Triangulated and Derived Categories in Algebra and Geometry

Lecture 14

0) More on (pre) sheaves

<u>Recall</u> $x \in X$, $\mathcal{F}$ – presheaf (always abelian groups)

$$\mathcal{F}_x = \varinjlim_{\mathcal{U} \ni x} \mathcal{F}(\mathcal{U})$$

Given $\mathcal{F} \in PSh(X)$, $\mathcal{U} \subset X \rightsquigarrow \mathcal{F}(\mathcal{U}) \to \mathcal{F}_x \ \forall x \in \mathcal{U}$ (from UP of $\varinjlim$).

Get $\mathcal{F}(\mathcal{U}) \xrightarrow{\iota_{\mathcal{U}}} \prod_{x \in \mathcal{U}} \mathcal{F}_x$.

<u>Def</u> $\mathcal{F}$ is separated if $\forall \mathcal{U}$ $\iota_{\mathcal{U}}$ is injective.

<u>Exc</u> Check that $\mathcal{F}$ is separated iff $\forall \mathcal{U}$, $\forall \mathcal{U} = \cup \mathcal{U}_i$, $f, g \in \mathcal{F}(\mathcal{U})$ $f = g \iff f|_{\mathcal{U}_i} = g|_{\mathcal{U}_i} \ \forall i$.

Since sheaves are seperated (check!), ι_U is always injective.

<u>Rmk</u> $\forall x \in U$ $\mathcal{F}(U) \to \mathcal{F}_x$ is a homomorphism of abelian groups.

Operations on (pre)sheaves

$f: X \to Y$ cont. map of topological spaces

Compare $PSh(X)$ & $PSh(Y)$.

<u>Def</u> The <u>pushforward functor</u> $f_*: PSh(X) \to PSh(Y)$ is given by

$$(f_* \mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

<u>Exc</u> 1) $f_* \mathcal{F}$ is naturally a presheaf.
2) f_* is a functor.
3) $\mathcal{F}$ - sheaf $\Rightarrow$ $f_* \mathcal{F}$ is also a sheaf.

Ex $\pi : X \to \{pt\}$, then

$\pi_* : PSh(X) \to PSh(\{pt\}) \simeq Ab$

$\pi_* = \Gamma$ — global section functor.

Exc $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $g_* \circ f_* \cong (g \circ f)_*$.

Very useful:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r] & \{pt\} \end{tikzcd}$$

$\Gamma \simeq \Gamma \circ f_*$ ⟵ smells like we could get a Grothendieck SS for $H^i(X, \mathcal{F})$!

In particular, if f_* happens to be exact,

$H^i(X, \mathcal{F}) \simeq H^i(Y, f_* \mathcal{F})$.

called the inverse image

Prop f_* has a left adjoint!

Constructive proof

$f^{-1} : PSh(Y) \to PSh(X)$

$\mathcal{F} \in PSh(Y)$

Would like: $(f^{-1}\mathcal{F})(U) = \mathcal{F}(f(U))$. But f in general is not open!

Let's mimic the stalk construction: if $Z \subset Y$ – arbitrary, put $\mathcal{F}(Z) = \varinjlim_{U \supset Z} \mathcal{F}(U)$.

Put $(f^{-1}\mathcal{F})(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V)$.

Exc
1) Put a presheaf structure.
2) f^{-1} is a functor.
3) Give an example of a sheaf $\mathcal{F} \in Sh(Y)$ s.t. $f^{-1}\mathcal{F}$ is not a sheaf (presheaf only)
4) Show that f^{-1} preserves stalks:
$$(f^{-1}\mathcal{F})_x \simeq \mathcal{F}_{f(x)}.$$
Conclude that f^{-1} is exact!
5) $f^{-1} \dashv f_*$ both for presheaves & sheaves (f^{-1} for sheaves $\rightsquigarrow$ sheafify).

Cor f_* is left exact (right adjoint).
f^{-1} is right exact (know that it's exact).

