

Triangulated and Derived Categories in Algebra and Geometry

Lecture 2

1. Functors \mathcal{C}, \mathcal{D} - categories

Covariant

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

$$\mathcal{C} \ni X \mapsto F(X) \in \mathcal{D}$$

$$f: X \rightarrow Y \text{ in } \mathcal{C}$$

$$F(X) \rightarrow F(Y)$$

$$F(Y) \rightarrow F(X)$$

Contravariant

subject:

- $F(\text{id}_X) = \text{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- respects composition

Warning Contravariant = covariant $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$.

$\forall X, Y \in \mathcal{C}$ a functor gives a map
 $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$

Types - F is faithful if it's injective on Hom's
- F is full if it's surjective on Hom's
- F is essentially surjective if $\forall Y \in \mathcal{D}$
 $\exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$.

Examples • Subcategory \rightsquigarrow embedding functor.
Always faithful, full \Leftrightarrow the subcategory is full

• Forgetful functors

$\text{Grp} \xrightarrow{\text{For}} \text{Sets} \quad G \mapsto G \text{ as a set}$

Since hom's of groups are maps of sets with...

- $\text{Vect-}k \xrightarrow{W} \text{Vect-}k$ W -fixed vect. space
 $V \longmapsto V \otimes_k W$

- Assume that $\forall x, y \in \mathcal{C}$ their product exists.
Exc Construct a product functor

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ (x, y) & \longmapsto & x \times y \end{array}$$

- homology groups are functors

$$\text{Top} \longrightarrow \text{Ab}$$

Def A presheaf on \mathcal{C} with values in \mathcal{A} is a functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{A}$.

Notation $\text{PSh}(\mathcal{C})$ - presheafs of sets on \mathcal{C} .

To every partially ordered set $P \rightsquigarrow$ a category:

- objects are P
- $\text{Hom}_P(x, y) = \begin{cases} \{x, y\}, & x \leq y, \\ \emptyset, & \text{otherwise.} \end{cases} \quad x \rightarrow y \text{ if } x \leq y$

To any topological space $X \rightsquigarrow \text{Op}(X)$ - partially ordered set of $\{U \subset X\}$, U -open. $U \leq W \Leftrightarrow U \subseteq W$.

Then a presheaf on X is a presheaf on $\text{Op}(X)$.

Most important examples of functors:

Given $X \in \mathcal{C}$ $h^X: \mathcal{C} \rightarrow \text{Sets}$ $Y \mapsto \text{Hom}_{\mathcal{C}}(X, Y)$

$h_X: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ $Y \mapsto \text{Hom}_{\mathcal{C}}(Y, X)$

\uparrow
a presheaf on \mathcal{C}

2. Morphisms of functors = natural transformations

also known as
natural transformation

Def $F, G: \mathcal{A} \rightarrow \mathcal{B}$ be functors. A morphism
 $\eta: F \rightarrow G$ is a collection $\eta_x: F(x) \rightarrow G(x)$
s.t. $\forall f: X \rightarrow Y$

$$\begin{array}{ccc} F(x) & \xrightarrow{\eta_x} & G(x) \\ F(f) \downarrow & \curvearrowright & \downarrow G(f) \\ F(y) & \xrightarrow{\eta_y} & G(y) \end{array} \quad \text{commutes.}$$

Prop The identity maps $F(x) \xrightarrow{\text{id}_{F(x)}} F(x)$ form a nat'l transformation $F \rightarrow F$. Morphisms of functors compose.

We get a category $\text{Fun}(\mathcal{A}, \mathcal{B})$ of functors.

Objects: $F: \mathcal{A} \rightarrow \mathcal{B}$.

Morphisms: nat'l transformations.

Set-theoretic issues.

Can now talk about isomorphisms of functors.

Warning Isomorphism of cat's is a useless notion.

Def \mathcal{C} and \mathcal{D} are called equivalent if $\exists F: \mathcal{C} \rightarrow \mathcal{D}$
and $G: \mathcal{D} \rightarrow \mathcal{C}$ s.t. $G \circ F \xrightarrow{\sim} \text{Id}_{\mathcal{C}}$, $F \circ G \xrightarrow{\sim} \text{Id}_{\mathcal{D}}$.

Examples • $\text{Sets}^{\mathcal{P}} \simeq$ full subcategory formed by
 $\emptyset, \{1\}, \{1,2\}, \dots$

• $\text{Vect}^f\text{-}k \simeq$ category with objects \mathbb{N} ,
 $\text{Map}(n, m) = \text{Mat}_{n \times m}(k)$.

