Triangulated and Derived Categories in Algebra and Geometry
Lecture 13

0) Sheaves

X - top space $\rightsquigarrow$ category of open $\mathcal{U} \subset X$
$Op(X)$

Objects: $\mathcal{U} \subset X$ - open
Morphisms: $Mor_{Op(X)}(\mathcal{U}, V) = \begin{cases} * & , \mathcal{U} \subseteq V \\ \emptyset & , \mathcal{U} \nsubseteq V \end{cases}$

<u>Def</u> A presheaf on X with values in a category $\mathcal{C}$ is a functor $Op(X) \longrightarrow \mathcal{C}$.

$\mathcal{U} \subset X \rightsquigarrow \mathcal{F}(\mathcal{U}) \in \mathcal{C}$
$V \subseteq \mathcal{U} \rightsquigarrow res^{\mathcal{U}}_{V} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V)$
Conditions: $res^{\mathcal{U}}_{\mathcal{U}} = id_{\mathcal{F}(\mathcal{U})}$
$res^{\mathcal{U}}_{W} = res^{V}_{W} \circ res^{\mathcal{U}}_{V} \quad \forall\, W \subseteq V \subseteq \mathcal{U}$

Most important cases for now:

1) Presheaves of sets
2) Presheaves of abelian groups
3) Presheaves of modules $\Leftarrow$ needs explanation

Observation: $PSh_{\mathcal{C}}(X)$ is just a functor category
$\Rightarrow$ category itself.

Morphism of presheaves

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ {\scriptstyle res}\downarrow & & \downarrow{\scriptstyle res} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

Keep two examples in mind:

1) Presheaves of "nice" functions
   X - smooth manifold $\rightsquigarrow$ $C^\infty(U)$ - smooth functions on U
   X - analytic $\rightsquigarrow$ $\mathcal{O}_X(U)$ - analytic ...

2) $\begin{array}{c} E \\ \downarrow p \\ X \end{array}$ $\rightsquigarrow$ presheaf of sections:

$$\mathcal{E}(\mathcal{U}) = \{ \mathcal{U} \xrightarrow{s} E \mid p \circ s = \mathrm{id}_{\mathcal{U}} \}.$$

Put $E = X \times \mathbb{R} \rightsquigarrow \mathcal{E}(\mathcal{U})$ - presheaf of cont. functions.

<u>Problem</u>: Want to define data locally.

$f \in \mathcal{F}(\mathcal{U})$ - same as giving $\{ f_i \in \mathcal{F}(\mathcal{U}_i) \}$, where $\mathcal{U} = \cup \mathcal{U}_i$ s.t. $f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = f_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$.

<u>Reformulation</u>: for any covering $\mathcal{U} = \cup \mathcal{U}_i$ the sequence

$$\mathcal{F}(\mathcal{U}) \longrightarrow \prod_{i \in I} \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$$

the equalizer

is exact.

Such presheaves are called <u>sheaves</u>.

We care about sheaves!

Why need presheaves? Look at $PSh_{Ab}(X)$. Ab - abelian
$\Rightarrow$ $PSh_{Ab}(X)$ also abelian:

take Ker & cokernels term-wise.

<u>Problem</u> While kernels of morphisms of sheaves are sheaves, cokernels are not in general sheaves.

<u>Solution</u> There is an adjoint to the inclusion functor of the full subcategory of sheaves into presheaves: <u>sheafification</u>.

<u>UP</u> $\mathcal{F}$ - presheaf on X (of Ab),
There is $\mathcal{F} \to \overline{\mathcal{F}}$ ← sheaf s.t.
$\forall$ $\mathcal{G}$ - sheaf, $\varphi : \mathcal{F} \to \mathcal{G}$

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \\ \overline{\mathcal{F}} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$$\mathrm{Hom}_{Sh_{Ab}}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{PSh_{Ab}}(\mathcal{F}, \mathcal{G})$$

Sheafification is left adjoint to the inclusion / forgetful functor. Should preserve cokernels (right exact!).

