Triangulated and Derived Categories in Algebra and Geometry
Lecture 18

Last time discussed derived functor / localized.

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q_A"'] \arrow[dr, phantom, "\Leftarrow"] & K^*(\mathcal{B}) \arrow[d, "Q_B"] \\ D^*(\mathcal{A}) \arrow[r, dashed, "RF"] & D^*(\mathcal{B}) \end{tikzcd}$$

universal

$$Q_B \circ F \to RF \circ Q_A$$

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "Q"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{T}' \\ & \mathcal{T}/\mathcal{N} \arrow[ur, dashed, "RF"'] & \end{tikzcd}$$

Put $\mathcal{N} = \mathrm{Acyc}^*(\mathcal{A})$,
$F = Q_B \circ F$

Main result: RF exists if $\mathcal{N}$ is left admissible
(maybe right)

Want a SOD: $\mathcal{T} = \langle \mathcal{N}, {}^{\perp}\mathcal{N} \rangle$ or $\langle \mathcal{N}^{\perp}, \mathcal{N} \rangle$.

In good situations:

$\mathcal{I} = K^+(\mathcal{A})$ if $\mathcal{A}$ has enough injectives $\Rightarrow$

$\Rightarrow$ $\mathcal{N} = \mathrm{Acyc}^+(\mathcal{A})$, then $\mathcal{N}^\perp = K^+(\mathrm{Inj}\,\mathcal{A})$

If $\mathcal{A}$ has enough projectives $\Rightarrow$

$\Rightarrow$ $K^-(\mathcal{A}) = \langle \mathrm{Acyc}^-(\mathcal{A}), K^-(\mathrm{Proj}\,\mathcal{A}) \rangle$ $\Rightarrow$

$\Rightarrow$ can compute (and define) LF as

1) take a projective resolution,
2) apply F term-wise.

Problem In many nice situations $\mathcal{A}$ does not have enough projectives! (Sheaves of Ab)

Still want to compute left derived functors.

$\mathcal{F}, \mathcal{G} \in \mathrm{AbSh}(X) \rightsquigarrow \mathcal{F} \otimes \mathcal{G}$ - sheafify $U \mapsto \mathcal{F}(U) \otimes_{\mathbb{Z}} \mathcal{G}(U)$.

Right exact in both arguments.

Comment $\mathcal{A} \longrightarrow D^*(\mathcal{A})$ – fully faithful embedding.

$F: \mathcal{A} \to \mathcal{B}$ (say, right exact), then if $LF: D^*(\mathcal{A}) \to D^*(\mathcal{B})$ exists, then for any SES $0 \to A' \to A \to A'' \to 0 \rightsquigarrow$

$$\rightsquigarrow \underbrace{LF(A') \to LF(A) \to LF(A'') \to LF(A')[1]}_{\text{exact}}$$

0. Back to category theory

Let I be a category, $\alpha: I \to \text{Sets}$.

One can define a binary relation on $\bigsqcup \alpha(i)$:

$(x, y) \in R$, $x \in \alpha(i)$, $y \in \alpha(j)$ if $\exists$ $f: i \to k$, $g: j \to k$

s.t. $\alpha(f)(x) = \alpha(g)(y)$.

<u>Obs</u> R is reflexive & symmetric.

Not transitive in general.

$\mathrm{colim}\,\alpha = \varinjlim \alpha \simeq \left(\bigsqcup \alpha(i)\right)/\sim$, where

$\sim$ is the transitive closure of R (smallest equiv relation refining R).

There are conditions on I when R is an equiv. relation.

<u>Def</u> A small category I is called filtered if

1) $I \neq \varnothing$,

2) $\forall\, i, j \in I \;\; \exists$
$$\begin{tikzcd} i \arrow[dr, dashed] & \\ & k \\ j \arrow[ur, dashed] & \end{tikzcd}$$

3) $\forall\; i \rightrightarrows j$ can be coeq'd: $i \rightrightarrows j \dashrightarrow k$

<u>Exc</u> If I is filtered $\Rightarrow$ R is an equivalence relation.

<u>Exc</u> Let I be filtered, J be finite, $\alpha : I \times J \to \mathrm{Sets}$.

Then $\varinjlim_i \varprojlim_j \alpha(i,j) \xrightarrow{\sim} \varprojlim_j \varinjlim_i \alpha(i,j)$.

(filtered colimits commute with finite limits).

(*Hint*: enough to check for finite products — $\mathcal{J}$ is discrete & equalizers — $\mathcal{J} = \bullet \rightrightarrows \bullet$)

Exc I -small is filtered $\Leftrightarrow$ colim over I commute with finite limits.

Remember Grothendieck categories?
AB5 axiom (intersection of chains of subobjects with a subobject).
AB5 $\Leftrightarrow$ filtered colimits are exact!