• Affine schemes^{op} \simeq Comm rings

Observation The rule $\mathcal{C} \ni X \mapsto h_X \in \text{Fun}(\mathcal{C}^{\text{op}}, \text{Sets}) = \text{PSh}(\mathcal{C})$
is a functor!

Exc Check it.

What can one say about this functor?

Most of the times not essentially surjective.

Def If $F \in \text{Psh}(\mathcal{C})$ and $F \simeq h_x$ for some $x \in \mathcal{C}$, then F is called representable.

But $\mathcal{C} \rightarrow \text{Psh}(\mathcal{C})$ is fully faithful. Follows immediately from the following.

Prop (Yoneda Lemma)

$\forall x \in \mathcal{C}$, $\forall F: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ there is a natural bijection

$$\text{Hom}_{\text{Psh}(\mathcal{C})}(h_x, F) \xrightarrow{1:1} F(x).$$

More precisely, there is an isomorphism of functors

$$\text{Hom}_{\text{Psh}(\mathcal{C})}(h_{-}, -), -(-): \mathcal{C}^{\text{op}} \times \text{Psh}(\mathcal{C}) \rightarrow \text{Sets}.$$

In diagrams:

$$\forall F \xrightarrow{f} G$$

$$\forall X \xrightarrow{f} Y$$

$$\text{Hom}(h_x, F) \xrightarrow{\sim} F(x)$$

$$\text{Hom}(h_x, F) \xrightarrow{\sim} F(x)$$

$$\downarrow$$

$$\hookrightarrow$$

$$\downarrow \eta_x$$

$$\uparrow$$

$$\hookrightarrow$$

$$\uparrow F(f)$$

$$\text{Hom}(h_x, G) \xrightarrow{\sim} G(x)$$

$$\text{Hom}(h_y, F) \xrightarrow{\sim} F(y)$$

Proof Need a map

$$\text{Hom}_{\text{PSH}(e)}(h_x, F) \longrightarrow F(x)$$

$$\downarrow$$

$$\downarrow$$

$$\eta \longmapsto \eta_x(\text{id}_x)$$

$$\eta_y : \text{Hom}_e(y, x) \longrightarrow F(y)$$

$$\eta_x : \text{Hom}_e(x, x) \longrightarrow F(x)$$

$$\downarrow$$

$$\downarrow$$

$$\text{id}_x \longmapsto \eta_x(\text{id}_x)$$

Exc Construct a map in the other direction and show that they are inverse to each other. \square

Cor \mathcal{C} embeds fully faithfully as a subcategory in $\text{PSh}(\mathcal{C})$.

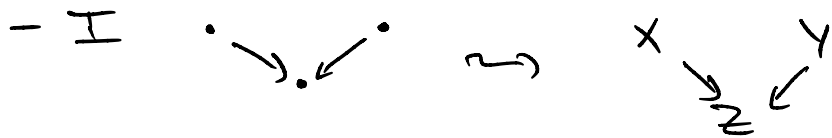
Def $F \in \text{PSh}(\mathcal{C})$ is representable if $F \simeq h_x$ for some $x \in \mathcal{C}$.

3. Limits & colimits

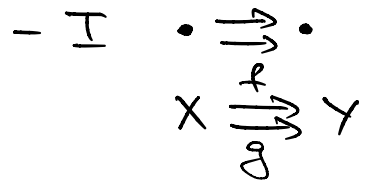
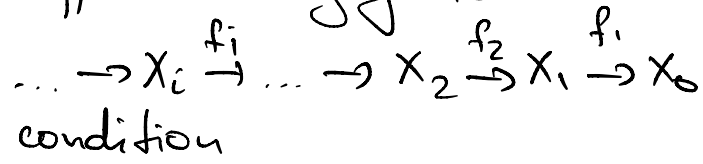
Def A diagram indexed by a category I in \mathcal{C} is just a functor $F: I \rightarrow \mathcal{C}$. ↙ index category

Examples - Take $I = \dots$

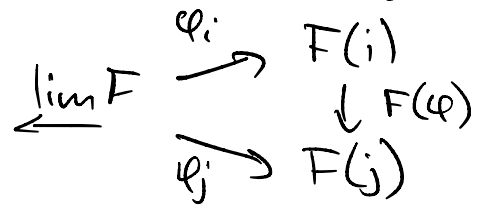
Diagram = pair of objects $X, Y \in \mathcal{C}$.
- $I \quad \bullet \rightarrow \bullet \quad \rightsquigarrow \quad X \rightarrow Y$



