# Triangulated and Derived Categories in Algebra and Geometry
# Lecture 17

## 1. Localization of functors

$\mathcal{C}$ - category, $S$ - multiplicative system

$\mathcal{C}[S^{-1}]$ - exists, $Q: \mathcal{C} \to \mathcal{C}_S = \mathcal{C}[S^{-1}]$ - localization functor

$F: \mathcal{C} \to \mathcal{A}$ - a functor

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{A} \\ & \mathcal{C}_S \arrow[ur, dashed] & \end{tikzcd}$$

when does this exist?

By the UP of $\mathcal{C}_S$ such a triangle exists $\iff$ $F(S) \subseteq \mathrm{Iso}_{\mathcal{A}}$.

What if $F(S) \not\subseteq \mathrm{Iso}_{\mathcal{A}}$?

Solution: take some kind of approximation.

__Def__ A right localization of $F$ is a functor $RF : \mathcal{C}_S \to \mathcal{A}$
+ a nat. transformation $\alpha : F \to RF \circ Q$

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "Q"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathcal{A} \\
& \mathcal{C}_S \arrow[ur, "RF"'] &
\end{tikzcd}$$

s.t. for any other $G : \mathcal{C}_S \to \mathcal{A}$ + $F \to G \circ Q$
$\exists!$ factorization $F \to RF \circ Q \to G \circ G$
In other words,

$$\mathrm{Hom}(RF, G) \xrightarrow{1:1} \mathrm{Hom}(F, G \circ Q).$$

Similarly one defines a left localization:

$$LF : \mathcal{C}_S \to \mathcal{A} \quad + \quad \beta : LF \circ Q \to F \quad + \quad \text{UP}.$$

__Lm__ If a right (left) localization exists $\Rightarrow$ unique up to $\simeq$ of functors.

**Observation** If $F(S) \subseteq \mathrm{Iso}_{\mathcal{A}}$, then $F_S \leftarrow$ the factorization is both RF & LF.

When do these exist / how to construct?

Look for a reasonable subcategory $\mathcal{T} \subset \mathcal{C}$ s.t. $F(T) \subseteq \mathrm{Iso}_{\mathcal{A}}$, where $T = S \cap \mathcal{T}$.

From the previous week: if $\mathcal{T} \subset \mathcal{C}$ is a full subcategory s.t. $\forall X \in \mathcal{C}$ $\exists$ $s: W \to X$ s.t. $W \in \mathcal{T}$, $s \in S$. Then $T = S \cap \mathcal{T}$ is a right localization system & $\mathcal{T}_T \to \mathcal{C}_S$ is an equivalence.

Assume further that $\forall s \in T = S \cap \mathcal{T}$ $F(s) \in \mathrm{Iso}_{\mathcal{A}}$.

Then the right localization exists by the following.

Let $\iota : \mathcal{T} \to \mathcal{C}$ be the embedding functor.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "Q_S"] \arrow[drr, bend left, "F"] & \mathcal{C}_S & \\
\mathcal{T} \arrow[u, "\iota"] \arrow[r, "Q_T"] \arrow[rr, bend right, "F\circ\iota"'] & \mathcal{T}_T \arrow[u, "j"', "\simeq"] \arrow[r, "F_T"] & \mathcal{A}
\end{tikzcd}$$

exists since $F(\mathcal{T}) \subseteq \mathrm{Iso}_{\mathcal{A}}$

Pick a quasi-inverse $j^{-1}$ to $j$, put $RF = F_T \circ j^{-1}$.

<u>Exc</u> Show that $F_T \circ j^{-1}$ is indeed the right localization.

## <u>2. Derived functors</u>

Let $F: \mathcal{A} \to \mathcal{B}$ be an additive b/w abelian.
$F$ extends to $F: K(\mathcal{A}) \to K(\mathcal{B})$.
Want to extend it to $F: D(\mathcal{A}) \to D(\mathcal{B})$.

<u>Lm</u> $F$ preserves quasi-iso's $\iff$ $F$ is exact (as $F: \mathcal{A} \to \mathcal{B}$).

Pf $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a SES in $\mathcal{A}$.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
& 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

$F$ preserves qis $\Rightarrow$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F(X) \arrow[r, "F(f)"] \arrow[d] & F(Y) \arrow[r] \arrow[d, "F(g)"] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & F(Z) \arrow[r] & 0
\end{tikzcd}$$

Thus, $\underbrace{F(X) \hookrightarrow F(Y)}$, $F(Y)/F(X) \simeq F(Z)$

compare $H^{-1}$ of the rows.