Given $\mathcal{F} \xrightarrow{f} \mathcal{G}$ a morphism of sheaves,

$\ker f$ = ker def in the category PSh_{Ab}.

$\mathrm{Coker}\, f$ = Coker in PSh_{Ab}, sheafified.

1) <u>How to sheafify?</u>

$$PSh_{Sets} \longrightarrow Sh_{Sets}$$

<u>Def</u> The <u>stalk</u> of $\mathcal{F}$ at x,

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

If you unwind the def,

$$\mathcal{F}_x = \{ (f, \mathcal{U}) \mid x \in \mathcal{U},\ f \in \mathcal{F}(\mathcal{U}) \} / \sim$$

$(f, \mathcal{U}) \sim (g, V)$ if $\exists\ x \in W \subseteq \mathcal{U} \cap V$ s.t. $f|_W = g|_W$.

Ex In the analytic case $\mathcal{O}_{X,x} \rightsquigarrow$ the ring of analytic expansions (series) of functions.

Consider $\coprod_{x \in X} \mathcal{F}_x = E \xrightarrow{p} X$. Put the following topology on E: the coarsest for which the following are open

$(f, \mathcal{U})$: $\mathcal{U} \subset X$, $f \in \mathcal{F}(\mathcal{U}) \rightsquigarrow \{ f_x \} \leftarrow$ want this set to be open.

$\forall\ x \in \mathcal{U}$ $f \leftarrow (f, \mathcal{U})$ gives an element in $\mathcal{F}_x$

Exc
- Every presheaf of sections is a sheaf.
- The sheaf of sections of $\coprod \mathcal{F}_x \to X$ gives sheafification.

- The natural map $\mathcal{F}_x \longrightarrow \overline{\mathcal{F}}_x$ is an isomorphism for presheaves of sets / ab. groups ...
- Check that a sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

of sheaves of abelian groups is exact if and only if $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ is exact $\forall x \in X$.

2) <u>Cohomology of sheaves (of Ab)</u>

$$Sh_{Ab}(X) \xrightarrow{\Gamma} Ab$$

$$\Gamma(\mathcal{F}) = \mathcal{F}(X)$$

<u>Exc</u> Γ is left exact!

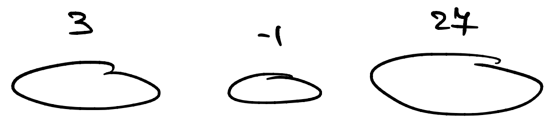
Would be nice to be able to take the right derived $R^i\Gamma$.

These are the cohomology functors $H^i(X, \mathcal{F})$.

Next time we will show that there are enough injectives.

Ex $A \in \mathrm{Ab} \rightsquigarrow \underline{A}$ - sheaf of locally constant functions on $U \subset X$ with values in A.

$\underline{\mathbb{Z}}$



Then $H^i(X, \underline{\mathbb{Z}}) \simeq H^i(X, \mathbb{Z})$. ← favorite topological way to define cohom.

sheaf cohomology

4) Triangulated categories

We will see that derived categories are almost never abelian. The structure of a triangulated category is what replaces abelianness.

Slogan A triangulated category – additive category with a collection of "triangles" ← substitute for SES's.

Def Let $\mathcal{T}$ – additive category. A shift functor is a collection $[n] : \mathcal{T} \to \mathcal{T}$ for $n \in \mathbb{Z}$ s.t. $[0] = \mathrm{Id}_{\mathcal{T}}$, $[n] \circ [m] = [n+m]$.

If $T \in \mathrm{Aut}(\mathcal{T})$ ← honest automorphisms $\Rightarrow [n] = T^n$.

Ex $C(\mathcal{A})$ comes with a shift functor, so does $K(\mathcal{A})$.

