Triangulated and Derived Categories in Algebra and Geometry
Lecture 19

0. Recap

One can partially define derived functors:

$\mathcal{C}$ - category, S - localization system (right)

$X \in \mathcal{C}$, $F : \mathcal{C} \to \mathcal{C}'$

$RF(X) = \varinjlim_{S^X} F(Z)$ ← if this exists, RF is defined on X

S^X - category of arrows $X \xrightarrow{s} Z$, $s \in S$
Morphisms are given by triangles

Exc $\mathcal{C}^F$ - strictly full subcategory of $\mathcal{C}[S^{-1}]$ on which RF is defined. If $\mathcal{C}$ - Δ, S associated to $\mathcal{N} \subset \mathcal{C}$, then $(\mathcal{C}/\mathcal{N})^F$ is a Δ subcategory in $\mathcal{C}/\mathcal{N}$.

Cor of the construction: if $G \dashv F$ between $\mathcal{T}$, $\mathcal{S}$ - Δ categories $\mathcal{N} \subset \mathcal{T}$, $\mathcal{M} \subset \mathcal{S}$, LG & RF are well defined, then $LG \dashv RF$!

1. Composition?

$\mathcal{T}_1 \xrightarrow{F} \mathcal{T}_2 \xrightarrow{G} \mathcal{T}_3$ exact b/w Δ categories

$\mathcal{N}_1 \subset \mathcal{T}_1$, $\mathcal{N}_2 \subset \mathcal{T}_2$ - Δ subcategories

$RF : \mathcal{T}_1/\mathcal{N}_1 \to \mathcal{T}_2/\mathcal{N}_2$, $RG : \mathcal{T}_2/\mathcal{N}_2 \to \mathcal{T}_3$

$R(G \circ F) : \mathcal{T}_1/\mathcal{N}_1 \to \mathcal{T}_3$

Is there an isomorphism?

Not in general.

$$\begin{array}{ccc} \Bbbk[x] & \longrightarrow & \Bbbk \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ P(x) & \longmapsto & P(0) \end{array}$$

Two functors: $F : \Bbbk\text{-mod} \to \Bbbk[x]\text{-mod}$
restriction
$G : \Bbbk[x]\text{-mod} \to \Bbbk\text{-mod}$
induction

$G(M) = \Bbbk \otimes_{\Bbbk[x]} M = M/xM$

<u>Rmk</u> F - exact functor $\Rightarrow RF = LF = F$

G - right exact (tensor product)

<u>Exc</u> Use the resolution

$$0 \to \Bbbk[x] \xrightarrow{x} \Bbbk[x] \to \Bbbk \to 0$$

to show that $LG(LF(V)) \simeq V \oplus V[1]$.

$G \circ F = Id$ functor $\Rightarrow L(G \circ F) = L\,id = id$!

<u>Q:</u> When is the composition of localizations the loc. of the composition?

Here is a sufficient condition.

$F: \mathcal{T} \to \mathcal{T}'$ - exact b/w Δ cat's, $\mathcal{N} \subset \mathcal{T}$ - strictly full Δ

$Q: \mathcal{T} \to \mathcal{T}/\mathcal{N}$ - localization functor $\qquad F \to RF \circ Q$

<u>Def</u> $X \in \mathcal{T}$ is <u>adjusted</u> to F if $F(X) \xrightarrow{\sim} RF(Q(X))$.

<u>Exc</u> Objects adjusted to F form a full triangulated subcategory.

<u>Exc</u> $F: \mathcal{A} \to \mathcal{B}$ - additive b/w abelian categories, $\mathcal{E} \subset \mathcal{A}$ - class of objects closed under $\oplus$ and cokernels of monomorphisms s.t.

$$\forall \quad 0 \to A_1 \to A_2 \to A_3 \to 0 \quad \text{with} \quad A_1 \in \mathcal{E}$$

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \to 0 \quad \text{is exact.}$$

If $\mathcal{A}$ has enough objects from $\mathcal{E}$ ($\forall A \in \mathcal{A}$ $\exists$ mono $A \hookrightarrow A'$, $A' \in \mathcal{E}$), then

1) any bounded from below complex with terms in $\mathcal{E}$ is adjusted to F in $K^+(\mathcal{A})$,

2) RF is defined on $D^+(\mathcal{A})$.

