Triangulated & Derived Categories in Algebra & Geometry
Lecture 12

0. Balancing Tor

Problem: Consider $M, N \in A\text{-Mod}$, A - comm. ring.

$P_\bullet \to M$, $Q_\bullet \to N$ - proj resolutions

$\mathrm{Tor}_i(M,N) = ?$ a) $H_i(P_\bullet \otimes_A N)$
b) $H_i(M \otimes_A Q_\bullet)$

Solution: Show that both are isom. to the same module.

$P_\bullet \to M$, $Q_\bullet \to N \rightsquigarrow$ double complex

Given any $X^\bullet, Y^\bullet \in C(A\text{-Mod})$ one defines

$X^\bullet \otimes Y^\bullet$ - bicomplex, (p,q) term is $X^p \otimes Y^q$,
$d_h : X^p \otimes Y^q \to X^{p+1} \otimes Y^q$, $d_h = d_X \otimes 1$

$$d_v : X^p \otimes Y^q \to X^p \otimes Y^{q+1}, \quad d_v = (-1)^p \, 1 \otimes d_Y$$

Koszul sign rule: $d_Y(x \otimes y) \overset{\text{exchange } d_Y \,\&\, x}{\rightsquigarrow} x \otimes d_Y(y)$

Whenever an exchange happens, multiply $(-1)^{\text{product of deg's}}$

$\deg d_Y = 1$, $\deg x = p \rightsquigarrow (-1)^p$

<u>Exc</u> $X^\bullet \otimes Y^\bullet$ is a double complex.

Back to Tor.

<u>Claim</u> $\mathrm{Tor}_i \simeq H_i(\mathrm{Tot}(P_\bullet \otimes Q_\bullet))$

defined either way

$$\begin{tikzcd}
 & 0 & 0 & \\
P_2 \otimes Q_0 \arrow[r] & P_1 \otimes Q_0 \arrow[u] \arrow[r] & P_0 \otimes Q_0 \arrow[u] \arrow[r] & 0 \\
\arrow[r] & P_1 \otimes Q_1 \arrow[u] \arrow[r] & P_0 \otimes Q_1 \arrow[u] & \\
 & \arrow[u] & P_0 \otimes Q_2 \arrow[u] & \\
 & & \arrow[u] &
\end{tikzcd}$$

Write two spectral sequences.

I^st page 1 $\rightsquigarrow$ take cohomology at every row.

i^{th} row:

$$\cdots \to P_2 \otimes Q_i \longrightarrow P_1 \otimes Q_i \longrightarrow P_0 \otimes Q_i \to 0$$

Q_i – projective $\Rightarrow$ Q_i is a direct summand of a free module. $\bigoplus A \underset{s}{\rightleftarrows} Q_i \to 0$

Tensoring with free modules is an exact functor
$\Rightarrow$ $- \otimes Q_i$ is also exact.

Thus, $H_j(P_\bullet \otimes Q_i) \simeq H_j(P_\bullet) \otimes Q_i = \begin{cases} 0, & j > 0 \\ M \otimes Q_i, & j = 0 \end{cases}$

The first page:

$$\begin{array}{ccc} & & 0 \\ & & \uparrow \\ 0 & 0 & M \otimes Q_0 \\ & & \uparrow \\ 0 & 0 & M \otimes Q_1 \\ & & \uparrow \\ 0 & 0 & M \otimes Q_2 \end{array}$$

Thus, the second page is nothing but

The spectral sequence degenerates at page 2,

$$\begin{array}{ccc}
0 & \mathrm{Tor}_0(M,N) & 0 \\
0 & \mathrm{Tor}_1(M,N) & 0 \\
0 & \mathrm{Tor}_2(M,N) & 0 \\
 & & 0 \\
 & & 0
\end{array}$$

← only one potent'ly nonzero term in each diag.

$H_i(\mathrm{Tot}(P_\cdot \otimes Q_\cdot)) \simeq$

$\simeq \mathrm{Tor}_i(M,N)$ ← defined using $Q_\cdot \to N$.

By symmetry, $H_i(\mathrm{Tot}(P_\cdot \otimes Q_\cdot)) \simeq H_i(P_\cdot \otimes N)$.

<u>Exc</u> If $\mathcal{A}$ has enough projectives and injectives, show that $\mathrm{Ext}^i(X,Y)$ can be defined/computed using either $P_\cdot \to X$ or $Y \to I^\cdot$.

You will need a double complex $\mathrm{Hom}(P_\bullet, I^\bullet)$.

1. Back to the Grothendieck SS

$F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{C}$, F, G are left exact, F takes injective objects to G-acyclic objects.

