Triangulated and Derived Categories in Algebra and Geometry
Lecture 15

0) Remarks

Fact Derived / triangulated categories are almost never abelian!

Pf Assume $\mathcal{T}$ - triangulated is abelian.
Let f be a monomorphism. Claim that f splits:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[l, bend left, "\exists p"] \end{tikzcd} \quad \text{s.t. } p \circ f = id_X.$$

Complete f to a dist Δ:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

Rotate:

$$Z[-1] \xrightarrow{-h[1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$

Dist. $\Rightarrow f \circ (-h[-1]) = 0 \overset{f\text{-mono}}{\Longrightarrow} h[-1] = 0 \Rightarrow h = 0$!

Thus, (Lemma from last time) $Y \simeq X \oplus Z$. $\square$

<u>Ex</u> $D^b(\mathbb{Z}\text{-mod})$

$\mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z}$. If g had a kernel, it would be mono! But $\mathbb{Z}/4\mathbb{Z}$ is indecomposable.
($\mathbb{Z}\text{-mod} \longrightarrow D^b(\mathbb{Z}\text{-mod})$ is fully faithful.)

Careful depiction of the octahedral axiom

Mimics the isomorphism $Z/Y \simeq (Z/X)/(Y/X)$
for $X \hookrightarrow Y \hookrightarrow Z$.

Assume given $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and
$Y \xrightarrow{f} Y' \xrightarrow{g} W \xrightarrow{h} Y[1]$ distinguished.

<u>TR4</u> $\exists$ a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal, "\mathrm{id}_X"'] & Y \arrow[r, "v"] \arrow[d, "f"] & Z \arrow[r, "w"] \arrow[d] & X[1] \arrow[d, equal, "\mathrm{id}_{X[1]}"] \\
X \arrow[r, "f\circ u"] & Y' \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d] & X[1] \\
& W \arrow[r, equal] \arrow[d, "h"] & W \arrow[d] & \\
& Y[1] \arrow[r, "v[1]"] & Z[1] &
\end{tikzcd}$$

the top two rows and the middle columns are dist. Δ's.

<u>Exc</u> Take any decent book on derived cat's, stare at the octahedron diagram.

1) <u>Back to the Δ on the homotopy category</u>

Two candidates for the dist Δ's:

$\underline{\mathrm{I}}$ $X \to Y \to Z \to X[1]$ isom to triangles associated with split exact sequences

$\underline{\mathrm{II}}$ $X \to Y \to Z \to X[1]$ isom to triangles of the form $X \xrightarrow{f} Y \to C(f) \to X[1]$.

<u>Last time</u>

- In $K(\mathcal{A})$ (or $K^*(\mathcal{A})$, $* \in \{+,-,b\}$) any Δ associated to a split exact one is $\simeq$ to the cone triangle.
- Given $X \xrightarrow{f} Y \to C(f) \to X[1]$, we almost showed that it's $\simeq$ to a Δ ass. to a split exact sequence

  <u>Need</u> "functoriality for the cone"

Thm $K^*(\mathcal{A})$ is triangulated with dist Δ being the Δ isom to cone Δ's / split exact seq. Δ's.

Pf (TR1) • isom – by definition

• $\underbrace{X \xrightarrow{id} X \to 0}_{\text{split exact}} \to X[1]$ $\quad$ ($0 \to X \xrightarrow{id_X} X \to 0[1]$ is a cone triangle)

• $X \xrightarrow{f} Y \dashrightarrow Z \dashrightarrow X[1] \leftarrow Z = C(f) \ldots$

(TR2) Assume $X \to Y \to Z \to X[1]$ – dist $\Rightarrow \simeq$

$$X \xrightarrow{f} Y \to C(f) \to X[1] \overset{\text{rotation}}{\rightsquigarrow}$$

$$\underbrace{Y \to C(f) \to X[1]}_{\text{split exact}} \xrightarrow{-f[1]} Y[1] \Rightarrow \simeq \text{ a cone triangle!}$$

(check that $-f[1]$ is what we get from the split exact sequence)

Rotation in the other direction?

Assume $Y \to Z \to X[1] \to Y[1]$ is distinguished

want: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is dist.

Rotate the 1st one twice to the right:

dist $X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$

The class of dist triangles in our def is closed under $[n]$ and change of sign!

(TR3)

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & C(f) \arrow[r] \arrow[d, dashed, "?"] & X[1] \arrow[d, "u[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r] & C(f') \arrow[r] & X'[1]
\end{tikzcd}$$

$vf - f'u \sim 0$ (left square commutes in $K^*(\mathcal{A})$)

$\exists\, h^i : X^i \to (Y')^{i-1}$ s.t.