Same holds for left localizations.

<u>Exc</u> Let $\mathcal{T}(F) \subset \mathcal{T}$ be the Δ subcategory of adjusted objects. Then $F(\mathcal{T}(F) \cap \mathcal{N}) = 0$. If $\mathcal{T}$ has sufficiently many adjusted: $\forall X \in \mathcal{T}$ $\exists X \xrightarrow{s} X_0 \in \mathcal{T}$ $s \in S(\mathcal{N})$, then RF is defined everywhere.

<u>Lm</u> $\mathcal{T}_1 \xrightarrow{F} \mathcal{T}_2 \xrightarrow{G} \mathcal{T}'$ - exact functors, $\mathcal{N}_1 \subset \mathcal{T}_1$, $\mathcal{N}_2 \subset \mathcal{T}_2$. Assume $\mathcal{T}_1$ has enough F-adjusted objects s.t. $F(X)$ is G-adjusted. Then RF is everywhere defined, RG is defined on $\operatorname{Im} RF$, $R(G \circ F) \simeq RG \circ RF$.

<u>Pf</u>
- RF is defined everywhere (previous exc).
- $\forall X \in \mathcal{T}_1$ $\exists\, s: X \to X_1$, $s \in S(\mathcal{N}_1)$ s.t. X_1 is F-adjusted, $F(X_1)$ is G-adjusted $\Rightarrow RF(X) \simeq RF(X_1) \simeq F(X_1)$ $\Rightarrow G$ is defined on $RF(X) \Rightarrow$ def'd on $\operatorname{Im} RF$.

- Let $\mathcal{T}_0 \subset \mathcal{T}$ be the strictly full Δ subcategory of F-adjusted s.t. $F(-)$ is G-adjusted.

$\mathcal{T}_0$ is Δ (the intersection of the Δ subcat of F-adjusted & the preimage of the Δ subcat of G-adjusted).

$$F(\mathcal{T}_0 \cap N_1) = 0 \quad (\text{exc}) \Rightarrow (G \circ F)(\mathcal{T}_0 \cap N_1) = 0$$

Use the exc again:

$$R(G \circ F)(X) = G(F(X_0)), \text{ where } X \xrightarrow{s} X_0$$

$X_0 \in \mathcal{T}_0$, $s \in S(N)$. $\square$

Example to keep in mind:

left exact functor, injectives are adjusted
right — " — , projectives — " —

2. Bifunctors

Want to consider $\mathrm{Hom}^\bullet(-,-)$ and $-\otimes-$ on derived categories.

Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}$ be triangulated categories.

<u>Def</u> $F: \mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}$ is exact if $\exists$ isomorphisms (given as data)

$\theta_1 : F \circ [1]_1 \xrightarrow{\sim} [1] \circ F$, $\theta_2 : F \circ [1]_2 \to [1] \circ F$

s.t. 1) the diagram (anti) commutes

$$\begin{tikzcd}
F \circ [1]_1 \circ [1]_2 \arrow[r, "\theta_1"] \arrow[d, "\theta_2"'] & {[1]} \circ F \circ [1]_2 \arrow[d, "\theta_2"] \\
{[1]} \circ F \circ [1]_1 \arrow[r, "\theta_1"] & {[2]} \circ F
\end{tikzcd}$$

depends whether in your def of a bicomplex

$$\begin{tikzcd}
X^{pq} \arrow[r, "d_1"] \arrow[d, "d_2"'] & X^{p+1,q} \arrow[d, "d_2"] \\
X^{p,q+1} \arrow[r, "d_1"] & X^{p+1,q+1}
\end{tikzcd}$$

