} \arrow[u, "d_v"'] \end{tikzcd}$$

$$d_v d_h + d_h d_v = 0$$

Warning I wrote (anti), sometimes commutes.

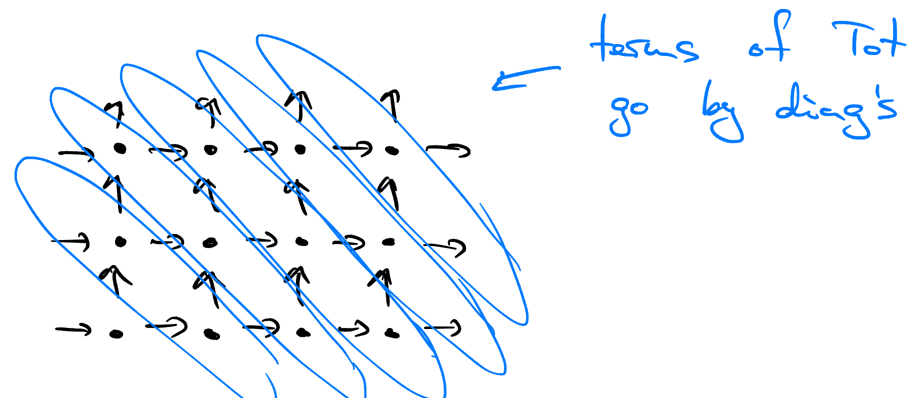
Pass from one to another by changing the sign of d_v in every even/odd column.

Observation Put $E^n = \bigoplus_{p+q=n} E^{pq}$, $d = d_h + d_v$. Then $d^2 = 0$.

$$(d_h + d_v)^2 = d_h^2 + d_v^2 + (d_h d_v + d_v d_h)$$

Thus, one gets a complex called the totalization of $E^{\bullet,\bullet}$, denoted by $\mathrm{Tot}(E^{\bullet,\bullet})$.

Rmk There are two types of totalization: could have taken $\prod E^{pq}$. If there are infinitely many nonzero terms on some diagonal, these are different, different cohomology as well!



Example Let $f: M^\cdot \to N^\cdot$ be a morphism in $\mathcal{C}(\mathcal{A})$.
Produce $E^{\cdot,\cdot}$:

$$
\begin{tikzcd}
 & 0 \arrow[r] & 0 \arrow[r] & 0 & & \\
\cdots \arrow[r] & N^n \arrow[u] \arrow[r, "d_N"] & N^{n+1} \arrow[u] \arrow[r, "d_N"] & N^{n+2} \arrow[u] \arrow[r] & \cdots & E^{n,0} \\
\cdots \arrow[r] & M^n \arrow[u, "f^n"] \arrow[r, "-d_M"] & M^{n+1} \arrow[u, "f^{n+1}"'] \arrow[r, "-d_M"] & M^{n+2} \arrow[u, "f^{n+2}"'] \arrow[r] & \cdots & E^{n,-1} \\
 & 0 \arrow[u] & 0 \arrow[u] & & &
\end{tikzcd}
$$

changed the signs of d in this row

$\mathrm{Tot}(E^{\cdot,\cdot})$ is nothing but $C(f)$ - cone

$C(f)^n = N^n \oplus M^{n+1}$, the diff is given by the same formula.

2. The spectral sequence of a double complex

Assumption the double complex $E^{\cdot,\cdot}$ is bounded "enough".

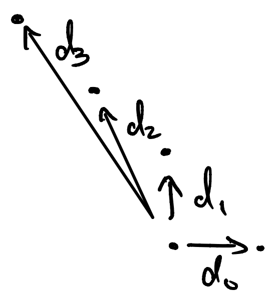
Will assume that all the terms out of the first quadrant are zero: $E^{pq} = 0$ if $p<0$ or $q<0$.

Def A spectral sequence (horizontal) is a collection of pages, where each page is a collection of objects E_r^{pq} (r - page number, $r \geq 0$).

Every page comes with a differential:

$$d_r : E_r^{pq} \longrightarrow E_r^{p-r+1,\, q+r}$$

Picture



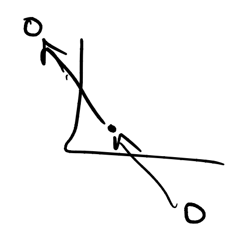
There is an isomorphism between E_{r+1}^{pq} & the coh at E_r^{pq} w/r to d_r.

<u>Thm</u> Let $E^{\cdot,\cdot}$ be a double complex concentrated in the I^{st} quadrant.