Ex $f : \{x\} \hookrightarrow X$

$f^{-1}\mathcal{F} = \mathcal{F}_x \leftarrow$ stalk.

Assume we are given an abelian group I_x for every $x \in X$.
Define a sheaf $\mathcal{J}$ by setting $\mathcal{J}(U) = \prod_{x \in U} I_x$.

Lm $\mathcal{F} \in Sh(X)$, then

$$\mathrm{Hom}_{Sh(X)}(\mathcal{F}, \mathcal{J}) \simeq \prod_{x \in X} \mathrm{Hom}_{Ab}(\mathcal{F}_x, I_x)$$

Pf Construct mutually inverse maps.

$\forall U \ni x$ $\quad \mathcal{F}(U) \to \mathcal{J}(U) = \prod_{y \in U} I_y \longrightarrow I_x$

Passing to $\varinjlim_{U \ni x}$ $\rightsquigarrow$ $\mathcal{F}_x \to I_x$.

<u>Warning</u> In general $\mathcal{J}_x \neq I_x$. Give an example.

Conversely, $\mathcal{F}(U) \longrightarrow \prod\limits_{x\in U} \mathcal{F}_x \longrightarrow \prod I_x$. $\square$

<u>Cor</u> If all I_x are are injective in Ab, then $\mathcal{J}$ is injective in $Sh(X)$.

<u>Pf</u> $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$, need to show that

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{G}, \mathcal{J}) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Hom}(\mathcal{F}, \mathcal{J}) \arrow[d, phantom, "\wr"] & \text{surj} \\
\prod \mathrm{Hom}(\mathcal{G}_x, \mathcal{J}_x) \arrow[r] & \prod \mathrm{Hom}(\mathcal{F}_x, \mathcal{J}_x) \arrow[r] & 0
\end{tikzcd}$$

<u>Prop</u> $Sh(X)$ has enough injectives!

<u>Pf</u> $\mathcal{F} \in Sh(X)$, embed every $\mathcal{F}_x \hookrightarrow I_x$ - injective.
Construct $\mathcal{J}$ as above.

Define $\mathcal{F} \to \mathcal{J}$ as above:

$$\mathcal{F}(\mathcal{U}) \longrightarrow \prod_{x \in \mathcal{U}} \mathcal{F}_x \longrightarrow \prod_{x \in \mathcal{U}} I_x = \mathcal{J}(\mathcal{U})$$

inj. since $\mathcal{F}$-sheaf $\Rightarrow$ separated

inj. by construction

$\mathcal{F}(\mathcal{U}) \hookrightarrow \mathcal{J}(\mathcal{U})$ for all $\mathcal{U}$ $\Rightarrow$ $\mathcal{F} \hookrightarrow \mathcal{J}$. $\blacksquare$

<u>Conclusion</u> Can define cohomology of sheaves of abelian groups:

$$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F}), \quad \Gamma(\mathcal{F}) = \mathcal{F}(\mathcal{U}).$$

<u>Comment</u> Almost never possible to compute anything using inj. resol. Injectives are very big. Need to look for a nice class of Γ-acyclic objects.

1) Properties (more) of (pre-)triangulated categories

<u>Lm</u> Let $\mathcal{T}$ be (pre-)triangulated, then given triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

$$\&$$

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X[1] \qquad (*)$$

$(*)$ are distinguished $\iff$

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow Z \oplus Z' \longrightarrow X \oplus X'[1] \quad \text{is dist.}$$

<u>Cor</u> $X \xrightarrow{\iota_X} X \oplus Y \xrightarrow{p_Y} Y \xrightarrow{0} X[1]$ is distinguished.

<u>Pf</u> $X \xrightarrow{id_X} X \xrightarrow{0} 0 \xrightarrow{0} X[1]$ is dist.

$0 \longrightarrow Y \xrightarrow{id_Y} Y \xrightarrow{0} 0[1]$ is dist.