Def A triangle in $\mathcal{T}$ is a sequence of morphisms

$$X \to Y \to Z \to X[1]$$

Alternative depiction:

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, "+1"] & \end{tikzcd}$$

A morphism of Δ:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

By definition, a triangulated category is an additive category with a shift functor a collection of triangles, called <u>distinguished</u>, satisfying the following properties:

(TR 1)
- Any triangle isom to a dist. one is distinguished.
- $\forall X \in \mathcal{T}$ $\quad X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished.
- Any $f : X \to Y$ can be completed to a distinguished

$$X \xrightarrow{f} Y \to Z \to X[1]$$

(If dist. triangle replaces SES $\Rightarrow$
- A seq $\simeq$ to a SES is SE
- $0 \simeq X/X$
- Y/X always exists $\leftarrow$ a bit strange)

$X \to Y \to Z \to X[1]$

<u>think</u> $Z \simeq Y/X$

(TR 2) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{h} X[1]$ is distinguished $\iff$

$Y \xrightarrow{v} Z \xrightarrow{h} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished

(get an infinite sequence

$$\dots \to Y[-1] \to Z[-1] \to X \to Y \to Z \to X[1] \to Y[1] \to \dots$$

any consecutive 4 terms are dist.)

(TR3) Given a diagram with dist. rows

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, dashed, "\exists h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

(Warning: h is not unique. Source of many problems.
h - "morphism b/w cokernels".)

(TR4) Octahedron axiom

Consider $X \xrightarrow{f} Y \xrightarrow{g} Z$ & dist $\triangle$ that complete f, g, $g \circ f$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, dashed] & X[1] \\
 & Z \arrow[r] \arrow[dr] & Y' \arrow[r] \arrow[d, dashed] & X[1] \\
 & & X' \arrow[r] \arrow[d, dashed] & Y[1] \\
 & & Z'[1] &
\end{tikzcd}$$

$\exists$ morphisms
$Z' \to Y' \to X'$
giving a dist. triangle
+ some diagrams commute

($X \hookrightarrow Y \hookrightarrow Z \quad \Rightarrow \quad Z/Y \simeq Z/X \big/ Y/X$

$0 \to Y/X \to Z/X \to Z/Y \to 0$ SES)

Fun fact: some are still trying to show that (TR4) does / does not follow from (TR1) – (TR3).

<u>Def</u> An additive category + shift functor + distinguished triangles satisfying (TR1) – (TR3) is called pre-triangulated.

<u>Exc</u> Show that the category of vector spaces / k with $[1] = \mathrm{id}$ & the collection of triangles

$$A \oplus B \to B \oplus C \to C \oplus A \to A \oplus B$$

is triangulated (every map is the comp of the proj to the second with the incl of the first).

<u>Later</u>: $K(\mathcal{A})$ & $D(\mathcal{A})$ are triangulated (give Δ structure).

Def A functor $H: \mathcal{T} \to \mathcal{A}$, $\mathcal{T}$ - triangulated, $\mathcal{A}$ - abelian is called cohomological if $\forall$ dist $X \to Y \to Z \to X[1]$ the sequence $H(X) \to H(Y) \to H(Z)$ is exact.

In particular, H - cohomological $\overset{\text{rotation axiom}}{\rightsquigarrow}$ LES "of cohomology"

$$\cdots \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to H(Y[1]) \to \cdots$$

5) Simple properties of Δ categories

Fix $\mathcal{T}$ - triangulated (pre-triangulated is enough)

Lm $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ - distinguished $\Rightarrow$ $v \circ u = 0$!

Pf

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed, "(TR3)"] & X[1] \arrow[d, "id_{X[1]}"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

$\leftarrow$ (TR1)

The only $0 \to Z$ is 0! $v \circ u = 0$. $\square$

Thus, $w \circ v = 0$. (Rotation)

Lm $W \in \mathcal{T} \Rightarrow \mathrm{Hom}_{\mathcal{T}}(W,-)$ & $\mathrm{Hom}_{\mathcal{T}}(-,W)$ are cohomological.