Claim There is a SS whose 2nd page is

$$R^pG(R^qF(A)) \Rightarrow R^{p+q}(G\circ F)(A).$$

Cartan - Eilenberg resolution

Lm $X^\bullet \in C^+(\mathcal{A})$ - bonded from below, $\mathcal{A}$ has enough injectives, there exists a double complex $I^{\bullet,\bullet}$ with the properties:

0) I^{pq} is injective for all p, q,

1) $I^{p,\bullet}$ is a inj. resolution for X^p,

2) Taking horizontal $Z/B/H$ gives <u>injective</u> resolutions for $Z^i(X^\bullet)/B^i(X^\bullet)/H^i(X^\bullet)$

$$\begin{tikzcd}
 & \uparrow & \uparrow & \\
\arrow[r] & I^{p1} \arrow[r] \arrow[u] & I^{p+1,1} \arrow[r] \arrow[u] & {} \\
\arrow[r] & I^{p0} \arrow[r] \arrow[u] & I^{p+1,0} \arrow[r] \arrow[u] & {} \\
\cdots \arrow[r] & X^p \arrow[r] \arrow[u] & X^{p+1} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

<u>Pf</u> Split $X^\bullet$ int SES, use induction. $\square$

Grothendieck spectral sequence comes from a C-E resolution of $F(\mathcal{Y}^\bullet)$, where $A \to \mathcal{Y}^\bullet$ - inj in $\mathcal{A}$.

Pick a C-E resol $I^{\bullet,\bullet}$ of $F(\mathcal{Y}^\bullet)$.

Consider the two spectral sequences associated with $G(I^{\bullet,\bullet})$

I^{st}: page 1 $\rightsquigarrow$ take cohomology of the columns

(flip) $\quad G(I^{p0}) \to G(I^{p1}) \to G(I^{p2}) \to \cdots$

$I^{p,\cdot}$ - inj resol of $F(J^p)$, we are computing

$$R^iG(F(J^p)) = \begin{cases} 0, & i>0 \\ GF(J^p), & i=0 \end{cases}$$

G-acyclic

$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow$

$0 \to GF(J^0) \to GF(J^1) \to \cdots$

<u>Conclude</u> the second page is $R^i(G\circ F)(A)$,
the SS degenerates, $R^i(G\circ F)(A) \simeq H^i(\mathrm{Tot}(G(I^{\cdot,\cdot})))$.

II^{nd}: page 1 $\rightsquigarrow$ take cohomology of the rows.

$$G(0 \to I^{0,q} \to I^{1,q} \to I^{2,q} \to \cdots)$$

$I^{\bullet,q}$ consists of injectives, all boundaries, cycles, cohomology are also injectives!

$\Rightarrow$ the complex splits: isom to

$$\cdots \to B_i \oplus H_i \oplus B_{i+1} \xrightarrow{\pi} B_{i+1} \oplus H_{i+1} \oplus B_{i+2} \to \cdots$$

Since G is additive, $I^{\bullet,q}$ splits, we conclude that $H^p(G(I^{\bullet,q})) = G(H^p(I^{\bullet,q}))$

II^{nd} page $\leadsto$ take cohomology of the complex

$G(H^p(I^{\bullet,\bullet}))$

↑ injective resolution of $H^p(F(J^\bullet))$

Thus, we are computing $R^qG(R^pF(A))$.

End of the construction.

2. Example

$A \to B$ - morphism of comm. rings

$M \in B\text{-Mod} \Rightarrow M$ is also an A-module

$$\begin{tikzcd}
A\text{-Mod} \arrow[rr, "\mathrm{Hom}_A(B,-)"] \arrow[dr, "\mathrm{Hom}_A(M,-)"'] & \arrow[d, phantom, "\circlearrowleft"] & B\text{-Mod} \arrow[dl, "\mathrm{Hom}_B(M,-)"] \\
& Ab &
\end{tikzcd}$$

Standard adjunction:

$$\mathrm{Hom}_B(M, \mathrm{Hom}_A(B,N)) \simeq \mathrm{Hom}_A(M \otimes_B B, N) = \mathrm{Hom}_A(M,N)$$

$\mathrm{Hom}_A(B,-)$ is right adjoint to an exact functor.

restriction $B\text{-Mod} \to A\text{-Mod}$

<u>Lm</u> $F \dashv G$, F - exact $\Rightarrow G$ preserves injectives.

