Triangulated and Derived Categories in Algebra and Geometry
Lecture 20

0. Recap

If $X^\bullet \in C(\mathcal{A})$ $\quad \dots \to X^n \to X^{n+1} \to X^{n+2} \to \dots$

$\hookrightarrow \tau_{\leq k} X^\bullet \in C(\mathcal{A})$ $\qquad \tau_{\leq k} X^\bullet \hookrightarrow X^\bullet$ (in $C(\mathcal{A}) \leftarrow$ abelian)

induced by those in $X^\bullet$

$$\begin{tikzcd}
\dots \arrow[r] & X^{k-2} \arrow[r] \arrow[d] & X^{k-1} \arrow[r] \arrow[d] & \operatorname{Ker} d^k \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & X^{k-2} \arrow[r] & X^{k-1} \arrow[r] & X^k \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

Property $\quad H^i(\tau_{\leq k} X^\bullet) = \begin{cases} H^i(X^\bullet), & i \leq k, \\ 0, & i > k. \end{cases}$

Alternatively $\tau_{\geq k} X^\bullet \in C(\mathcal{A})$ $\quad X^\bullet \twoheadrightarrow \tau_{\geq k} X^\bullet$ (in $C(\mathcal{A})$)

$$\begin{tikzcd}
\cdots \arrow[r] & X^{k-1} \arrow[r] \arrow[d, two heads] & X^k \arrow[r] \arrow[d] & X^{k+1} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \operatorname{Im} d^{k-1} \arrow[r, hook] & X^k \arrow[r] & X^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Property $\quad H^i(\tau_{\geq k} X^\bullet) = \begin{cases} H^i(X^\bullet), & i \geq k, \\ 0, & i < k. \end{cases}$

In $C(\mathcal{A})$: SES's for all k:

$$0 \to \tau_{\leq k} X^\bullet \to X^\bullet \to \tau_{\geq k+1} X^\bullet \to 0$$

These are functors, called the canonical truncations.

Descend on $K(\mathcal{A})$ (trivial).

Preserve qis $\Rightarrow$ Descend on $D(\mathcal{A})$ (and all its versions).

In $\mathcal{D}(\mathcal{A})$ $\forall\ X \in \mathcal{D}(\mathcal{A})$, $k \in \mathbb{Z}$ $\rightsquigarrow$ dist Δ

$$\tau_{\leq k} X \to X \to \tau_{\geq k+1} X \to \tau_{\leq k} X[1].$$

Side remark: there are other truncations.

Stupid truncation: $\sigma_{\leq k} X^{\cdot} = \left(\to X^{k-2} \to X^{k-1} \to X^{k} \to 0 \to \dots \right)$

<u>Problem</u>

$$H^i(\sigma_{\leq k} X^{\cdot}) = \begin{cases} H^i(X^{\cdot}), & i < k, \\ X^k / \mathrm{Im}\, d^{k-1}, & i = k, \\ 0, & i > k. \end{cases}$$

<u>Exc</u> $\sigma_{\leq k}$ does not preserve qis's.

t-structures abstract the properties of the canonical truncation

<u>Def</u> $\mathcal{T}$ – Δ category, a <u>t-structure</u> is a pair of strictly full (not Δ!!!) subcategories

$(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ s.t.

(1) $\mathcal{T}^{\le -1} = \mathcal{T}^{\le 0}[1] \subset \mathcal{T}^{\le 0}$
$\mathcal{T}^{\ge 1} = \mathcal{T}^{\ge 0}[-1] \subset \mathcal{T}^{\ge 0}$

$\mathcal{T}^{\le k} = \mathcal{T}^{\le 0}[-k]$
$\mathcal{T}^{\ge k} = \mathcal{T}^{\ge 0}[-k]$

(2) $\mathrm{Hom}(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 1}) = 0$

(3) $\forall\ X \in \mathcal{T}$ $\exists$ a dist Δ

$$\underset{\mathcal{T}_{\le 0}}{\underset{\cap}{X_{\le 0}}} \longrightarrow X \longrightarrow \underset{\mathcal{T}_{\ge 1}}{\underset{\cap}{X_{\ge 1}}} \longrightarrow X_{\le 0}[1].$$

It's non-degenerate if $\bigcap \mathcal{T}^{\le k} = \bigcap \mathcal{T}^{\ge k} = 0.$

Relation with truncations?

Def The standard t-structure on $\mathcal{D}(\mathcal{A})$ is given by

$\mathcal{D}(\mathcal{A})^{\le 0} = \{X^\cdot \mid H^i(X^\cdot) = 0,\ i > 0\}$, non-degenerate
$\mathcal{D}(\mathcal{A})^{\ge 0} = \{Y^\cdot \mid H^i(Y^\cdot) = 0,\ i < 0\}$.

1. The standard t-structure

<u>Recall</u> $\mathcal{A} \hookrightarrow C(\mathcal{A})$

$$A \mapsto \dots \to 0 \to 0 \to A \to 0 \to 0 \to \dots$$

Since preserves qis $\Rightarrow$ $\mathcal{A} \to D(\mathcal{A})$.

<u>Lm</u> $\operatorname{Hom}_{D(\mathcal{A})}(X, Y[t]) \simeq \begin{cases} 0, & t < 0 \\ \operatorname{Ext}^t_Y(X, Y), & t \geq 0, \end{cases}$

where Ext^t_Y – Yoneda ext!