Non-example $\bullet \rightrightarrows \bullet$ $\leftarrow$ not filtered!

Def Let $\varphi: \mathcal{J} \to I$ be a functor b/w small categories.
We say that φ is cofinal if
1) $\forall i \in I \ \exists \ i \to \varphi(j)$ for some j,

2) $\forall i$, $i \to \varphi(j)$, $i \to \varphi(k)$ $\exists\ j \xrightarrow{f} t$, $k \xrightarrow{g} t$

$$\begin{tikzcd}
 & \varphi(j) \arrow[dr, dashed, "\varphi(f)"] & \\
i \arrow[ur] \arrow[dr] & & \varphi(t) \\
 & \varphi(k) \arrow[ur, dashed, "\varphi(g)"'] &
\end{tikzcd}$$

<u>Lm</u> Let $\varphi : \mathcal{J} \to I$ be cofinal, $\alpha : I \to \mathcal{C}$. If $\varinjlim \alpha$ exists, then $\varinjlim \alpha \simeq \varinjlim \alpha\circ\varphi$.

<u>Exc</u> Define final functors, get that $\varprojlim$ are the same.

<u>Exc</u> I is filtered $\iff$ $\Delta : I \to I \times I$ is cofinal.

1. Rethinking localization

Recall that $S \subset \mathrm{Mor}\,\mathcal{C}$ is a right localization system if

1) $\forall x \in \mathcal{C}$ $\mathrm{id}_x \in S$,

2) $s, t \in S \Rightarrow t \circ s \in S$ (if compose)

3)
$$\begin{tikzcd} X' \arrow[r, dashed, "g"'] & Y' \\ X \arrow[u, "S \ni s"] \arrow[r, "f"'] & Y \arrow[u, dashed, "t \in S"'] \end{tikzcd}$$

4) $W \xrightarrow{s} X \overset{f}{\underset{g}{\rightrightarrows}} Y \overset{t}{\dashrightarrow} Z.$

Given $\mathcal{C}$, $S \subset \mathrm{Mor}\,\mathcal{C}$ - right localization system, define the category S^X for $X \in \mathcal{C}$ by

$$\mathrm{Ob}\, S^X = \{ s : X \to X' \mid s \in S \}$$

$$\mathrm{Hom}(X \xrightarrow{s} X', X \xrightarrow{t} X'') = \left\{ \begin{tikzcd}[row sep=small] & X' \arrow[dd, "f"] \\ X \arrow[ur, "s"] \arrow[dr, "t"'] & \\ & X'' \end{tikzcd} \right\}$$

<u>Lm</u> S^X is filtered.

<u>Pf</u> (Issue about small, but ignore it.)

1) $S^X \neq \emptyset$: $X \xrightarrow{id_X} X \in S^X$ since $id_X \in S$.

2) $\forall (X \xrightarrow{s} X')$, $(X \xrightarrow{t} X'')$ $\exists$ a third object & morphisms into it from both.

$$\begin{tikzcd} & X' \arrow[dr] & \\ X \arrow[ur, "s"] \arrow[rr, "s' \circ t"] \arrow[dr, "t"'] & & X''' \\ & X'' \arrow[ur, "s'"'] & \end{tikzcd}$$

$s' \in S$, $t \in S \Rightarrow s' \circ t \in S \Rightarrow$

$\Rightarrow X \to X'''$ is our object

3) finish yourself.

$\square$

Remark that if $Y \xrightarrow{s} X$, $s \in S$ $\rightsquigarrow$

$\rightsquigarrow$ get a functor $S^X \longrightarrow S^Y$

$$(X \xrightarrow{t} X') \longmapsto \left(\begin{tikzcd} Y \arrow[r, "s"] \arrow[rr, bend right] & X \arrow[r, "t"] & X' \end{tikzcd} \right).$$

<u>Exc</u> Show that $S^X \xrightarrow{\circ s} S^Y$ is cofinal.

There is a natural "forgetful" functor

$$S^X \xrightarrow{\pi} \mathcal{C}, \qquad (X \to X') \longmapsto X'.$$

Given $X, Y \in \mathcal{C}$, set

$$\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \varinjlim_{S^Y} \mathrm{Hom}(X, Y') = \varinjlim \left(\mathrm{Hom}(X, -) \circ \pi \right).$$

Since S^Y is filtered, one can compute $\varinjlim$ via the equivalence relation:

$$\coprod_{Y \xrightarrow{s} Y'} \mathrm{Hom}(X, Y') \Big/ \sim$$

$$\left(X \xrightarrow{f} Y' \xleftarrow{s} Y \right) \sim \left(X \xrightarrow{g} Y'' \xleftarrow{t} Y \right) \iff$$

$$\iff \begin{tikzcd} & Y' \arrow[d] & \\ X \arrow[ur] \arrow[r] \arrow[dr] & Y''' & Y \arrow[ul] \arrow[l] \arrow[dl] \\ & Y'' \arrow[u] & \end{tikzcd}$$

all in S

This $\mathrm{Hom}_{\mathcal{C}'}(X,Y)$ is exactly what we defined for localization!