There is a spectral sequence with the property that $E_0^{pq} = E^{pq}$, $d_0 = d_h$, $d_1 = d_v$
(E_1^{pq} by def must be $H^p(E^{\cdot,q})$ q-th row)

Moreover, there is a filtration on $H^{p+q}(\mathrm{Tot}(E^{\cdot,\cdot}))$ with subquotient isomorphic to E_∞^{pq}.

(Rmk that for $N \gg 0$ d_N in the spectral sequence goes out of bounds of the quadrant.
Thus, for $N \gg 0$
E_N^{pq} stabilizes & we denote it by E_∞^{pq}).



One says that $E_r^{pq} \Rightarrow H^{p+q}(\mathrm{Tot}(E^{\bullet\bullet}))$ converges to cohom of Tot.

<u>Observation</u> We can flip the picture, define vertical SS's. The same should hold!

We can deduce information about things by comparing these spectral sequences!

3. Examples

1) Cartesian squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \iff \begin{array}{c} 0 \to A \to B \oplus C \to D \\ \text{is left exact} \end{array}$$

<u>Rmk</u>: up to a sign change $0 \to A \to B\oplus C \to D = \mathrm{Tot}\left(\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \right)$

Assume the square is Cartesian, then Tot can have cohomology only in deg 2.

Spectral sequence:

E_0^{pq}
$$\begin{array}{cc} A & B \\ f\downarrow & g\downarrow \\ C & D \end{array}$$

E_1^{pq}
$$\ker f \xrightarrow{d_1} \ker g$$
$$\operatorname{Coker} f \xrightarrow{d_1'} \operatorname{Coker} g$$

E_2^{pq}
$$\begin{array}{cccccc}
 & 0 & 0 & 0 & 0 & \\
0 & \ker d_1 & \operatorname{Coker} d_1 & 0 & 0 & \\
0 & 0 & \ker d_1' & \operatorname{Coker} d_1' & 0 & \\
0 & 0 & 0 & 0 & 0 &
\end{array}$$

d_2

$E_r^{pq} = E_2^{pq}, \; r \geq 2$

say that the SS degenerates at page 2

<u>But</u> We know that there is a filtration on $H^{p+q}(\mathrm{Tot})$ with ass. quotients $E_\infty^{pq} = E_2^{pq}$.

$0 = H^0(\mathrm{Tot}) \simeq E_2^{00} = \ker d_1$. $\Rightarrow$ $\ker d_1 = 0$

$\ker f \hookrightarrow \ker g$

$0 = H^1(\mathrm{Tot})$ has a filtration with quotients $\mathrm{Coker}\, d_1$ & $\ker d_1'$

Thus, $\ker d_1' = 0 \Rightarrow \mathrm{Coker}\, f \hookrightarrow \mathrm{Coker}\, g$

$\mathrm{Coker}\, d_1 = 0 \Rightarrow \ker f \twoheadrightarrow \ker g$.

<u>Combining</u>

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd} \Rightarrow \begin{array}{l} \ker f \simeq \ker g \\ \mathrm{Coker}\, f \hookrightarrow \mathrm{Coker}\, g \end{array}$$

2) Snake lemma

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d, "g"'] & M'' \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}$$

Begin with the SS

$E_1^{pq} = 0$ for all p, q : rows are exact!

Thus, $H^\bullet(\mathrm{Tot}) = 0$.

The other spectral sequence.

E_1^{pq} $\quad 0 \to \operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h \to 0$

$0 \to \operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h \to 0$

$$E_2^{pq}\qquad \begin{tikzcd} 0 & ? & ? & ?? & 0 \\ 0 & ?? \arrow[urr] & ? & ? & 0 \end{tikzcd}$$

We already know that $H^\bullet(\mathrm{Tot}) = 0$. All ? terms will survive / are stable. Thus, they must be zero! Also $d_2 : ?? \to ??$ must be an isom!

$$d_2 : \ker(\operatorname{Coker} f \to \operatorname{Coker} g) \xrightarrow{\sim} \operatorname{Coker}(\ker g \to \ker h).$$

You get exactly the Snake lemma sequence!

3) LES of cohomology

<u>Exc</u> Deduce the cohomology LES given $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$.

4) 5-lemma (<u>Exc</u>)

5) Grothendieck spectral sequence

<u>Thm</u> Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian, $\mathcal{A}$ & $\mathcal{B}$ have enough injectives, F takes injective objects into G-acyclic objects ($B \in \mathcal{B}$ s.t. $R^iG(B) = 0$, $i > 0$).

Then there is a spectral sequence converging to $R^{p+q}(G \circ F)$ whose second page is $R^pF \circ R^qG$.