Pf Let's show that $\forall$ dist $X \xrightarrow{u} Y \xrightarrow{v} Z \to X[1]$

$$\mathrm{Hom}_{\mathcal{T}}(W,X) \xrightarrow{u\circ} \mathrm{Hom}_{\mathcal{T}}(W,Y) \xrightarrow{v\circ} \mathrm{Hom}_{\mathcal{T}}(W,Z)$$

is exact.

$$\begin{tikzcd}
W \arrow[r, "id"] \arrow[d, dashed, "\exists g"'] & W \arrow[r] \arrow[d, "f"] \arrow[dr, "0"] & 0 \arrow[r] \arrow[d] & W[1] \arrow[d, dashed, "g[1]"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

$f = u \circ g$!

$\square$

Lm Given a diagram with dist rows

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

if f & g are iso, then h is iso!

<u>Pf</u> Apply $\mathrm{Hom}_{\mathcal{T}}(W, -)$ to both rows:
$$\mathrm{Hom}_{\mathcal{T}}(W,-) = [W,-]$$

$$\begin{tikzcd}
{[W,X]} \arrow[r] \arrow[d, "\wr"] & {[W,Y]} \arrow[r] \arrow[d, "\wr"] & {[W,Z]} \arrow[r] \arrow[d] & {[W,X[1]]} \arrow[r] \arrow[d, "\wr"] & {[W,Y[1]]} \arrow[d, "\wr"] \\
{[W,X']} \arrow[r] & {[W,Y']} \arrow[r] & {[W,Z']} \arrow[r] & {[W,X'[1]]} \arrow[r] & {[W,Y'[1]]}
\end{tikzcd}$$

Apply 5-lemma. Get an isom of functors $h_Z \xrightarrow[\sim]{h} h_{Z'}$.

Yoneda $\Rightarrow$ $Z \xrightarrow[\sim]{h} Z'$. $\square$

<u>Warning</u> Isom not unique!

Good news though.

<u>Lm</u> Given a morphism of dist. triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

g is uniquely completing f & h if $\mathrm{Hom}(Y, X') = 0$.

Pf Enough to show that if $\mathrm{Hom}(Y, X') = 0$, then 0 is the only morphism completing

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "0"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "0"] \arrow[dl, "0"'] & X[1] \arrow[d, "0"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

$\mathrm{Hom}(Y, -)$ – cohomological $\Rightarrow$ $g = Y \xrightarrow{0} X' \to Y' \Rightarrow g = 0$. ☐

By similar reasoning g is unique if $\mathrm{Hom}(Z, Y') = 0$.

Lm TFAE in $\mathcal{T}$

1) $f : X \to Y$ – iso
2) $X \xrightarrow{f} Y \to 0 \to X[1]$ is distinguished
3) $\forall$ $X \xrightarrow{f} Y \to Z \to X[1]$ dist $Z = 0$.

Pf $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "id"', "s"] & X \arrow[r] \arrow[d, "s", "f"'] & 0 \arrow[r] \arrow[d, dashed, "s"] & X[1] \arrow[d, "id_{X[1]}"] \\ X \arrow[r, "f"'] & Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

$\Rightarrow Z \simeq 0$ $\left(1) \Rightarrow 3) \right)$

3) $\Rightarrow$ 2) trivial

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id"'] & Y \arrow[r] \arrow[d, dashed, "\exists g"] & 0 \arrow[r] \arrow[d, "0"] & X[1] \arrow[d, "id"] \\ X \arrow[r, "id"'] & X \arrow[r] & 0 \arrow[r] & X[1] \end{tikzcd}$$

$g \circ f = id$

g must be iso!

$\square$

$\left(X \hookrightarrow Y, \; X' \hookrightarrow Y', \; X \simeq X', \; Y \simeq Y' \Rightarrow Y/X \simeq Y'/X' . \right)$

Tomorrow More properties of Δ categories

\+ Δ structure on $K(\mathcal{A})$.