Their direct sum is iso to what we want! $\square$

Pf (of the lemma)

Consider the $\oplus$ of the two dist Δ.

$$\begin{array}{ccccccc} X\oplus X' & \xrightarrow{f\oplus f'} & Y\oplus Y' & \longrightarrow & Z\oplus Z' & \longrightarrow & X\oplus X'[1] \\ \| & & \| & & \downarrow c & & \| \\ X\oplus X' & \xrightarrow{f\oplus f'} & Y\oplus Y' & \longrightarrow & W & \longrightarrow & X\oplus X'[1] \end{array}$$

migh not be dist.

Rmk there is a morphism $c: Z\oplus Z' \xrightarrow{t\oplus t'} W$

Coming from
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \iota_X\downarrow & & \downarrow \iota_Y & & \downarrow t & & \downarrow i_X \\ X\oplus X' & \longrightarrow & Y\oplus Y' & \longrightarrow & W & \longrightarrow & X\oplus X'[1] \end{array}.$$

If you look at the proof of Δ 5-lemma, it only needed the LES property!

Both triangles satisfy this property: the lower one is dist, the upper one is $\oplus$ of dist $\Rightarrow$ satisfies the LES property! $\Rightarrow$ c-iso!

Assume $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \xrightarrow{h \oplus h'} X \oplus X'[1]$ is dist.

$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a triangle, satisfies the LES property (as a direct summand of a dist triangle).

Complete $X \xrightarrow{f} Y \to W \to X[1]$, look at

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook, "\iota"'] \arrow[dd, bend left, "id", blue] & Y \arrow[r] \arrow[d, hook, "\iota"'] \arrow[dd, bend left, "id", blue] & Z \arrow[r] \arrow[d, hook, "\iota"'] & X[1] \arrow[d, hook, "\iota"'] \arrow[dd, bend left, "id", blue] \\
X \oplus X' \arrow[r] \arrow[d, "p"'] & Y \oplus Y' \arrow[r] \arrow[d, "p"'] & Z \oplus Z' \arrow[r] \arrow[d, dashed, "\exists c"] & X \oplus X'[1] \arrow[d, "p"] \\
X \arrow[r, "f"] & Y \arrow[r] & W \arrow[r] & X[1] & \arrow[l, phantom, "\longleftarrow \text{dist}"]
\end{tikzcd}
$$

Conclude: $c \circ \iota$ - iso! $\square$

Lm If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is dist. & $h = 0$, then g has a right inverse $s : Z \to Y$ $(g \circ s = id_Z)$.

Pf Apply $\mathrm{Hom}(Z,-)$:

$$\mathrm{Hom}(Z,Y) \xrightarrow{g\circ} \mathrm{Hom}(Z,Z) \xrightarrow{h\circ = 0} \mathrm{Hom}(Z,X[1])$$

Exact $\Rightarrow$ $\mathrm{Hom}(Z,Y) \xrightarrow{g\circ} \mathrm{Hom}(Z,Z)$ is surjective. $\square$

Exc If $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ is dist, $g \circ s = \mathrm{id}_Z$ for $s: Z \to Y$, then

$$X \oplus Z \xrightarrow{(f,s)} Y \text{ is an isomorphism!}$$

2) Exact functors

Def $F: \mathcal{T} \to \mathcal{T}'$ is exact if there exists an isom: (part of data)

$$F(X[1]) \xrightarrow{\sim} F(X)[1] \quad \text{functorial}$$

such that if $X \to Y \to Z \to X[1]$ is dist in $\mathcal{T}$, then $F(X) \to F(Y) \to F(Z) \to F(X[1]) \xrightarrow{\sim} F(X)[1]$ is dist. in $\mathcal{T}'$.

Lm $F : \mathcal{T} \to \mathcal{T}'$ is exact $\Rightarrow$ F is additive.