Recover composition?

Observation: $s: X \to X' \in S \Rightarrow$

$$\Rightarrow \mathrm{Hom}_{\mathcal{C}'}(X', Y) \xrightarrow{\circ s} \mathrm{Hom}_{\mathcal{C}'}(X, Y) \leftarrow \text{isomorphism}$$

(Check it using cofinality.)

Define the composition:

$$\mathrm{Hom}_{\mathcal{C}'}(X,Y) \times \mathrm{Hom}_{\mathcal{C}'}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}'}(X,Z)$$

$$\varinjlim_{Y \to Y'} \mathrm{Hom}(X, Y') \times \varinjlim_{Z \to Z'} \mathrm{Hom}(Y, Z') \simeq$$

$$\simeq \varinjlim_{Y \to Y'} \left(\mathrm{Hom}(X, Y') \times \varinjlim_{Z \to Z'} \mathrm{Hom}(Y, Z') \right) \xleftarrow{\sim}$$

$$\xleftarrow{\sim} \varinjlim_{Y \to Y'} \left(\operatorname{Hom}(X, Y') \times \varinjlim_{Z \to Z'} \operatorname{Hom}(Y', Z') \right) \longrightarrow$$

$$\longrightarrow \varinjlim_{Y \to Y'} \varinjlim_{Z \to Z'} \operatorname{Hom}(X, Z') \simeq \varinjlim_{Z \to Z'} \operatorname{Hom}(X, Z') =$$

$$= \operatorname{Hom}_{\mathcal{C}'}(X, Z)$$

<u>Exc</u> This composition is associative.

<u>Exc</u> $\mathcal{C}'$ together with the natural functor

$\mathcal{C} \to \mathcal{C}'$ $(\operatorname{Ob} \mathcal{C}' = \operatorname{Ob} \mathcal{C})$

is the localization.

<u>Exc</u> Formulate everything for left localization systems.

2. Localization of functors: Deligne's construction

$\mathcal{T}$, $\mathcal{T}'$ – triangulated, $\mathcal{N} \subset \mathcal{T}$ – full Δ, S corresponds to $\mathcal{N}$, $F : \mathcal{T} \to \mathcal{T}'$ – exact functor.

Assume RF exists $\Rightarrow$ $\forall\, G : \mathcal{T}/\mathcal{N} \to \mathcal{T}'$ & any $F \to G \circ Q$ we get a factorization

$F \to RF \circ Q \to G \circ Q$. $\quad s : X \to Y \quad s \in S$

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d, "F(s)"'] & RF(Q(X)) \arrow[r, dashed] \arrow[d, "s"] & G(Q(X)) \arrow[d, "s"] \\
F(Y) \arrow[r] \arrow[ur] & RF(Q(Y)) \arrow[r, dashed] & G(Q(Y))
\end{tikzcd}$$

Get a morphism $\varinjlim_{S^X} F(Y) \to (RF \circ Q)(X)$.

UP of the right derived functor $\rightsquigarrow$

$\rightsquigarrow$ conclud $\varinjlim_{S^X} F(Y) \xrightarrow{\sim} (RF \circ Q)(X)$.

Let us actually define

$$RF(X) \quad \text{as} \quad \varinjlim_{X \to Y \in S^X} F(Y)$$

<u>Problems</u> 1) might not exist,
2) define on morphisms.

Solution: put

$$rF(X) = \varinjlim_{X \to Y \in S^X} h_{F(Y)} \in \mathrm{Fun}(\mathcal{T}^{op}, Ab)$$

<u>Check</u> $F(\mathcal{N}) = 0$, then $rF(X) \simeq h_{F(X)}$

<u>Prop</u> Assume that $rF(X)$ is representable for all $X \in \mathcal{T}$ $\Rightarrow$ F has a right derived.

<u>Cor</u> Assume $\mathcal{T}$ has a strictly full $\mathcal{T}_0 \subset \mathcal{T}$ s.t. $F(\mathcal{N}_0) = 0$, where $\mathcal{N}_0 = \mathcal{N} \cap \mathcal{T}_0$. Assume

that $\forall X \in \mathcal{T}$ $\exists X_0 \in \mathcal{T}_0$ and $X \xrightarrow{f} X_0$ s.t. Cone $(f) \in \mathcal{N}$. Then F has a right derived and $RF(X) \simeq F(X_0)$

<u>Pf</u> Put $S_0 = S \cap \mathcal{T}_0$. Then $S_0^{X_0} \subset S^X$ ← cofinal

$\Rightarrow$ colimits of the functors are the same

$\Rightarrow$ (if any repres $\Rightarrow$ second also).