$\square$

Define the right derived functor as the right localization:

$$\begin{tikzcd}
K(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] \arrow[dr, bend left, ""{name=U}] \arrow[dr, bend right, ""'{name=D}] & K(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
D(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B})
\arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

$\alpha : Q_{\mathcal{B}} \circ F \to RF \circ Q_{\mathcal{A}}$

+ UP.

Similarly, define $LF$ – left derived functor.

Obs $F$ – exact $\Rightarrow$ both exist and are isomorphic to the functor obtained by termwise application of $F$.

Want to construct these in general.

## 3. Localization of functors & semiorthogonal decompositions

Instead of localizing, let's take quotients!

$F : \mathcal{T} \to \mathcal{T}'$ – exact functor b/w triangulated cat's

$\mathcal{N} \subset \mathcal{T}$ – strictly full $\Delta$ subcategory

Want

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{T}' \\ \mathcal{T}/\mathcal{N} \arrow[ur] & \end{tikzcd}$$

$\Leftarrow$ happens only if $F(\mathcal{N}) = 0$!

Approximation: right localization (as before).

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "Q"'] & \arrow[d, phantom, "\alpha \Downarrow"] & \mathcal{T}' \\ & \mathcal{T}/\mathcal{N} \arrow[ur, "RF"'] & \end{tikzcd}$$

$\alpha$ is universal / initial

<u>Prop</u> Assume that $\iota : \mathcal{N} \longrightarrow \mathcal{T}$ has a right adjoint. Then every exact $F : \mathcal{T} \to \mathcal{T}'$ exact has a right localization.

<u>Def</u> Let $\mathcal{T}$ be a $\Delta$ category. A pair of strictly full $\Delta$ subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ is called a <u>semiorthogonal decomposition</u> of $\mathcal{T}$

$$\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$$

if 1) $\forall B \in \mathcal{B}, A \in \mathcal{A}$ $\quad \mathrm{Hom}_{\mathcal{T}}(B, A) = 0$,

no hom's from right to left

2) $\forall X \in \mathcal{T}$ $\exists$ a dist $\Delta$

$$B_X \longrightarrow X \longrightarrow A_X \longrightarrow B_X[1]$$

with $A_X \in \mathcal{A}$, $B_X \in \mathcal{B}$.

Analogy with linear algebra: $V$ - vector space, $(-,-)$ - nonsymm bilinear form. If $(-,-)$ were symmetric, one would look for $V \simeq U \oplus W$, $U, W \subseteq V$ s.t. $U \perp W$: $\forall u \in U, w \in W$ $(u,w) = (w,u) = 0$. If $(-,-)$ is non-symm, you can only get $(w,u) = 0$ $\forall w \in W, u \in U$.

<u>Lm</u> Let $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. Then $\alpha : \mathcal{A} \hookrightarrow \mathcal{T}$ & $\beta : \mathcal{B} \to \mathcal{T}$ have a left & right adjoint respectively $\alpha^* : \mathcal{T} \to \mathcal{A}$, $\beta^! : \mathcal{T} \to \mathcal{B}$. Moreover, the SOD $\Delta$ is functorial & is isomorphic to the $\Delta$

$$\beta\beta^! X \longrightarrow X \longrightarrow \alpha\alpha^* X \longrightarrow \beta\beta^! X[1]$$

induced by adjointness

<u>Pf</u> We need to construct $\mathcal{T} \to \mathcal{A}$ & $\mathcal{T} \to \mathcal{B}$.
For every $X \in \mathcal{T}$ fix a triangle as in the definition of a SOD:

$$B_X \to X \to A_X \to B_X[1].$$

On objects put $\alpha^*(X) = A_X$, $\beta^!(X) = B_X$. Need to extend to morphisms. $f: X \to Y$ in $\mathcal{T}$.

$$\begin{tikzcd} B_X \arrow[r, "u_X"] \arrow[d, dashed, "B_f"'] & X \arrow[r, "v_X"] \arrow[d, "f"] & A_X \arrow[r] \arrow[d, dashed, "A_f"] & B_X[1] \\ B_Y \arrow[r, "u_Y"] & Y \arrow[r, "v_Y"] & A_Y \arrow[r] & B_Y[1] \end{tikzcd}$$

$v_Y \circ f \circ u_X \in Hom(B_X, A_Y) = 0$

By the LES assoc. to $Hom(B_X, -)$ & the lower $\Delta$ we get $B_f: B_X \to B_Y$.

From the LES assoc. to $Hom(-, A_Y)$ & the upper $\Delta$ we get $A_f: A_X \to A_Y$.

