Triangulated and Derived Categories in Algebra and Geometry
Lecture 16

0) Answers to some questions

Def $\mathcal{D} \subset \mathcal{C}$ is strictly full if $\forall X \in \mathcal{D}$, $Y \in \mathcal{C}$ if $X \simeq Y \Rightarrow Y \in \mathcal{D}$.

Lm $\mathcal{D} \subset \mathcal{T}$ strictly full in $\mathcal{T}$, $\mathcal{T}$-triangulated. Then $\mathcal{D}$ is a Δ subcat. $\Leftrightarrow$ $\mathcal{D}$ is closed under $[\ ]$ & taking cones:

$$X \to Y \to Z \to X[1] \quad \text{dist in } \mathcal{T},$$
$$X, Y \in \mathcal{D} \Rightarrow Z \in \mathcal{D}.$$

Want to appy to $K^*(\mathcal{A}) \subset K(\mathcal{A})$, where $* \in \{+, -, b\}$.

Exc These might not be in general strictly full.

However Rmk that if $\mathcal{D} \subset \mathcal{T}$ is a full subcategory s.t.

$$\forall X \in \mathcal{T} \ \exists\, Y \in \mathcal{D} \text{ s.t. } Y \simeq X \quad (\mathcal{D} \to \mathcal{T} \text{ is an equiv.}).$$

One can put the Δ structure on $\mathcal{D}$ (for simplicity assume that $\mathcal{D}$ closed $[\,]$) by saying

$$X \to Y \to Z \to X[1] \text{ is dist} \iff$$
$$\text{isomorphic to a dist (dist in } \mathcal{T}).$$

Works since:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \quad \text{is dist in } \mathcal{T}, \quad Z \xrightarrow[\sim]{h} Z$$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "id"] \\
X \arrow[r, "u"] & Y \arrow[r, "hv"] & Z' \arrow[r, "wh^{-1}"] & X[1]
\end{tikzcd}$$

$\leftarrow$ isom of $\Delta \Rightarrow$

$\Rightarrow$ lower one is dist.

How to use it in our situation?

Define $K(\mathcal{A})^*$, $* \in \{+,-,b\}$ by the full subcategory containing $*$ cohomologically bounded objects.

$K(\mathcal{A})^- = \{ X^\cdot \in K(\mathcal{A}) \mid H^i(X^\cdot) = 0 \text{ for } i \gg 0 \}$

<u>Remark</u> $K^-(\mathcal{A}) \subset K(\mathcal{A})^-$ ← equivalence of categories.

$K(\mathcal{A})^-$ is a strictly full subcategory of $K(\mathcal{A})$.

There was a question about why $D^b(\mathbb{Z}\text{-mod})$ is not abelian.

$\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z}$ if had a kernel $\Rightarrow$

$\Rightarrow$ ker is a direct summand in $\mathbb{Z}/4\mathbb{Z}$

(showed that now split)

$\mathbb{Z}/4\mathbb{Z}$ is indecomposable in $\mathbb{Z}$-mod. Why in $D^b(\mathbb{Z}\text{-mod})$?

<u>Exe</u> $\mathcal{A} \longrightarrow D^b(\mathcal{A})$ is fully faithful

(do it by hand).

Decompositions in $\oplus \xleftrightarrow{1:1}$ idempotents in End.

1) Important variants of homotopy / derived categories

$\mathcal{E} \subset \mathcal{A}$, $\mathcal{E}$ closed under $\oplus$ and $\simeq$'s

$K^*(\mathcal{E}) \subset K^*(\mathcal{A})$ – strictly full subcategory of objects (complexes) whose terms are in $\mathcal{E}$.

Know (lemma) $K^*(\mathcal{E})$ – triangulated subcategory.

$\mathcal{B} \subset \mathcal{A}$ - Serre subcategory:

$$0 \to B' \to A \to B'' \to 0, \quad B', B'' \in \mathcal{B} \Rightarrow A \in \mathcal{B}.$$

Def $K^*_{\mathcal{B}}(\mathcal{A})$ – full subcategory in $K^*(\mathcal{A})$ consisting of complexes whose cohomology belongs to $\mathcal{B}$.

Typical situation: $\mathcal{A} = R\text{-Mod}$ (abelian category of left R-modules), $\mathcal{B} = R\text{-mod}$ (abelian category of finitely generated R-modules). Want to compare

D or $K^*_{\mathcal{B}}(\mathcal{A})$ with D or $K^*(\mathcal{B})$.
(Need some Noetherian assumptions at least.)

