Triangulated and Derived Categories in Algebra and Geometry
Lecture 2

1. Functors

$\mathcal{C}, \mathcal{D}$ — categories

Covariant　　　　　　　　　　Contravariant

$$F : \mathcal{C} \to \mathcal{D}$$

$$\mathcal{C} \ni X \longmapsto F(X) \in \mathcal{D}$$

$$f : X \to Y \text{ in } \mathcal{C}$$

$$F(X) \to F(Y) \qquad\qquad F(Y) \to F(X)$$

subject:

- $F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$
- respects composition

Warning Contravariant = covariant $\mathcal{C}^{op} \to \mathcal{D}$.

$\forall\ X, Y \in \mathcal{C}$ a functor gives a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

Types
- F is faithful if it's injective on Hom's
- F is full if it's surjective on Hom's
- F is essentially surjective if $\forall\ Y \in \mathcal{D}$ $\exists\ X \in \mathcal{C}$ s.t. $Y \simeq F(X)$.

Examples
- Subcategory $\rightsquigarrow$ embedding functor.
  Always faithful, full $\Leftrightarrow$ the subcategory is full
- Forgetful functors

$$\mathrm{Grp} \xrightarrow{\mathrm{For}} \mathrm{Sets} \qquad G \longmapsto G \text{ as a set}$$

Since hom's of groups are maps of sets with...

- $\text{Vect-}k \longrightarrow \text{Vect-}k$ $\quad W$ - fixed vect. space

$$V \longmapsto V \otimes_k W$$

- Assume that $\forall\, X, Y \in \mathcal{C}$ their product exists.

Exc Construct a product functor

$$\begin{aligned} \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (X, Y) &\longmapsto X \times Y \end{aligned}$$

- homology groups are functors

$$\mathcal{T}op \longrightarrow \mathcal{A}b$$

Def A presheaf on $\mathcal{C}$ with values in $\mathcal{A}$ is a functor $\mathcal{C}^{op} \to \mathcal{A}$.

Notation $PSh(\mathcal{C})$ - presheafs of sets on $\mathcal{C}$.

To every partially ordered set $P \rightsquigarrow$ a category:

- objects are P
- $\mathrm{Hom}_P(x,y) = \begin{cases} \{*\}, & x \le y, \\ \emptyset, & \text{otherwise.} \end{cases}$ $\qquad x \longrightarrow y$ if $x \le y$

To any topological space $X \rightsquigarrow Op(X)$ – partially ordered set of $\{U \subset X\}$, U – open. $U \le W \iff U \subseteq W$.

Then a presheaf on X is a presheaf on $Op(X)$.

Most important examples of functors:

Given $X \in \mathcal{C}$ $\qquad h^X : \mathcal{C} \to \mathrm{Sets} \qquad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$

$\qquad h_X : \mathcal{C}^{op} \to \mathrm{Sets} \qquad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y,X)$

a presheaf on $\mathcal{C}$

2. Morphisms of functors = natural transformations

also known as natural transformation

<u>Def</u> $F, G: \mathcal{A} \to \mathcal{B}$ be functors. A <u>morphism</u> $\eta: F \to G$ is a collection $\eta_X: F(X) \to G(X)$ s.t. $\forall f: X \to Y$

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

commutes.

<u>Rmk</u> The identity maps $F(X) \xrightarrow{id_{F(X)}} F(X)$ form a nat'l transformation $F \to F$. Morphisms of functors compose.

We get a category $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ of functors.
Objects: $F: \mathcal{A} \to \mathcal{B}$.
Morphisms: nat'l transformations.

Set-theoretic issues.

Can now talk about isomorphisms of functors.

Warning Isomorphism of cat's is a useless notion.

Def $\mathcal{C}$ and $\mathcal{D}$ are called equivalent if $\exists\, F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ s.t. $G \circ F \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}$, $F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$.