$vf - f'u = dh + hd$

Exc Check that maps $C(f)^n \to C(f')^n$

given by

$$\begin{array}{ccc} Y^n \oplus X^{n+1} & \xrightarrow{\begin{pmatrix} v & h \\ 0 & u \end{pmatrix}} & (Y')^n \oplus (X')^{n+1} \\ \shortparallel & & \shortparallel \\ C(f)^n & & C(f')^n \end{array}$$

produce the desired morphism.

(TR4) Try to do this using split exact triangles. $\square$

2) <u>Localization of triangulated categories</u>

<u>Goal</u> Construct a Δ structure on $D^*(\mathcal{A})$ $(* \in \{+,-,b\})$ from the fact that

$$D(\mathcal{A}) = K(\mathcal{A})[qis^{-1}].$$

<u>Q</u> Let $\mathcal{T}$ be a triangulated category, $S \subset Mor(\mathcal{T})$ be a class of morphisms.

What are the natural conditions on S enough to have a Δ structure on $\mathcal{T}[S^{-1}]$?

S are those morphisms that we want to invert.

(1) $s \in S \Rightarrow s[n] \in S$ for all $n \in \mathbb{Z}$

$(X \xrightarrow{\sim} Y \Leftrightarrow X[1] \xrightarrow{\sim} Y[1])$

$s \in S \Leftrightarrow s[1] \in S$

(2) If

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"'] & Z \arrow[r] \arrow[d, dotted, "u"] & X[1] \arrow[d, "s[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

$s, t \in S$, rows dist $\Rightarrow \exists$ a completion

$Z \xrightarrow{u} Z'$, $u \in S$

<u>Def</u> A localization system S is compatible with the Δ structure on $\mathcal{T}$ if it satisfies (1) & (2).

Thm If $\mathcal{T}$ - triangulated, S is a localization system (left & right) compatible with the Δ structure on $\mathcal{T}$, then $\mathcal{T}[S^{-1}]$ has a natural Δ structure: $\mathcal{T} \longrightarrow \mathcal{T}[S^{-1}]$ is triangulated.

Pf Need the Δ structure. Since $\mathrm{Ob}\,\mathcal{T}[S^{-1}] = \mathrm{Ob}\,\mathcal{T}$, put $X[1]$ - same as in $\mathcal{T}$, for morphisms

$$\begin{tikzcd}[column sep=small] & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \overset{[1]}{\longmapsto} \quad \begin{tikzcd}[column sep=small] & X'[1] \arrow[dl, "{s[1]}"'] \arrow[dr, "{f[1]}"] & \\ X[1] & & Y[1] \end{tikzcd}$$

use (1) from compatibility

$X \to Y \to Z \to X[1]$ is dist in $\mathcal{T}[S^{-1}]$ iff it's isom to the image under $\mathcal{T} \to \mathcal{T}[S^{-1}]$ of a dist. triangle ($\mathcal{T} \to \mathcal{T}[S^{-1}]$ will be exact as soon as $\mathcal{T}[S^{-1}]$ is triangulated).

Verify the axioms:

(TR1) • isom to dist - dist by definition

• $X \xrightarrow{id} X \to 0 \to X[1]$ dist by definition

• $X \xrightarrow{s^{-1}\circ f} Y$, want to complete to a dist

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal, "id_X"'] & Y' \arrow[r, "g"] & Z \arrow[r] \arrow[d, equal, "id_Z"] & X[1] \arrow[d, equal, "id_X"] \\
X \arrow[r, "s^{-1}\circ f"] & Y \arrow[r, "g\circ s"] \arrow[u, "s"'] & Z \arrow[r] & X[1]
\end{tikzcd}$$

$\leftarrow$ dist in $\mathcal{T}$

$\leftarrow \simeq$ of Δ's

$\leftarrow$ dist in $\mathcal{T}[S^{-1}]$

(TR2) by definition

(TR3) Assume given an almost morphism of Δ's :

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists"] & X[1] \arrow[d, "f[1]"] \\
X'' \arrow[r] & Y'' \arrow[r] & Z'' \arrow[r] & X''[1] \\
X' \arrow[r, "u'"] \arrow[u, "s"] \arrow[ur] & Y' \arrow[r, "v'"] \arrow[u, "t"] & Z' \arrow[r] \arrow[u, dashed, "\exists \in S"'] & X'[1] \arrow[u, "s[1]"']
\end{tikzcd}$$

see below

use (2) from compatibility

Using the localization system conditions:

$$\begin{tikzcd} & Y''' \\ X'' \arrow[ur] & Y'' \arrow[u, "qr"'] \\ X' \arrow[u, "s"] \arrow[ur, "tu'"'] & \end{tikzcd}$$

replacing Y'' with Y''', g with rg, t with rt we may assume that there is a morphism $X'' \longrightarrow Y''$

(TR4) <u>Exc</u> Attempt or look in a book. $\square$

<u>Cor</u> The derived category is triangulated.