Complexes with nice terms vs complexes with nice cohomology.

<u>Ex</u> $\mathcal{B} \subset \mathcal{A}$ – Serre $\Rightarrow$ $K^*_{\mathcal{B}}(\mathcal{A}) \subset K^*(\mathcal{A})$ is a Δ subcat.

2) <u>Localization of subcategories</u>

<u>Lm</u> $\mathcal{T}' \subset \mathcal{T}$ - strictly full Δ subcategory, S - left localization system (comp w Δ) & $\forall$ $\underset{\mathcal{T}'}{\overset{\cap}{X}} \xrightarrow{s} \underset{\mathcal{T}}{\overset{\cap}{Y}}$, $s \in S$ $\exists t : Y \to \underset{\mathcal{T}'}{\overset{\cap}{X'}}$ s.t.
$X \to Y \to X' \in S$. Then $S \cap \mathcal{T}'$ is a left localization system compatible with the Δ structure,
$$\mathcal{T}'[S^{-1}] \hookrightarrow \mathcal{T}[S^{-1}] \text{ - strictly full } \Delta \text{ subcategory.}$$
Similar statement for right localization.

Cor Define $D^*_{\mathcal{B}}(\mathcal{A}) = K^*_{\mathcal{B}}(\mathcal{A})[\mathrm{Qis}^{-1}]$. Then

$D^*_{\mathcal{B}}(\mathcal{A}) \subset D^*(\mathcal{A})$ is a strictly full Δ subcategory.

Pf Know that Qis is a localization system in $K^*(\mathcal{A})$ compatible w/ the Δ structure.

Apply the lemma to $K^*_{\mathcal{B}}(\mathcal{A}) \subset K^*(\mathcal{A})$

If
$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Qis}} & Y \\ \cap & & \cap \\ K^*_{\mathcal{B}}(\mathcal{A}) & & K^*(\mathcal{A}) \end{array}$$
$\Rightarrow$ cohomology of Y is in $\mathcal{B}$

$\Rightarrow Y \in K^*_{\mathcal{B}}(\mathcal{A})$!

$Y \xrightarrow{\mathrm{id}} Y$ works. $\square$

Lm $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Assume that for any

$A \twoheadrightarrow B$, $A \in \mathcal{A}$, $B \in \mathcal{B}$ $\exists$ $B' \hookrightarrow A$, $B' \in \mathcal{B}$ s.t.

$$\begin{tikzcd} B' \arrow[r, hook] \arrow[rr, bend left] & A \arrow[r, two heads] & B \end{tikzcd}$$

is epi. Then $D^-_{\mathcal{B}}(\mathcal{A}) \xrightarrow{\sim} D^-(\mathcal{B})$

and $D^b_{\mathcal{B}}(\mathcal{A}) \xrightarrow{\sim} D^b(\mathcal{B})$.

Cor A - Noetherian ring $\Rightarrow$ $D^b_{A\text{-mod}}(A\text{-Mod}) \xleftarrow{\sim} D^b(A\text{-mod})$.

Pf Indeed:

$$\begin{tikzcd} & A^{\oplus n} \arrow[dl, dashed, "f"'] \arrow[d, two heads] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

N f.g.

$\operatorname{Im} f$ - f.g. submodule like in the lemma! $\square$

Pf (Lemma)

$K^*(\mathcal{B}) \subset K^*(\mathcal{A})$ - strictly full Δ subcategory in $K^*_{\mathcal{B}}(\mathcal{A})$.

Will check that for $* \in \{-, b\}$ for any complex $A^\bullet \in K^*_{\mathcal{B}}(\mathcal{A})$ there exists a q.is subcomplex $B^\bullet \subset A^\bullet$ whose terms are in $\mathcal{B}$.

(Assume that $\mathcal{T}' \subset \mathcal{T}$ is a strictly full Δ subcategory, S - localization system comp w/ the Δ structure.

If $\forall X \in \mathcal{T}$ $\exists$ $Y \xrightarrow{s} X$, $Y \in \mathcal{T}'$, $s \in S$ $\Rightarrow$

1) $S' = S \cap \mathcal{T}'$ is a righ localization system,
2) $\mathcal{T}'[S'^{-1}] \xrightarrow{\sim} \mathcal{T}\{S\}$.