Pf $\quad 0 \to A \to B \to C \to 0 \qquad$ Appy $\mathrm{Hom}(-, G(I)) \simeq$

$$\simeq \mathrm{Hom}(F(-), I)$$

preserves SES

if I is injective $\Rightarrow$ RHS is exact. $\square$

Grothendieck SS:

$$\mathrm{Ext}^p_B(M, \mathrm{Ext}^q_A(B, N)) \Rightarrow \mathrm{Ext}^{p+q}_A(M, N)$$

for all $M \in B\text{-Mod}$, $N \in A\text{-Mod}$.

If B - proj as A-mod, then $\mathrm{Ext}^{>0}_A(B, -) = 0$

and

$$\mathrm{Ext}^p_B(M, \mathrm{Hom}_A(B, N)) \simeq \mathrm{Ext}^p_A(M, N).$$

4. Yoneda Ext

Standard fact from homological algebra

$$\operatorname{Ext}^1(X,Y) \longleftrightarrow 0 \to Y \to Z \to X \to 0 \,/\sim$$

extension of A by B

<u>Construction</u> Given $0 \to B \to C \to A \to 0$,

apply $\operatorname{Hom}(A,-)$:

$$0 \to \operatorname{Hom}(A,B) \to \operatorname{Hom}(A,C) \to \operatorname{Hom}(A,A) \xrightarrow{\delta} \operatorname{Ext}^1(A,B)$$

$$\mathrm{id}_A \longmapsto \varepsilon$$

To a SES $0 \to B \to C \to A \to 0 \rightsquigarrow \varepsilon \in \operatorname{Ext}^1(A,B)$.

Do the reverse. Let $\varepsilon \in \operatorname{Ext}^1(A,B]$. Want: a SES.

$P_\bullet \to A$ is a projective resolution.

$\operatorname{Ext}^1(A,B)$ is cohomology of the complex $\operatorname{Hom}(P_\bullet, B)$

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d"] & P_1 \arrow[r] \arrow[d, "f"] & P_0 \arrow[r] & A \arrow[r] & 0 \\ & & B & & & \end{tikzcd}$$

ε is represented by $f : P_1 \to B$

$f \circ d = 0$

f factors through the cokernel of $P_2 \to P_1$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, "f"] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & C \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

SES of the form that we wanted.

More general definition: fix $n > 0$, a length n extension of A by B is an exact complex

$$0 \to B \to C_{n-1} \to C_{n-2} \to \cdots \to C_0 \to A \to 0$$

Put an equivalence relation: $\sim$ if $\exists$ a diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & C'_{n-1} \arrow[r] & \cdots \arrow[r] & C'_0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, "id_B"] \arrow[d, "id_B"'] & C_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & C_0 \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, "id_A"'] \arrow[d, "id_A"] & 0 \\
0 \arrow[r] & B \arrow[r] & C''_{n-1} \arrow[r] & \cdots \arrow[r] & C''_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Exc Check that $\sim$ is an equiv. relation.

Exc Identify (at least, construct map b/w)

$$\left\{ \substack{\text{extensions} \\ \text{of len. } n} \text{ of } A \text{ by } B \right\}/\sim \quad \longleftrightarrow \quad \mathrm{Ext}^n(A, B)$$

Returning to our example

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \\ \shortparallel & & \shortparallel \\ A & & B \end{array}$$

$$\mathrm{Ext}^p_{\mathbb{Z}/n\mathbb{Z}}(M, \mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbb{Z}}(M, N)$$

$\mathrm{Ext}^q_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N)$ — computed using $\underbrace{\boxed{0 \to \mathbb{Z} \to \mathbb{Z}}}_{P_\bullet} \to \mathbb{Z}/n\mathbb{Z} \to 0$

For any N: $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, N)$

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) \simeq N/nN$$

$$\mathrm{Ext}^{>1} = 0$$

If N has no n-torsion, $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, N) = 0$.

$$\mathrm{Ext}^{p+1}_{\mathbb{Z}}(M, N) = \mathrm{Ext}^p_{\mathbb{Z}/n\mathbb{Z}}\big(M, \underbrace{\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N)}_{\wr\ N/nN}\big)$$

Put $p = 0$:

$$\mathrm{Ext}^1_{\mathbb{Z}}(\underbrace{M}_{n\text{-torsion}}, \underbrace{N}_{\text{no } n\text{-torsion}}) \simeq \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(M, N/nN).$$

<u>Exc</u> Work out by hand.

Associate a morphism to every ext. & vice versa.