Examples
- $\mathrm{Sets}^f \simeq$ full subcategory formed by $\emptyset, \{1\}, \{1,2\}, \ldots$
- $\mathrm{Vect}^f\text{-}k \simeq$ category with objects $\mathbb{N}$, $\mathrm{Map}(n, m) = \mathrm{Mat}_{n\times m}(k)$.
- Affine schemes$^{op} \simeq$ Comm rings

Observation The rule $\mathcal{C} \ni X \longmapsto h_X \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets}) = \mathrm{PSh}(\mathcal{C})$ is a functor!

Exc Check it.

What can one say about this functor?
Most of the times not essentially surjective.

<u>Def</u> If $F \in PSh(\mathcal{C})$ and $F \simeq h_X$ for some $X \in \mathcal{C}$, then F is called representable.

But $\mathcal{C} \to PSh(\mathcal{C})$ is fully faithful. Follows immediately from the following.

<u>Prop</u> (Yoneda Lemma)

$\forall X \in \mathcal{C}$, $\forall F : \mathcal{C}^{op} \to Sets$ there is a natural bijection

$$\mathrm{Hom}_{PSh(\mathcal{C})}(h_X, F) \overset{1:1}{\longleftrightarrow} F(X).$$

More precisely, there is an isomorphism of functors

$$\mathrm{Hom}_{PSh(\mathcal{C})}(h_{-}, -), \; -(-) : \mathcal{C}^{op} \times PSh(\mathcal{C}) \to Sets.$$

In diagrams:

$\forall\, F \xrightarrow{\eta} G$

$$\begin{tikzcd}
\mathrm{Hom}(h_X, F) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \arrow[d, "\eta_X"] \\
\mathrm{Hom}(h_X, G) \arrow[r, "\sim"] & G(X)
\end{tikzcd}$$

$\forall\, X \xrightarrow{f} Y$

$$\begin{tikzcd}
\mathrm{Hom}(h_X, F) \arrow[r, "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(X) \\
\mathrm{Hom}(h_Y, F) \arrow[r, "\sim"] \arrow[u] & F(Y) \arrow[u, "F(f)"']
\end{tikzcd}$$

<u>Proof</u> Need a map

$$\begin{array}{ccc}
\mathrm{Hom}_{PSh(\mathcal{C})}(h_X, F) & \longrightarrow & F(X) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\eta & \longmapsto & \eta_X(\mathrm{id}_X)
\end{array}$$

$\eta_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \longrightarrow F(Y)$

$$\begin{array}{cccc}
\eta_X : & \mathrm{Hom}_{\mathcal{C}}(X, X) & \longrightarrow & F(X) \\
& \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
& \mathrm{id}_X & \longmapsto & \eta_X(\mathrm{id}_X)
\end{array}$$

Exc Construct a map in the other direction and show that they are inverse to each other. $\square$

Cor $\mathcal{C}$ embeds fully faithfully as a subcategory in $PSh(\mathcal{C})$.

Def $F \in PSh(\mathcal{C})$ is *representable* if $F \simeq h_X$ for some $X \in \mathcal{C}$.

3. Limits & colimits

index category

Def A *diagram* indexed by a category I in $\mathcal{C}$ is just a functor $F: I \to \mathcal{C}$.

Examples - Take $I = \bullet \quad \bullet$

Diagram = pair of objects $X, Y \in \mathcal{C}$.

- $I \quad \bullet \longrightarrow \bullet \quad \rightsquigarrow \quad X \longrightarrow Y$

- I $\bullet \to \bullet \leftarrow \bullet$ $\rightsquigarrow$ $X \to Z \leftarrow Y$

- I – opposite category to $\mathbb{N}$ ← as a partially ordered set

$$\dots \to X_i \xrightarrow{f_i} \dots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

condition

- I $\bullet \rightrightarrows \bullet$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

<u>Def</u> A limit of $F: I \to \mathcal{C}$ is an object $\varprojlim F \in \mathcal{C}$ and a collection of $p_i : \varprojlim F \to F(i)$ for all $i \in I$. Satisfying: $\forall \varphi : i \to j$ in I

$$\begin{tikzcd} & F(i) \arrow[d, "F(\varphi)"] \\ \varprojlim F \arrow[ur, "\varphi_i"] \arrow[r, "\varphi_j"'] & F(j) \end{tikzcd}$$

And it's universal: $\forall X$, $\{ f_i : X \to F(i) \}$ s.t.