$d_1 d_2 + d_2 d_1 = 0$ ← preferable

or $d_1 d_2 - d_2 d_1 = 0$

2) $\forall$ $X_1 \to X_2 \to X_3 \to X_1[1]$ dist in $\mathcal{T}_1$, $Y \in \mathcal{T}_2$

$F(X_1, Y) \to F(X_2, Y) \to F(X_3, Y) \to F(X_1, Y)[1]$ dist

3) $\forall\ Y_1 \to Y_2 \to Y_3 \to Y_1[1]$ dist in $\mathcal{T}_2$, $X \in \mathcal{T}_1$

$$F(X, Y_1) \to F(X, Y_2) \to F(X, Y_3) \to F(X, Y_1)[1] \text{ dist.}$$

Def $F: \mathcal{T}_1 \times \mathcal{T}_2 \to \mathcal{T}$ – exact bifunctor, $\mathcal{N}_1 \subset \mathcal{T}_1$, $\mathcal{N}_2 \subset \mathcal{T}_2$ Δ subcategories. The right localization:

$$RF: \mathcal{T}_1/\mathcal{N}_1 \times \mathcal{T}_2/\mathcal{N}_2 \to \mathcal{T} \quad \text{with } F \to RF \circ Q$$

$\&$ universal (initial among $F \to G \circ Q$).

3. Canonical filtration

No chance to have naive filtrations

$$F^p X \hookrightarrow F^{p-1} X \hookrightarrow \cdots \hookrightarrow X \text{ in triangulated}$$

categories:

any monomorphism splits (same for epimorphism).

Def A filtration on X is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F^{p-1}X \arrow[r] \arrow[d] & F^pX \arrow[r] \arrow[d] & F^{p+1}X \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r] & \cdots
\end{tikzcd}$$

A filtration is finite if for $p \gg 0$ $F^pX \xrightarrow{\sim} X$ and for $p \ll 0$ $F^pX = 0$.

$gr_F^p(X) = C(F^{p-1}X \to F^pX)$ are called the associated quotients (defined up to a <u>noncanonical</u> isomorphism).

Consider the homotopy category $K(\mathcal{A})$, $X^\bullet \in K(\mathcal{A})$.

$$(\tau_{\leq p} X)^k = \begin{cases} X^k & \text{if } k < p, \\ \operatorname{Ker} d^p & \text{if } k = p, \\ 0 & \text{if } k > p. \end{cases}$$

<u>Exc</u> $gr_\tau^p(X) \simeq H^p(X)$.

$$\boxed{F^pX = \tau_{\leq p} X}$$

Observation $\tau_{\le p}$ is a functor.

Lm If $s: X \to Y$ is a qis in $K(\mathcal{A})$, then $\tau_{\le p} s : \tau_{\le p} X \to \tau_{\le p} Y$ is a qis.

Pf

$$\begin{tikzcd} \tau_{\le p} X \arrow[r, "\tau_{\le p} s"] \arrow[d] & \tau_{\le p} Y \arrow[d] \\ X \arrow[r, "qis"] & Y \end{tikzcd}$$

Apply H^k. Then if $k \le p$, then

$$H^k(\tau_{\le p} X) \xrightarrow{\sim} H^k(X).$$

If $k > p \Rightarrow H^k(\tau_{\le p} X) = 0$.

Same for $Y \Rightarrow$ follows the statement. $\square$

Get a functor $\tau_{\le p} : D(\mathcal{A}) \to D(\mathcal{A})$

Exc Construct $\tau_{\geq q} : K(\mathcal{A}) \to K(\mathcal{A})$, show that $\tau_{\geq q}$ preserve qis's (should have $X \to \tau_{\geq q} X$, morphism of functors $id \to \tau_{\geq q}$ and $H^k(\tau_{\geq q} X) = \begin{cases} H^k(X), & k \geq q, \\ 0, & k < q \end{cases}$).