- \underline{I} - opposite category to \mathbb{N} ← as a partially ordered set



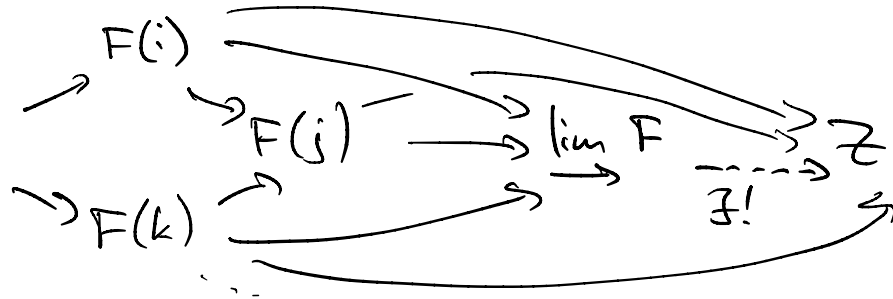
Def A limit of $F: I \rightarrow \mathcal{C}$ is an object $\varprojlim F \in \mathcal{C}$ and a collection of $p_i: \varprojlim F \rightarrow F(i)$ for all $i \in I$.
Satisfying: $\forall \varphi: i \rightarrow j$ in I



And it's universal: $\forall X, \{ f_i: X \rightarrow F(i) \}$ s.t.
 $\forall i \xrightarrow{\varphi} j$

$$\begin{array}{ccc}
 X & \xrightarrow{f_i} & F(i) \\
 & \searrow G & \downarrow F(\varphi) \\
 & & F(j) \\
 & \nearrow f_j & \\
 X & & F(j)
 \end{array}$$
 $\exists! X \xrightarrow{\eta} \lim F$ s.t.
 $f_i = \varphi \circ \eta$.

A colimit:



Example - $I = \dots \rightsquigarrow$ limit = product
 colimit = coproduct

I - discrete \rightsquigarrow products indexed by I

- $I = \cdot \rightrightarrows \cdot \rightsquigarrow$ limit = equalizer
 colimit = coequalizer

- In algebra: I - category of \mathbb{Z}_{50} ,
 $n \rightarrow m \Leftrightarrow n | m$.

$$\begin{array}{ccc}
 I & \rightarrow & \text{Ab} \\
 \downarrow & & \downarrow \\
 n & \hookrightarrow & \mathbb{Z}/n\mathbb{Z}
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ccc}
 \mathbb{Z}/n\mathbb{Z} & \hookrightarrow & \mathbb{Z}/m\mathbb{Z} \\
 \downarrow & & \downarrow \\
 i & \hookrightarrow & \frac{m}{n}i
 \end{array}$$

Exc $\varinjlim \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z} \cong \mu_{\infty} \leftarrow \text{complex roots of unity}$

- $\varprojlim \left(\dots \rightarrow \mathbb{Z}/p^3 \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \right) = \mathbb{Z}_p$ p-adic numbers

Prop In Sets all limits and colimits exist.

Pf Via products and equalizers.

□

Prop Assume all products & equalizers exist in \mathcal{C} .
Then all limits exist in \mathcal{C} .

Pf Consider $F: I \rightarrow \mathcal{C}$. Construct two morphisms

$$\prod_{i \in I} F(i) \begin{array}{c} \xrightarrow{\psi} \\ \xrightarrow{\varphi} \end{array} \prod_{f: i \rightarrow j} F(j)$$

In order to construct $\rightarrow \prod$ enough (same thing)
as to construct morphisms into its elements.

$\forall f: i \rightarrow j$ need $\prod_{i \in I} F(i) \rightrightarrows F(j)$.

$$\varphi_f: \prod_{i \in I} F(i) \xrightarrow{P_j} F(j) \xrightarrow{\text{id}} F(j)$$

$$\varphi_f: \prod_{i \in I} F(i) \xrightarrow{P_i} F(i) \xrightarrow{F(f)} F(j)$$

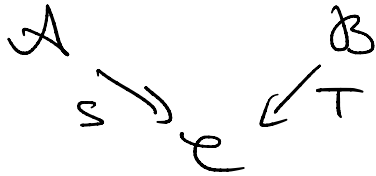
Finish the proof: from the equalizer $\leadsto \varprojlim F$. \square

Exc State and prove a similar statement for colimits.