4. Grothendieck spectral sequence

Cartan - Eilenberg resolutions

<u>Def</u> Let $A^\bullet \in C(\mathcal{A})$. A <u>Cartan - Eilenberg resolution</u> is a double complex $I^{\bullet,\bullet}$ s.t. every term is an injective object, if we fix the first index, then

$I^{n,\bullet}$ is an injective resolution for A^n.

Want something like:

$$
\begin{tikzcd}
 & {} \arrow[r] & I^{n,1} \arrow[r] & {} & {} & & \\
 & {} \arrow[r] & I^{n,0} \arrow[r] \arrow[u] & I^{n+1,0} \arrow[r] \arrow[u] & {} & & \\
\cdots \arrow[r] & {} \arrow[r] & A^n \arrow[r] \arrow[u] & A^{n+1} \arrow[r] \arrow[u] & A^{n+2} \arrow[r] & {} & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

<u>Prop</u> If $\mathcal{A}$ has enough injectives, $A^\bullet \in C^+(\mathcal{A})$, then a Cartan-Eilenberg resolution exists.

<u>Proof</u> WLOG $A^n = 0$, $n < 0$.

Split $A^\bullet$ into SES's:

$$0 \to Z^n \to A^n \to B^{n+1} \to 0$$

$$0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$$

Inductive construction: pick injective resol's

$$\begin{tikzcd} 0 \arrow[r] & I_n^\bullet \arrow[r] & J_n^\bullet \arrow[r] & K_{n+1}^\bullet \arrow[r] & 0 \\ 0 \arrow[r] & Z^n \arrow[r] \arrow[u, hook] & A^n \arrow[r] \arrow[u, hook] & B^{n+1} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Same for

$$\begin{tikzcd} 0 \arrow[r] & K_{n+1}^\bullet \arrow[r] & I_{n+1}^\bullet \arrow[r] & ? \arrow[r] & 0 \\ 0 \arrow[r] & B^{n+1} \arrow[r] \arrow[u, hook] & Z^{n+1} \arrow[r] \arrow[u, hook] & H^{n+1} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Stitch $J_n^\bullet$ via $J_n^\bullet \twoheadrightarrow K_{n+1}^\bullet \hookrightarrow I_{n+1}^\bullet \hookrightarrow J_{n+1}^\bullet$. ∎

Application to the Grothendieck exact sequence.

Pick an injective resolution $I^\bullet$ for $M \in \mathcal{A}$.

Then consider the complex $F(I^\bullet)$

Cohomology of this complex $= R^\bullet F(M)$.

Pick a C-E resolution for $F(I^\bullet)$. Say, $\mathcal{J}^{\bullet,\bullet}$.

Apply G to $\mathcal{J}^{\bullet,\bullet}$.

What can we say about the resulting complex?

$$\begin{tikzcd}
 & \uparrow & \uparrow & \\
0 \arrow[r] & \mathcal{J}^{0,1} \arrow[r] \arrow[u] & \mathcal{J}^{1,1} \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & \mathcal{J}^{0,0} \arrow[r] \arrow[u] & \mathcal{J}^{1,0} \arrow[r] \arrow[u] & \mathcal{J}^{2,0} \arrow[r] & \cdots \\
0 \arrow[r] & F(I^0) \arrow[r] \arrow[u] & F(I^1) \arrow[r] \arrow[u] & F(I^2) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

Recall: $\mathcal{J}^{n,\bullet}$ is an injective resolution for $F(I^n)$.

In particular, $G(\mathcal{J}^{n,\bullet})$, then it computes $R^\bullet G(F(I^n))$

But $F(I^n)$ is G-acyclic. Thus, only $R^0 G(F(I^n)) =$

$= (G \circ F)(I^{\cdot})$ is non-zero. When we look at the SS of the double complex, E_1^{pq}:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \\
GF(I^0) \arrow[r] & GF(I^1) \arrow[r] & GF(I^2) \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

E_2^{pq}

$$\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & 0 & 0 & & \\
 & GF(A) \arrow[dr] & R^1GF(A) \arrow[dr] & R^2GF(A) \arrow[dr] & & \\
 & & 0 & 0 & 0 & 0
\end{tikzcd}$$

Thus, the spectral sequence degenerates,

$$H^\circ(\mathrm{Tot}) = R^{p+q}(G\circ F)(A).$$

Look at the second spectral sequence:

The first differential is horizontal:

Recall that last time we said that F-acyclic resolutions can be used to compute derived functors of F. This argument proves it via spectral sequences: check that Tot of a Cartan-Eilenb. resolution is qis to your original complex.

Tomorrow
- Finish the construction of the Grothendieck SS.
- A few words about the construction of SS's.
- Geometric examples: sheaves of Ab.