<u>Pf</u>

$$\begin{tikzcd} X \arrow[r, "t"] & Z \\ & \mathcal{T}' \arrow[u] \end{tikzcd} \qquad t \in S$$

consider $Y \xrightarrow{s} X$, $s \in S$, $Y \in \mathcal{T}' \Rightarrow$

$$\begin{tikzcd} Y \arrow[r] \arrow[rr, bend left, "ts"] & X \arrow[r] & Z \end{tikzcd} \qquad ts \in S \text{ since } s, t \in S!$$

$\mathcal{T}'[s'^{-1}] \hookrightarrow \mathcal{T}[s^{-1}]$ is fully faithful. But our cond implies essentially surjective! $\square$)

Let's build this complex B by induction:

$A^n = 0$ for $n \gg 0$ $\Rightarrow$ put $B^n = 0$.

Inductive step: assume that B^k are constructed for $k \geq n$, $H^k(B^\bullet) = H^k(A^\bullet)$ for $k \geq m+1$, $H^m(B^\bullet) \twoheadrightarrow H^k(A^\bullet)$.

$$\begin{tikzcd}
 & & B^n \arrow[r] \arrow[d, hook] & B^{n+1} \arrow[r] \arrow[d, hook] & B^{n+2} \arrow[r] \arrow[d, hook] & \cdots \\
A^{n-2} \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & A^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

Need to find $B^{n-1} \subset A^{n-1}$ & solve two problems: should be surjective on H^{n-1} & fix H^n.

2^{nd} problem: find a subobject $B_0 \hookrightarrow A^{n-1}$ s.t.
$B_0 \hookrightarrow A^{n-1} \longrightarrow I^n \hookleftarrow \mathrm{Im}(A^{n-1} \to A^n)$ be surjective on $I^n \cap B^n$: apply the condition to the epi

$$B_0 \hookrightarrow \underbrace{\mathrm{Ker}\left(A^{n-1} \oplus I^n \cap B^n \longrightarrow I^n\right)} \longrightarrow I^n \cap B^n$$

1^{st} problem:

$$\begin{tikzcd}
B_1 \arrow[r, hook] \arrow[rr, bend right, two heads] & \mathrm{Ker}(A^{n-1} \to A^n) \arrow[r, two heads] & H^{n-1}(A^\circ) \in \mathcal{B}
\end{tikzcd}$$

Put $B^{n-1} = B_0 + B_1 \leftarrow$ sum of subobjects in A^{n-1}. $\square$

3) Quotient & localization

In the case of abelian categories quotients could be defined via localization:

$$A \xrightarrow{s} A' \quad \text{is in } S \text{ if } \ker s \in \mathcal{B},\ \operatorname{coker} s \in \mathcal{B}.$$

Def Let $\mathcal{N} \subset \mathcal{T}$ be a strictly full Δ subcategory.
triangulated
The Δ quotient $\mathcal{T}/\mathcal{N}$ is a Δ category + a exact functor $\mathcal{T} \xrightarrow{Q} \mathcal{T}/\mathcal{N}$ s.t. $Q(\mathcal{N}) = 0$, given any $\mathcal{T} \xrightarrow{F} \mathcal{T}'$ s.t. $F(\mathcal{N}) = 0$ $\exists!$ up to isom $F_Q : \mathcal{T}/\mathcal{N} \to \mathcal{T}'$ s.t. $F_Q \circ Q \simeq F$:

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{T}' \\ & \mathcal{T}/\mathcal{N} \arrow[ur, dashed, "\exists!\, F_Q"'] & \end{tikzcd}$$

Thm $\mathcal{T}/\mathcal{N}$ always exists.

Prop Let $\mathcal{N} \subset \mathcal{T}$ be a strictly full Δ subcategory. Put $S \subset \mathrm{Mor}\,\mathcal{T}$ be those $f: X \to Y$ s.t.

$$X \xrightarrow{f} Y \to Z \to X\{1\} \text{ dist } \Rightarrow Z \in \mathcal{N}.$$

Then S is a localization system and $\mathcal{T} \to \mathcal{T}[S^{-1}]$ is the quotient $\mathcal{T}/\mathcal{N}$.

Before the proof, nice applications.

$$D^*(\mathcal{A}) = K^*(\mathcal{A}) / Acyc^*(\mathcal{A}), \text{ where}$$

$Acyc^*(\mathcal{A})$ - full subcategory of acyclic objects (LES of cohomology).

We can also reformulate our subcategory lemma:

Lm $F: \mathcal{T}' \to \mathcal{T}$ - exact functor, $\mathcal{N} \subset \mathcal{T}$ is a strictly full subcategory. Put $\mathcal{N}' = F^{-1}(\mathcal{N})$.