Pf Let's check that $F(0)$ is 0!

$$0 \xrightarrow{id} 0 \xrightarrow{id} 0 \xrightarrow{0} 0 = 0[1] \leftarrow \text{dist}$$

Apply F:

$$F(0) \xrightarrow{id} F(0) \xrightarrow{id} F(0) \xrightarrow{?} F(0)[1] \leftarrow \text{dist}$$

$$0 \overset{\leftarrow \text{ since dist}}{=} id_{F(0)} \circ id_{F(0)} = id_{F(0)} \Rightarrow F(0) = 0.$$

Conclude also that $F(X \xrightarrow{0} Y) = F(X) \xrightarrow{0} F(Y)$.

As for $X \oplus Y$: apply F to $X \xrightarrow{\iota} X \oplus Y \xrightarrow{p} Y \xrightarrow{0} X[1]$

$$F(X) \to F(X \oplus Y) \xrightarrow{F(p)} F(Y) \xrightarrow{0} F(X[1]) \simeq F(X)[1] \quad \text{dist}$$

by the prev. comment

$\Rightarrow$ $F(p)$ has a right inverse (Lemma)

$\Rightarrow$ $F(X \oplus Y) \simeq F(X) \oplus F(Y)$ (Excer.) $\square$

Lm $F : \mathcal{T} \to \mathcal{T}'$ is exact and fully faithful, then
$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is dist $\iff$

$F(X) \to F(Y) \to F(Z) \to F(X)[1]$ is dist.

Pf $\Rightarrow$ by def

$\Leftarrow$ Assume $F(X) \xrightarrow{F(f)} F(Y) \to F(Z) \to F(X)[1]$ dist.

Complete $X \xrightarrow{f} Y \to W \to X[1]$ to a dist. Apply F!

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, phantom, "\shortparallel"] & F(Y) \arrow[r] \arrow[d, phantom, "\shortparallel"] & F(W) \arrow[r] \arrow[d, dashed, "\exists e"] & F(X)[1] \arrow[d, phantom, "\shortparallel"] & \text{dist} \\
F(X) \arrow[r, "F(f)"] & F(Y) \arrow[r] & F(Z) \arrow[r] & F(X)[1] &
\end{tikzcd}$$

$\exists\, e : F(W) \to F(Z)$ it is iso! F is f.f $\Rightarrow$

$\Rightarrow$ e lifts to an iso $e : W \to Z$, gives an iso of triangles. $\square$

4) Homotopy category is triangulated

$K(\mathcal{A})$ comes with a shift functor $[1] : K(\mathcal{A}) \to K(\mathcal{A})$

Want to say that distinguished triangles are those isomorphic to triangles

cone.

$$X^\bullet \xrightarrow{f} Y^\bullet \to C(f) \to X[1]$$

Alternatively: distinguished triangles are those isomorphic to triangles coming from split exact SES's of complexes.

Def A SES of complexes $0 \to A \to B \to C \to 0$ is split exact if $\forall n \in \mathbb{Z}$ $0 \to A^n \to B^n \to C^n \to 0$ is split exact:

$$0 \to A^n \underset{p^n}{\overset{f^n}{\rightleftarrows}} B^n \underset{s^n}{\overset{g^n}{\rightleftarrows}} C^n \to 0 \qquad \begin{aligned} g^n \circ s^n &= id_{C^n} \\ p^n \circ f^n &= id_{A^n} \end{aligned}$$

Construction $0 \to X \to Y \to Z \to 0$ is a split exact SES of complexes. Put

$$Z^n \xrightarrow{h^n} X^{n+1} \qquad h^n = -p^{n+1} \circ d \circ s^n$$

$$\begin{tikzcd}
0 \arrow[r] & X^n \arrow[r] \arrow[d, "d"] & Y^n \arrow[r] \arrow[d, "d"] & Z^n \arrow[r] \arrow[d, "d"] \arrow[l, bend right, "s^n"'] & 0 \\
0 \arrow[r] & X^{n+1} \arrow[r] & Y^{n+1} \arrow[r] \arrow[l, bend left, "p^{n+1}"] & Z^{n+1} \arrow[r] & 0
\end{tikzcd}$$

Need to check that h is a morphism of complexes.