Back a couple of lectures: since $Hom(B_X, A_Y[-1]) = 0$, these are unique! Get functors.

Let's check the adjointness: $\forall B \in \mathcal{B}$ apply $Hom(B, -)$ to $B_X \to X \to A_X \to B_X[1]$.

$$\underset{\overset{\parallel}{0}}{\mathrm{Hom}(B, A_X[-1])} \to \mathrm{Hom}(B, B_X) \xrightarrow{\sim} \mathrm{Hom}(B, X) \to \underset{\overset{\parallel}{0}}{\mathrm{Hom}(B, A_X)}$$

$\Rightarrow$ $B_X$ is right adjoint to $\beta : \mathcal{B} \to \mathcal{T}$.

Similarly for $\mathcal{A}$. $\square$

How to construct SOD's?

$\mathcal{N} \subset \mathcal{T}$ - full $\Delta$ subcategory $\Rightarrow$ if $\mathcal{T} = \langle ?, \mathcal{N} \rangle$, then $\mathcal{N} \hookrightarrow \mathcal{T}$ must have a right adjoint.

**Def** $\mathcal{N} \subset \mathcal{T}$ - full $\Delta$ subcategory is left (right) admissible if $\iota : \mathcal{N} \hookrightarrow \mathcal{T}$ has a left (right) adjoint.

Given $\mathcal{N} \subset \mathcal{T}$ - $\Delta$ subcategory define

$$\mathcal{N}^{\perp} = \{ X \in \mathcal{T} \mid \forall N \in \mathcal{N} \ \mathrm{Hom}_{\mathcal{T}}(N, X) = 0 \},$$

$$^{\perp}\mathcal{N} = \{ Y \in \mathcal{T} \mid \forall N \in \mathcal{N} \ \mathrm{Hom}_{\mathcal{T}}(Y, N) = 0 \}.$$

$\mathcal{N}^\perp$ – left orthogonal, ${}^\perp\mathcal{N}$ – right orthogonal.

<u>Exc</u> $\mathcal{N}^\perp$ & ${}^\perp\mathcal{N}$ are strictly full $\Delta$ subcategories.

<u>Lm</u> If $\mathcal{N} \subset \mathcal{T}$ is right admissible, then $\mathcal{T} = \langle \mathcal{N}^\perp, \mathcal{N} \rangle$.
Similarly, if $\mathcal{N} \subset \mathcal{T}$ is left admissible, then
$\mathcal{T} = \langle \mathcal{N}, {}^\perp\mathcal{N} \rangle$.

<u>Pf</u> Let $\alpha : \mathcal{N} \hookrightarrow \mathcal{T}$, $\alpha^!$ – right adjoint.

$\mathrm{Hom}(\mathcal{N}, \mathcal{N}^\perp) = 0$ by the def. of $\mathcal{N}^\perp$.
Let $X \in \mathcal{T}$, consider the counit & complete to $\Delta$:

$$\alpha\alpha^! X \to X \to Y \to \alpha\alpha^! X[1]$$

$\forall\, N \in \mathcal{N}$:

$$\mathrm{Hom}(\alpha N, \alpha\alpha^! X) \cong \mathrm{Hom}(N, \alpha^! X) \cong \mathrm{Hom}(\alpha N, X) \cong \mathrm{Hom}(N, X)$$

$\Rightarrow \mathrm{Hom}(N, Y) = 0$ (LES assoc to $\mathrm{Hom}(N, -)$ & the $\Delta$). $\square$

Pf (Proposition)

$\beta : \mathcal{N} \hookrightarrow \mathcal{T}$ — strictly full $\Delta$ subcategory, $\beta$ has a right adjoint $\rightsquigarrow$ SOD

$$\mathcal{T} = \langle \mathcal{N}^\perp, \mathcal{N} \rangle .$$

$$\begin{tikzcd} \mathcal{T} \arrow[r, "F"] \arrow[dr, "Q"'] & \mathcal{T}' \\ & \mathcal{T}/\mathcal{N} \end{tikzcd}$$

Put $\alpha : \mathcal{N}^\perp \hookrightarrow \mathcal{N}$.

Let's apply $Q$ to the $\Delta$

$$\beta\beta^! X \longrightarrow X \longrightarrow \alpha\alpha^* X \longrightarrow \beta\beta^! X[1]$$

$Q(\mathcal{N}) = 0$, $\beta : \mathcal{N} \hookrightarrow \mathcal{T} \Rightarrow Q(\beta\beta^! X) = 0$.

$$0 \longrightarrow Q(X) \xrightarrow{\sim} Q(\alpha\alpha^* X) \longrightarrow 0[1]$$

$\rightsquigarrow$ isomorphism of functors $Q \simeq Q\alpha\alpha^*$.