$\forall\, i \xrightarrow{\varphi} j$

$$\begin{tikzcd}
 & F(i) \arrow[d, "F(\varphi)"] \\
X \arrow[ur, "f_i"] \arrow[dr, "f_j"'] \arrow[r, phantom, "\circlearrowleft"] & F(j) \\
 &
\end{tikzcd}$$

$\exists!\; X \xrightarrow{\eta} \lim F$ s.t.

$f_i = \varphi_i \circ \eta$.

A colimit:

$$\begin{tikzcd}
 & F(i) \arrow[dr] \arrow[rr, bend left] \arrow[rrr, bend left] & & & \\
 & & F(j) \arrow[r] \arrow[rr, bend left] & \lim F \arrow[r, dashed, "\exists!"'] & Z \\
 F(k) \arrow[uur] \arrow[urr] \arrow[urr, bend right] \arrow[urrr, bend right] & & & &
\end{tikzcd}$$

$\dots$

Example — $I = \bullet \quad \bullet$ $\rightsquigarrow$ limit = product

colimit = coproduct

I - discrete $\rightsquigarrow$ products indexed by I

- $I = \bullet \rightrightarrows \bullet$ $\rightsquigarrow$ limit = equalizer
  colimit = coequalizer

- In algebra: I - category of $\mathbb{Z}_{>0}$,
  $n \to m \iff n \mid m$.

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Ab} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ n & \longmapsto & \mathbb{Z}/n\mathbb{Z} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/n\mathbb{Z} & \hookrightarrow & \mathbb{Z}/m\mathbb{Z} \\ \rotatebox{90}{$\in$} & & \\ i & \longmapsto & \frac{m}{n} i \end{array}$$

<u>Exc</u> $\varinjlim \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z} \simeq \mu \leftarrow$ complex roots of unity

- $\varprojlim \left(\cdots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \right) = \mathbb{Z}_p$ p-adic numbers

<u>Prop</u> In Sets all limits and colimits exist.

<u>Pf</u> Via products and equalizers. $\square$

Prop Assume all products & equalizers exist in $\mathcal{C}$. Then all limits exist in $\mathcal{C}$.

Pf Consider $F: I \to \mathcal{C}$. Construct two morphisms

$$\prod_{i\in I} F(i) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \prod_{f: i\to j} F(j)$$

In order to construct $\to \prod$ enough (same thing) as to construct morphisms into its elements.
$\forall f: i \to j$ need $\prod_{i\in I} F(i) \rightrightarrows F(j)$.

$$\varphi_f : \prod_{i\in I} F(i) \xrightarrow{p_j} F(j) \xrightarrow{id} F(j)$$

$$\psi_f : \prod_{i\in I} F(i) \xrightarrow{p_i} F(i) \xrightarrow{F(f)} F(j)$$

Finish the proof: from the equalizer $\rightsquigarrow \underleftarrow{\lim} F$. $\square$

Exc State and prove a similar statement for colimits.

Cor Sets has all products & equalizers $\Rightarrow$ limits
— " — coproduct & coequalizers $\Rightarrow$ colimits.