Cor Sets has all products & equalizers \Rightarrow limits.
 — u — coproduct & coequalizers \Rightarrow colimits.

Alternative definition through comma categories

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ — categories, $S: \mathcal{A} \rightarrow \mathcal{C} \leftarrow \mathcal{B}: T$



Def The comma category $(S \downarrow T)$ is the category whose objects are triples $A \in \mathcal{A}, B \in \mathcal{B}, f: S(A) \rightarrow T(B)$. Morphisms are $(\varphi, \psi): A \times B \rightarrow A' \times B'$ (in $\mathcal{A} \times \mathcal{B}$) such that

$$\begin{array}{ccc}
 S(A) & \xrightarrow{f} & T(B) \\
 S(\varphi) \downarrow & & \downarrow T(\psi) \\
 S(A') & \xrightarrow{f'} & T(B')
 \end{array}$$

Example Take $\mathcal{A} = \mathcal{A}$, $\mathcal{C} = \mathcal{A}$, $\mathcal{B} = \{*\}$. $S = \text{Id}$, $T = A \in \mathcal{C}$.
 Get the slice category \mathcal{A}/A :
 objects are $A' \rightarrow A$, morphisms are

$$\begin{array}{ccc} A' & \xrightarrow{S} & A'' \\ & \searrow & \swarrow \\ & A & \end{array}$$

Given an object $X \in \mathcal{C}$, define the constant diagram
 $\underline{X} : \mathcal{I} \rightarrow \mathcal{C}$, sends all objects to X , all morphisms
 to the identity morphism.

Exc Check that you get a functor $\text{const} : \mathcal{C} \rightarrow \text{Func}(\mathcal{I}, \mathcal{C})$.

Consider the comma category of

$$\begin{array}{ccc} \mathcal{C} & & \{*\} \\ \text{const} \searrow & & \swarrow F \\ & \text{Func}(\mathcal{I}, \mathcal{C}) & \end{array}$$

Exc Check that $\varprojlim F$ is the same as a terminal object in this comma category.

Exc Do the same for colimits.

Alternative definition via representable functors

For every $X \in \mathcal{C}$ consider the composition

$$\begin{array}{ccccc} \mathcal{I} & \xrightarrow{F} & \mathcal{C} & \xrightarrow{h^X} & \mathbf{Sets} \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\ i & \hookrightarrow & F(i) & \hookrightarrow & \mathbf{Hom}(X, F(i)) \end{array}$$

Since all limits exist in \mathbf{Sets} , we can compute (pick one)

$$\varprojlim (h^X \circ F) = \varprojlim \mathbf{Hom}(X, F(i)).$$

Ex This limit, $\varprojlim (h^X \circ F)$ produces a functor $F^X : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$.

Exe $\varprojlim F$ exists in $\mathcal{C} \iff \tilde{F} \in \text{PSh}(\mathcal{C})$ is representable.

4. Adjoint functors

Def $F: \mathcal{A} \rightleftarrows \mathcal{B} : G$ are called adjoint (F is left adjoint to G , G - right adjoint to F). If there exists a natural bijection of sets

$$\text{Hom}_{\mathcal{B}}(F(x), y) \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(x, G(y))$$

for all $x \in \mathcal{A}, y \in \mathcal{B}$.

Both sides are functors $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \text{Sets}$.
As functors must be isomorphic.

Examples 1) $V \in \text{Vect}^{\mathbb{F}}-k$

$$-\otimes_k V : \text{Vect}^{\mathbb{F}}-k \rightleftarrows \text{Vect}^{\mathbb{F}}-k : \text{Hom}_k(V, -),$$

2) free objects

Free is left adjoint to For:

For: $\text{Grp} \rightarrow \text{Sets}$

$$\text{Hom}_{\text{Grp}}(\text{Free}(X), G) = \text{Map}_{\text{Sets}}(X, G).$$

Same for Comm, Ab, ...

Notation $F \dashv G$.

Problem 1

Show that $F: \mathcal{A} \rightarrow \mathcal{B}$ has a right adjoint
($G: \mathcal{B} \rightarrow \mathcal{A}$ + an isom of bifunctors)
if and only if the functor

$$\text{Hom}_{\mathcal{B}}(F(-), Y): \mathcal{A}^{\text{op}} \rightarrow \text{Sets}$$

is representable $\forall Y \in \mathcal{B}$.

Want a proof with all the details.