Given $F : \mathcal{T} \longrightarrow \mathcal{T}'$ exact, put

$\tilde{F} = F\alpha\alpha^*$

$$\alpha : \mathcal{N}^{\perp} \underset{\longrightarrow}{\overset{\alpha^*}{\longleftarrow}} \mathcal{T}$$

$F\alpha\alpha^*$ factors throug $\mathcal{T}/\mathcal{N}$!

Once applied to $N \in \mathcal{N}$, $\alpha\alpha^* N = 0$! $\Rightarrow$ $F\alpha\alpha^* N = 0$!

$$\begin{array}{ccccccc} N & \xrightarrow{\sim} & N & \longrightarrow & \alpha\alpha^* N & \longrightarrow & N[1] \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} & & \\ \mathcal{N} & & & & \mathcal{N}^{\perp} & & \end{array}$$

$\tilde{F} = F\alpha\alpha^* = RF \circ Q$

From the unit $\mathrm{Id} \to \alpha\alpha^* \rightsquigarrow F \xrightarrow{\varepsilon} F\alpha\alpha^* = RF \circ Q$.

<u>Claim</u> $RF + \varepsilon$ — right localization.

Check the UP. $G : \mathcal{T}/\mathcal{N} \to \mathcal{T}'$, $\eta : F \to G \circ Q$.

Compose it with $\alpha\alpha^*$:

$$\underset{\substack{\shortparallel \\ RF\circ Q}}{F\alpha\alpha^*} \longrightarrow G\underbrace{Q\alpha\alpha^*}_{\substack{\cong \\ Q}} \rightsquigarrow RF\circ Q \longrightarrow G\circ Q$$

<u>Exc</u> Check uniqueness of this morphism. $\square$

## 4. Apply to derived categories

**Lm** Let $\mathcal{A}$ be abelian with enough injectives. Then

$$K^+(\mathcal{A}) = \langle K^+(\mathrm{Inj}(\mathcal{A})), \mathrm{Acyc}^+(\mathcal{A}) \rangle$$

– SOD. In particular, any additive $F: \mathcal{A} \to \mathcal{B}$ gives a right derived $D^+(\mathcal{A}) \to D^+(\mathcal{B})$.

Similarly, if $\mathcal{A}$ has enough projectives,

$$K^-(\mathcal{A}) = \langle \mathrm{Acyc}^-(\mathcal{A}), K^-(\mathrm{Proj}(\mathcal{A})) \rangle \ - \ \mathrm{SOD}.$$

Any additive has a left derived.

**Pf** Need to check the SOD conditions.

$X^\bullet$ is acyclic, $I^\bullet$ is injective, both bounded from below $\Rightarrow$ $\mathrm{Hom}_{K^+(\mathcal{A})}(X^\bullet, I^\bullet) = 0$.

If $X^\bullet$ is arbitrary (bounded from below),

$\exists$ a quis'm $X^\bullet \xrightarrow{f} I^\bullet$ ← nothing but an injective resolution

$$C(f)^\bullet[-1] \to X^\bullet \xrightarrow{f} I^\bullet \to C(f)^\bullet$$

↑ acyclic since $f$ – q.i.s.

Thus, $K^+(\mathcal{A}) = \langle K^+(\mathrm{Inj}\,\mathcal{A}), \mathrm{Acyc}^+(\mathcal{A}) \rangle$. $\square$

How is this related to classical derived functors?

How is RF defined?

$$\begin{tikzcd}
K^+(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_{\mathcal{A}}"'] \arrow[dr, "F"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
D^+(\mathcal{A}) \arrow[r, "RF"'] \arrow[ur, phantom, "\Swarrow"] & D^+(\mathcal{B})
\end{tikzcd}$$

$RF \circ Q_{\mathcal{A}}$ — apply $F$ term-wise to an injective resolution of $X^\bullet$

<u>Cor</u> $F : \mathcal{A} \to \mathcal{B}$, $\mathcal{A}$ has enough injectives, $F$ is left exact

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[rrr, bend right, "R^iF"'] & D^+(\mathcal{A}) \arrow[r, "RF"] & D^+(\mathcal{B}) \arrow[r, "H^i"] & \mathcal{B} \end{tikzcd}$$

classical derived functor

$$R^iF(X) = H^i(RF(X)).$$