<u>Pf</u> $D(\mathcal{A}) = K(\mathcal{A})[qis^{-1}]$, enough to show that Qis is compatible with the Δ structure on $K(\mathcal{A})$.

(1) is trivial: $[n]$ preserves / shifts maps on cohomology.

(2) Assume given

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

Step 1: $f = \mathrm{id}_X$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X \arrow[r] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \\
& W \arrow[r, equal] \arrow[d] & W \arrow[d] & \\
& Y[1] \arrow[r] & Z[1] &
\end{tikzcd}$$

enough to show that h is in Qis.

g is Qis $\Rightarrow$

$\Rightarrow$ W is acyclic

$\Rightarrow$ $h \in$ Qis !

A very similar argument applies to $g = \mathrm{id}_Y$.

<u>Exc</u> Finish the proof. $\square$

Obs $\mathcal{D}^*(\mathcal{A})$ is triangulated for $* \in \{+,-,b\}$.

3) Triangulated subcategories

[**Recall** When discussing abelian categories $\rightsquigarrow$ relation b/w quotients & localization $\boxed{\text{Later}}$]

Def $\mathcal{S} \subset \mathcal{T}$ is triangulated if $\mathcal{S}$ is triangulated, $\mathcal{S} \hookrightarrow \mathcal{T}$ is exact.

Lm $\mathcal{S} \subset \mathcal{T}$ be a full subcategory s.t. with every object $\mathcal{S}$ cotains all isomorphic ones. Then $\mathcal{S}$ is triangulated iff $\mathcal{S}$ is closed under $[\;]$ & taking the cone ($X \to Y \to Z \to X[1]$ dist in $\mathcal{T}$, $X, Y \in \mathcal{S} \Rightarrow Z \in \mathcal{S}$).

Pf $\mathcal{S} \subset \mathcal{T}$ is triangulated $\Rightarrow$ $\forall$ $X \in \mathcal{S}$

$$X[1]_{\mathcal{S}} \simeq X[1]_{\mathcal{T}} \Rightarrow \mathcal{S} \text{ is closed under } [\;]$$

Let $X \xrightarrow{f} Y$ be a morphism in S.

$X \xrightarrow{f} Y \to Z \to X[1]$ – dist in S.

$\Rightarrow$ dist in $\mathcal{T}$ $\Rightarrow$ $Z \simeq$ any cone of $X \to Y$ in $\mathcal{T}$

$\Rightarrow$ closed under taking cones.

The other direction: define $[1]$ on S as the restr of $[1]$ on $\mathcal{T}$, dist. triangles – those dist in $\mathcal{T}$. $\square$

<u>Cor</u> $K^*(\mathcal{A})$ are full triangulated subcategories in $K(\mathcal{A})$.

Let $\mathcal{E} \subset \mathcal{A}$ be a class of objects closed under direct sums & isomorphisms. $K^*(\mathcal{E}) \subset K^*(\mathcal{A})$ – full subcategory with terms in $\mathcal{E}$.

<u>Cor</u> $K^*(\mathcal{E})$ – full triang in $K^*(\mathcal{A})$.

4) <u>Interplay with localization</u>

General situation

$\mathcal{T}' \subset \mathcal{T}$ – subcategory in $\mathcal{T}$, $S \subset Mor(\mathcal{T})$ – class of morphisms. Put $S' = S \cap \mathcal{T}'$. Want

$$\mathcal{T}'_{S'} \longrightarrow \mathcal{T}_S \quad \leftarrow \text{functor b/w localizations}$$

<u>Lm</u> Let $\mathcal{T}' \subset \mathcal{T}$ be a full subcategory, $S \subset Mor\,\mathcal{T}$ be a right localization system, $S' = S \cap \mathcal{T}'$.

(i) if S' is a right loc. system $\Rightarrow$ $\mathcal{T}'_{S'} \longrightarrow \mathcal{T}_S$ is well-defined,

(ii) assume $\forall$ $\underset{\in S}{s} : \underset{\in \mathcal{T}'}{Y} \longrightarrow X$ $\quad \exists\, g : X \longrightarrow \underset{\in \mathcal{T}'}{W}$

s.t. $gs \in S$. Then S' is a right localization system &

$$\mathcal{T}_{S'} \longrightarrow \mathcal{T}_S \quad \text{is fully faithful!}$$

<u>Problem</u> Prove the lemma!