For any $X^\bullet \in K(\mathcal{A})$ (thus, in $D(\mathcal{A})$) one gets an exact triangle

$$\tau_{\leq p} X \longrightarrow X \longrightarrow \tau_{\geq p+1} X \longrightarrow \tau_{\leq p} X[1]$$

has the same H^i up to p, then 0

has the same H^i above p, 0 at p & below

$p+1$ & more

t-structures generalize this situation.

4. t-structures

<u>Def</u> A t-structure on $\mathcal{T}$ is a pair of strictly full subcategories $\mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 0}$ in $\mathcal{T}$ (<u>not</u> Δ subcat's !!!) s.t. $(\mathcal{T}^{\leq p} = \mathcal{T}^{\leq 0}[-p]$, $\mathcal{T}^{\geq p} = \mathcal{T}^{\geq 0}[-p])$

1) $\mathcal{T}^{\leq -1} \subset \mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$,

complexes with H^i only in deg -1 & less are simultaneously complexes with H^i in deg 0 & less

2) $\mathrm{Hom}(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$

looks like SO but the subcat's are not Δ'd.

3) $\forall\ X \in \mathcal{T}\ \exists$ a dist Δ

$$\underset{\mathcal{T}^{\leq 0}}{\overset{}{\underset{\cap}{X_{\leq 0}}}} \longrightarrow X \longrightarrow \underset{\mathcal{T}^{\geq 1}}{\underset{\cap}{X_{\geq 1}}} \longrightarrow X_{\leq 0}[1].$$

A t-structure is non-degenerate if $\bigcap \mathcal{T}^{\leq p} = \bigcap \mathcal{T}^{\geq q} = 0$.

Lm / Example Let $\mathcal{T} = \mathcal{D}(\mathcal{A})$. Put

$$\mathcal{D}(\mathcal{A})^{\leq 0} = \{ X \in \mathcal{D}(\mathcal{A}) \mid H^i(X) = 0 \,, i > 0 \}$$

$$\mathcal{D}(\mathcal{A})^{\geq 0} = \{ X \in \mathcal{D}(\mathcal{A}) \mid H^i(X) = 0 \,, i < 0 \}.$$

Then $(\mathcal{D}(\mathcal{A})^{\leq 0}, \mathcal{D}(\mathcal{A})^{\geq 0})$ is a non-degenerate t-structure called standard.

Pf 1) trivial from the def,
3) follows from our truncation construction,
2) $\mathrm{Hom}(X, Y) = 0$ if $H^i(X) = 0 \ \forall i > 0$, $H^j(Y) = 0$, $j \leq 0$.

Any morphism $X \to Y$ in $\mathcal{D}(\mathcal{A})$ is an equiv class of

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

qis

$Y \to Z$ is a qis

$Y \in \mathcal{D}(\mathcal{A})^{\geq 1} \Rightarrow Z \in \mathcal{D}(\mathcal{A})^{\geq 1}$

$Z \to \tau_{\geq 1} Z$ is a qis

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"'] & & Y \arrow[ul, "s"'] \end{tikzcd} \quad \sim \quad \begin{tikzcd} & \tau_{\geq 1} Z & \\ & Z \arrow[u] & \\ X \arrow[ur] \arrow[uur, bend left, "g"] & & Y \arrow[ul] \arrow[uul, "t"'] \end{tikzcd}$$

$\tau_{\geq 1} Z$ has 0 terms in non-positive degrees

Replace X with $\tau_{\leq 0} X \leftarrow$ isomorphic in $\mathcal{D}(x)$.
May assume X has 0 terms in positive degrees.

Then $X \xrightarrow{g} \tau_{\geq} Z$ must be 0! $\Rightarrow s^{-1}f \sim t^{-1}0 = 0$!

<u>Motivation</u> If $(\tau^{\leq 0}, \tau^{\geq 0})$ is a non-degenerate t-structure, then $\tau^{\leq 0} \cap \tau^{\geq 0}$ is an abelian category!

$\mathcal{A}$ - abelian $\rightsquigarrow \mathcal{D}(\mathcal{A}) \rightsquigarrow$ t-structure (non-standard) $\rightsquigarrow$ a new abelian category!