1) $\exists$ exact $\mathcal{T}'/\mathcal{N}' \to \mathcal{T}/\mathcal{N}$.

2) Assume that F is fully faithful & $\forall X \in \mathcal{T}'$, $N \in \mathcal{N}$ $\quad N \to F(X)$ can be decomposed as $N \to F(N') \to F(X)$ for some $\underset{\mathcal{N}'}{N'} \to X$.

Then $\mathcal{T}'/_{\mathcal{N}'} \to \mathcal{T}/_{\mathcal{N}}$ is fully faithful.

<u>Cor</u> Assume that $\mathcal{A}$ has enough projectives. Then

$$K^-(\mathrm{Proj}\,\mathcal{A}) \longrightarrow K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A}) \leftarrow \text{equivalence!}$$

full subcat of complexes with projective terms

$$K^-(\mathrm{Proj}\,\mathcal{A})^b \xrightarrow{\sim} D^b(\mathcal{A}).$$

bounded above, bounded cohomology

<u>Pf</u> In $K^-(\mathcal{A})$ $\quad \mathrm{Acyc}^-(\mathcal{A}) \cap K^-(\mathrm{Proj}\,\mathcal{A}) = 0$:

$\dots \to P^{n-1} \to P^n \to 0 \to 0$ acyclic $\Rightarrow$

$\Rightarrow$ a resolution of 0, any two proj. resol's are isom in $K^-(\mathcal{A}) \Rightarrow P^\bullet \simeq 0$

<u>Exc</u> Check the condition / dual one from the lemma.

<u>Pf</u> (Proposition)

Need to check that S

$$\begin{tikzcd} X \arrow[r, "s"] & Y \arrow[r] & Z \arrow[r] & X[1] \\ & & \mathcal{N} \arrow[u, phantom, "\in" rotate=90] & \end{tikzcd}$$

satisfies the localization system conditions.

(1) f, g, gf if 2 are in S, then so is the 3rd.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, phantom, "\parallel"] & Y \arrow[r] \arrow[d, "g"] & U \arrow[r] \arrow[d] & X[1] \arrow[d, phantom, "\parallel"] \\ X \arrow[r, "gf"] & Z \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & X[1] \\ & W \arrow[r, equal] \arrow[d] & W \arrow[d] & \\ & Y[1] \arrow[r] & U[1] & \end{tikzcd}$$

$f \in S \Rightarrow U \in \mathcal{N}$

$g \in S \Rightarrow W \in \mathcal{N}$

$\mathcal{N}$ - Δ $\Rightarrow$

$V \in \mathcal{N}$

same for other cases

(2)
$$\begin{tikzcd} & X \arrow[dr, dashed, "f"] & \\ W \arrow[ur, "t"] \arrow[dr, "g"'] & & Z \\ & Y \arrow[ur, dashed, "s"'] & \end{tikzcd}$$

$t \in S \Rightarrow \exists\, f, s : s \in S$

Complete $W \xrightarrow{\binom{t}{g}} X \oplus Y \xrightarrow{(-f\ s)} Z \to W[1]$

Want: check that $s \in S \Leftrightarrow$ cone of $Y \xrightarrow{s} Z$ is in $\mathcal{N}$.

$$\begin{tikzcd} Y \arrow[r, "\iota_Y"] \arrow[d, equal] & X \oplus Y \arrow[r, "p_X"] \arrow[d] & X \arrow[r, "0"] \arrow[d] & Y[1] \arrow[d, equal] \\ Y \arrow[r, "s"] & Z \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & Y[1] \\ & W[1] \arrow[r, equal] \arrow[d] & W[1] \arrow[d, "t[1]"] & \\ & X[1] \oplus Y[1] \arrow[r, "p_{X[1]}"'] & X[1] & \end{tikzcd}$$

$t[1] \in S \Rightarrow$
$\Rightarrow N \in \mathcal{N} \Rightarrow$
$\Rightarrow s \in S$.

(3) $X \xrightarrow{s} Y \xrightarrow{f} Z$ $\quad fs = 0 \Rightarrow \exists\ t : Z \to W$ s.t. $tf = 0$, $t \in S$.

$$\begin{tikzcd} X \arrow[r, "s"] & Y \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[dl, dashed, "h"', "t"] & X[1] \\ & Z & \mathcal{N} \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

$fs = 0 \Rightarrow$ (LES of Hom's for dist Δ's)

Take the cone of h:

$t \leftarrow$ that we want.

$$N \xrightarrow{h} Z \to W \to N[1]$$

Remains to check compatibility with the Δ structure. Do it yourself. $\square$