5. A few words about how the SS is constructed

Want a spectral sequence of a double complex $E^{\bullet,\bullet}$

Rmk 1) $E_0^{pq} = E^{pq}$ (by definition)

2) $\forall r$ E_r^{pq} is a subquotient of E_{r-1}^{pq}

Thus, if we want a SS, we "need" a chain of submodules

$$0 = B_0^{pq} \subset B_1^{pq} \subset \dots \qquad \dots \subset Z_2^{pq} \subset Z_1^{pq} \subset Z_0^{pq} = E^{pq}$$

boundaries $\qquad$ cycles

Want $E_r^{pq} = Z_r^{pq} / B_r^{pq}$, also the differentials.

Consider the "$p+q$" element of $\underbrace{\mathrm{Tot}(E^{\bullet,\bullet})}_{E^{n\bullet}}$: $\bigoplus_{p+q=n} E^{pq}$

Define the (p,q)-strip $E^{p+q} \supset S^{pq} = \bigoplus_{r \geq 0} E^{p-r,q+r}$

$$\begin{array}{ccc} E^{p-1,q+1} & E^{p,q+1} & \\ E^{p-1,q} & E^{pq} & E^{p+1,q} \\ & & E^{p+1,q-1} \end{array}$$

← take

← do not take

There is a filtration on E^n:

$$S^{pq} \subset S^{p+1,q-1} \subset \dots$$

Say that an element of S^{pq} is <u>r-closed</u> if its image under d is in $S^{p+1-r,\, q+r}$.

every element of $S^{p,q}$ is 0-closed: $d : S^{p,q} \to S^{p+1,q}$.

Being r-closed for all r is the same as closed.

Denote by S^{pq}_r the submodule of r-closed elements

Put $X^{pq}_r = S^{pq}_r / S^{p-1,q+1}_{r-1}$, $\quad$ subobject $\quad Y^{pq}_r = \dfrac{d\, S^{p+r-2,\,q-r+1}_{r-1} + S^{p-1,q+1}_{r-1}}{S^{p-1,q+1}_{r-1}}$.

<u>Big exercise</u> 1) $Y_r^{pq} \subset X_r^{pq}$, 2) d induces a differential in the candidate for page r: $E_r^{pq} = X_r^{pq} / Y_r^{pq}$.
3) this is our SS!

6. Long running geometric example

<u>Def</u> A presheaf of abelian groups $\mathcal{F}$ on a top. space X is a functor $Op(X)^{op} \to Ab$.

To every $U \subset X \rightsquigarrow \mathcal{F}(U) \in Ab$,
to every $V \subset U \rightsquigarrow res^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$,
if $W \subset V \subset U \Rightarrow res^V_W \circ res^U_V = res^U_W$,
$\forall U \quad res^U_U = id$.

Any geometric theory $\rightsquigarrow$ notion of "good" functions on $U \subset X$.

X - top space, $\mathcal{U} \subset X \rightsquigarrow \mathcal{F}(\mathcal{U}) = \mathrm{Map}(\mathcal{U}, \mathbb{R})$.
Naturally an abelian group (add functions point-wise).

Can restrict $f: \mathcal{U} \to \mathbb{R}$, $V \hookrightarrow \mathcal{U} \Rightarrow f|_V$ is cont's.

Rmk Presheaves (of abelian groups) on X form an abelian category.

Exc
1) $\mathrm{PSh}(X)$ is additive,
2) ker & coker can be taken point-wise,
3) Abelian.

What people often want is an extra property:

$f: \mathcal{U} \to \mathbb{R}$ is nice, $\mathcal{U} = \bigcup \mathcal{U}_i$
$g: \mathcal{U} \to \mathbb{R}$ is nice

then $f = g \iff f|_{\mathcal{U}_i} = g|_{\mathcal{U}_i}$ for all i.

Moreover, people like to build things of smaller ones:

$f_i : \mathcal{U}_i \to \mathbb{R}$, nice s.t. $f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = f_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$,

then $\exists$ $f : \mathcal{U} \to \mathbb{R}$ nice s.t. $f|_{\mathcal{U}_i} = f_i$.

<u>Exc</u> Formulate the last two properties in terms of exactness of some diagram

$$\mathcal{F}(\mathcal{U}) \overset{?}{\to} \prod \mathcal{F}(\mathcal{U}_i) \overset{?}{\to} \prod \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$$

<u>Problem</u> Let $E^{\bullet,\bullet}$ be a double complex situated in the I^{st} quadrant: $E^{pq} = 0$ if $p<0$ or $q<0$. $E^{\bullet} = \mathrm{Tot}(E^{\bullet,\bullet})$

Easy: $H^0(E^{\bullet}) = E_2^{0,0}$.

Construct an exact sequence

$$0 \to E_2^{0,1} \to H^1(E^{\bullet}) \to E_2^{1,0} \xrightarrow{d_2} E_2^{0,2} \to H^2(E^{\bullet}).$$