Alternative definition through comma categories

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ — categories, $S : \mathcal{A} \to \mathcal{C} \leftarrow \mathcal{B} : T$

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "S"'] & & \mathcal{B} \arrow[dl, "T"] \\
& \mathcal{C} &
\end{tikzcd}$$

Def The comma category $(S \downarrow T)$ is the category whose objects are triples $A \in \mathcal{A}$, $B \in \mathcal{B}$, $f : S(A) \to T(B)$.
Morphisms are $(\varphi, \psi) : A \times B \to A' \times B'$ (in $\mathcal{A} \times \mathcal{B}$) such that

$$\begin{tikzcd}
S(A) \arrow[r, "f"] \arrow[d, "S(\varphi)"'] & T(B) \arrow[d, "T(\psi)"] \\
S(A') \arrow[r, "f'"] & T(B')
\end{tikzcd}$$

<u>Examples</u> Take $\mathcal{A} = \mathcal{A}$, $\mathcal{C} = \mathcal{A}$, $\mathcal{B} = \{*\}$. $S = \mathrm{Id}$, $T = A \in \mathcal{C}$.
Get the slice category $\mathcal{A}/_A$:
objects are $A' \to A$, morphisms are

$$
\begin{tikzcd}
A' \arrow[rr] \arrow[dr] & \circlearrowleft & A'' \arrow[dl] \\
& A &
\end{tikzcd}
.
$$

Given an object $X \in \mathcal{C}$, define the constant diagram $\underline{X} : I \to \mathcal{C}$, sends all objects to X, all morphisms to the identity morphism.

<u>Exc</u> Check that you get a functor $\mathrm{Const} : \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$.

Consider the comma category of

$$
\begin{tikzcd}
\mathcal{C} \arrow[dr, "\mathrm{Const}"'] & & \{*\} \arrow[dl, "F"] \\
& \mathrm{Func}(I, \mathcal{C}) &
\end{tikzcd}
.
$$

Exc Check that $\varprojlim F$ is the same as a terminal object in this comma category.

Exc Do the same for colimits.

Alternative definition via representable functors

For every $X \in \mathcal{C}$ consider the composition

$$\begin{array}{ccccc} I & \xrightarrow{F} & \mathcal{C} & \xrightarrow{h^X} & Sets \\ \cup & & \cup & & \cup \\ i & \longmapsto & F(i) & \longmapsto & Hom(X, F(i)) \end{array}$$

Since all limits exist in Sets, we can compute (pick one)
$\varprojlim (h^X \circ F) = \varprojlim Hom(X, F(i))$.

Ex This limit, $\varprojlim (h^X \circ F)$ produces a functor
$\hat{F} : \mathcal{C}^{op} \longrightarrow Sets$.

<u>Exc</u> $\varprojlim F$ exists in $\mathcal{C}$ $\iff$ $\tilde{F} \in PSh(\mathcal{C})$ is representable.

4. Adjoint functors

<u>Def</u> $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ are called <u>adjoint</u> (F is <u>left adjoint</u> to G, G - <u>right adjoint</u> to F). If there exists a natural bijection of sets

$$Hom_{\mathcal{B}}(F(X), Y) \xrightarrow{\sim} Hom_{\mathcal{A}}(X, G(Y))$$

for all $X \in \mathcal{A}$, $Y \in \mathcal{B}$.

Both sides are functors $\mathcal{A}^{op} \times \mathcal{B} \to Sets$.
As functors must be isomorphic.

<u>Examples</u> 1) $V \in Vect^f\text{-}k$

$$- \otimes_k V : Vect^f\text{-}k \rightleftarrows Vect^f\text{-}k : Hom_k(V, -),$$

2) free objects

Free is left adjoint to For:

For : $Grp \to Sets$

$$\mathrm{Hom}_{Grp}(\mathrm{Free}(X), G) = \mathrm{Map}_{Sets}(X, G).$$

Same for Comm, Ab, ...

<u>Notation</u> $F \dashv G$.

<u>Problem 1</u> Show that $F : \mathcal{A} \to \mathcal{B}$ has a right adjoint ($G : \mathcal{B} \to \mathcal{A}$ + an isom of bifunctors) if and only if the functor

$$\mathrm{Hom}_{\mathcal{B}}(F(-), Y) : \mathcal{A}^{op} \to Sets$$

is representable $\forall\, Y \in \mathcal{B}$.

Want a proof with all the details.