$RF|_{\mathcal{T}_0}(X_0) = RF(X_0)$. $\square$

<u>Cor</u> Let $F: \mathcal{A} \to \mathcal{B}$ be left exact. If $\mathcal{A}$ has enough F-acyclic objects (any A can be embedded in $A' \in F$-acyclic) $\Rightarrow$ F has a right derived on $\mathcal{D}^+(\mathcal{A})$.

<u>Pf</u> $Acyc^F(\mathcal{A})$ – F-acyclic object. $\mathcal{T}_0 = K^+(Acyc^F(\mathcal{A}))$

$\Rightarrow$ the cond's of the previous hold. $\square$

Exc The object X for which $rF(X)$ is representable form a triangulated subcategory in $\mathcal{X}/\mathcal{N}$.
The domain of definition of RF.

Analogous statements for left derived functors.

$$\underset{\in \mathrm{Fun}(\mathcal{T}'^{op}, Ab)}{\ell F(X)} = \varprojlim_{S_X} h_{F(x)}, \quad \text{where} \quad \mathrm{Ob}\, S_X = \{ X' \xrightarrow{s} X \mid s \in S \}$$

Mor:
$$\begin{tikzcd} X' \arrow[d, "f"'] \arrow[dr, "s"] & \\ X'' \arrow[r, "t"'] & X \end{tikzcd}$$

Lm If $\ell F(X)$ is repr. $\forall X \Rightarrow$
$\Rightarrow LF$ exists!

Lm If $F: \mathcal{A} \to \mathcal{B}$ — right exact, $\mathcal{A}$ has enough F-acyclic objects ($\forall A \in \mathcal{A}\ \exists A' \to A \to 0 \mid A'$ — F-acyclic), then LF exists on $D^-(\mathcal{A})$, can be computed using F-acyclic resolutions.

3. Properties of derived functors

<u>Lm</u> $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor. If $G \dashv F$, then G is also exact (same for $F \dashv G$).

<u>Pf</u> $F \dashv G$ $\quad X \to Y \to Z \to X[1]$ – dist in $\mathcal{T}'$

$$\begin{tikzcd}
G(X) \arrow[r] \arrow[d, "id"] & G(Y) \arrow[r] \arrow[d, "id"] & W \arrow[r] \arrow[d, dashed, "?"] & G(X)[1] \arrow[d, "id"] \\
G(X) \arrow[r] & G(Y) \arrow[r] & G(Z) \arrow[r] & G(X)[1]
\end{tikzcd}$$

check this $[1]G \simeq G[1]$

Look at the counit $F \circ G \xrightarrow{\varepsilon} Id$

$$\begin{tikzcd}
F\circ G(X) \arrow[r] \arrow[d, "\varepsilon_X"] & F\circ G(Y) \arrow[r] \arrow[d, "\varepsilon_Y"] & F(W) \arrow[r] \arrow[d, dashed, "\varphi"] & F\circ G(X)[1] \arrow[d, "\varepsilon_{X[1]}"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

dist

Put $W \to G(Z)$ to be adj. to $F(W) \xrightarrow{\varphi} Z$.

Apply $\mathrm{Hom}(u, -)$ to check that the Δ are isomorphic. $\square$

Assume $F : \mathcal{S} \rightleftarrows \mathcal{T} : G$ are exact & adjoint.

$\mathcal{M} \subset \mathcal{S}$, $\mathcal{N} \subset \mathcal{T}$ — Δ subcategories

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift right, "F"'] \arrow[d] & \mathcal{T} \arrow[l, shift right, "G"'] \arrow[d] \\ \mathcal{S}/\mathcal{M} \arrow[r, shift right, "RF"'] & \mathcal{T}/\mathcal{N} \arrow[l, shift right, "LG"'] \end{tikzcd}$$

LF – left derived of $\mathcal{S} \to \mathcal{T}/\mathcal{N}$

RG – right derived of $\mathcal{T} \to \mathcal{S}/\mathcal{M}$

Lm If RG is defined on $X \in \mathcal{T}/\mathcal{N}$, LF is defined at $Y \in \mathcal{S}/\mathcal{M}$ then the Hom adjunction isom holds:

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(LF(Y), X) \simeq \mathrm{Hom}_{\mathcal{S}/\mathcal{M}}(Y, RG(X)).$$

Pf Look at Deligne's formula! $\square$

Next week

Examples (modules & sheaves), further properties of derived functors (composition vs. the Grothendieck SS.)