Claim The triangle associated with a split SES is isomorphic in $K(\mathcal{A})$ to the triangle

$$X \xrightarrow{f} Y \to C(f) \to X[1].$$

Pf Will construct isomorphism

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, phantom, "\wr"] & Y \arrow[r, "g"] \arrow[d, phantom, "\wr"] & Z \arrow[r, "h"] \arrow[d, shift left, "v"] & X[1] \arrow[d, phantom, "\wr"] \\
X \arrow[r, "f"] & Y \arrow[r] & C(f) \arrow[r] \arrow[u, shift left, "u"] & X[1]
\end{tikzcd}$$

Need to define $u: Z \to C(f)$

$$Z^n \xrightarrow{\begin{pmatrix} s^n \\ h^n \end{pmatrix}} C(f)^n = Y^n \oplus X^{n+1}$$

In the opposite: $\quad v = (g^n, 0)$

$$C(f)^n = Y^n \oplus X^{n+1} \longrightarrow Z^n.$$

Things to check: 1) u & v are morphisms of complexes,

2) the squares commute (in $K(\mathcal{A})$!)

3) $v \circ u = id_Z$

4) $u \circ v \sim id_{C(f)}$ (in $K(\mathcal{A})$ $=$) $\square$

Triangles associated with SE SES's $\simeq$ to cone triangles.

<u>Lm</u> Cone triangles are isomorphic to triangles associated with SE SES's!

<u>Construction</u> $f : X^\cdot \to Y^\cdot$ a morphism in $C(\mathcal{A})$.

Decompose f:

$$\begin{tikzcd} X^\cdot \arrow[r, "\tilde{f}"] \arrow[rr, bend right, "f"'] & \tilde{Y}^\cdot \arrow[r, "\pi"] & Y^\cdot \end{tikzcd}$$

where $\tilde{f}$ is termwise-split injective, π has a right inverse $s : Y^\cdot \to \tilde{Y}^\cdot$ $(\pi \circ s = id)$ such that $s \circ \pi \sim id$ (in $K(\mathcal{A})$ $Y^\cdot \simeq \tilde{Y}^\cdot$).

Put $\tilde{Y}^\cdot = Y^\cdot \oplus C(id_{X^\cdot})$

$\tilde{f} = \begin{pmatrix} f \\ \iota \end{pmatrix} : X \longrightarrow \tilde{Y}^\cdot = Y^\cdot \oplus C(id_{X^\cdot})$

$\pi : \tilde{Y}^\cdot = Y^\cdot \oplus C(id_{X^\cdot})$ – projection

$s : Y^\cdot \hookrightarrow Y^\cdot \oplus C(id_{X^\cdot})$ – inclusion

All the rest follows trivially.

Pf (of Lemma)

$\tilde{f}$ & $\tilde{\pi}$ are as in our construction

$$\begin{tikzcd}
X^\bullet \arrow[r, "\tilde{f}"] \arrow[d, equal] & \tilde{Y}^\bullet \arrow[r] \arrow[d, "\tilde{\pi}"] & Z \arrow[r] \arrow[d, dotted, "\text{need this!}"] & X^\bullet[1] \arrow[d, equal, "\mathrm{id}"] \\
X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r] & C(f) \arrow[r] & X^\bullet[1]
\end{tikzcd}$$

The top triangle $\simeq$ to the cone triangle!

It will all follow once we establish some sort of functoriality for cones. $\square$

Next week
1) Finish with $\triangle$ structure on $K(\mathcal{A})$.
2) Discuss how $\triangle$ cat's localize.
3) Pass to $D(\mathcal{A})$!

Problem No problem / holiday!