<u>Reminder</u> $\operatorname{Ext}^t_Y(X, Y)$ is given by equiv classes of diagrams

$$0 \to Y \to Z_{t-1} \to \dots \to Z_0 \to X \to 0 \;/\!\sim$$

$\sim$ is generated by

$$\begin{array}{ccccccccccccc} 0 & \to & Y & \to & Z_{t-1} & \to & \dots & \to & Z_0 & \to & X & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \to & Y & \to & Z'_{t-1} & \to & \dots & \to & Z'_0 & \to & X & \to & 0 \end{array}$$

We discussed: $\mathcal{A}$ has enough injectives / projectives

$$\mathrm{Ext}^t_Y(X,Y) \simeq \mathrm{Ext}^t(X,Y) \leftarrow \text{derived functor.}$$

Put $\mathrm{Ext}^0_Y(X,Y) = \mathrm{Hom}(X,Y)$.

<u>Pf</u> (Lemma)

<u>Case $t<0$</u>: $X \in D(\mathcal{A})^{\leq 0}$, $Y[t] \in D(\mathcal{A})^{\geq -t} \subset D(\mathcal{A})^{\geq 1}$.

t-structure $\Rightarrow$ $\mathrm{Hom}(X, Y[t]) = 0$.

<u>Case $t>0$</u>: take any element from $\mathrm{Hom}(X, Y[t])$ represented by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Z \arrow[d] & Y[t] \arrow[l, "s"'] \arrow[dl] \\ & \tau_{\geq -t} Z & \end{tikzcd}$$

$\leftarrow$ equivalent

May assume that Z is concentrated in degrees $-t$ and greater: $0 \to 0 \to Z^{-t} \to Z^{-t+1} \to \ldots$

$$Y[t] \xrightarrow{qis} Z$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z^{-t} \arrow[r] & Z^{-t+1} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

pass to the cone $\rightsquigarrow$

$$\cdots \to 0 \to Y \to Z^{-t} \to Z^{-t+1} \to \cdots$$

truncate from above: $\tau_{\leq 0}$

$$0 \to Y \to Z^{-t} \to Z^{-t+1} \to \cdots \to Z^{-1} \to \ker d^0 \to 0$$

Gives us an element in $\mathrm{Ext}^t_{\mathcal{Y}}(\ker d^0, Y)$

$X \to Z \;\to$ gives a morphism in $\mathcal{A}$ $X \to \ker d^0$.

Compose and get an element in $\mathrm{Ext}^t_{\mathcal{Y}}(X, Y)$.

In the other direction:

$$0 \to Y \to Z^{-t} \to Z^{-t+1} \to \cdots \to Z^{-1} \to \underset{\substack{\shortparallel \\ X}}{Z^0} \to 0$$

$$\wr$$

$$X \to (0 \to Z^{-t} \to Z^{-t+1} \to \cdots \to Z^0 \to 0) \xleftarrow{\sim} Y[t].$$

Need to chek that

1) the maps are well-def (do not depend on the choice),
2) mutually inverse.

Both are left as an exercise.

<u>Case $t=0$:</u>

$$X \to Z^\bullet \xleftarrow[s]{\sim} Y \qquad \text{May assume that } Z^i=0,\ i<0!$$

$$\mathrm{Cone}(s) \rightsquigarrow \underbrace{0 \to Y \to Z^0 \to Z^1}\to \dots$$

Apply $\mathrm{Hom}(X,-)$:

$$0 \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z^0) \to \mathrm{Hom}(X,Z^1)$$

Moreover, $\mathcal{A} \xrightarrow{\sim} \mathcal{D}^0(\mathcal{A}) = \mathcal{D}(\mathcal{A})^{\le 0} \cap \mathcal{D}(\mathcal{A})^{\ge 0}$

complexes with cohomology in 0^{th} term only

Enough to check that $\mathcal{A} \to \mathcal{D}^0(\mathcal{A})$ essentially surjective.

Qis

$$X \xrightarrow{\sim} \tau_{\geq 0} X \xrightarrow{\sim} \tau_{\leq 0}\tau_{\geq 0} X \quad \text{for } X \in \mathcal{D}^0(\mathcal{A})$$

$$\operatorname{Im}\,(\mathcal{A} \to \mathcal{D}^0(\mathcal{A})). \qquad \square$$

It turns out, if $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a non-degenerate t-structure, then its heart $\mathcal{T}^0 = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category!

We've just seen: $\mathcal{D}^0(\mathcal{A}) \simeq \mathcal{A}$.

2. Properties of t-structures

Recall that if $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ – SOD, then

$\mathcal{A} \to \mathcal{T}$ has a left adjoint, $\mathcal{B}$ is determined by $\mathcal{A}$ as ${}^\perp\mathcal{A} = \{ X \in \mathcal{T} \mid \operatorname{Hom}(X, Y) = 0 \ \forall\, Y \in \mathcal{A} \}$.

<u>Lm</u> If $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ is a t-structure on $\mathcal{T}$, then there are functors $\tau_{\le t}: \mathcal{T} \to \mathcal{T}^{\le t}$ and $\tau_{\ge t}: \mathcal{T} \to \mathcal{T}^{\ge t}$, morphisms of functors

$$\tau_{\le t} \to \mathrm{id} \to \tau_{\ge t+1} \to [1]\circ \tau_{\le t} \quad \text{s.t.}$$

$\forall\, X \in \mathcal{T}$ the corresp. Δ is distinguished. Moreover, $\tau_{\le t}$ is right adjoint to $\mathcal{T}^{\le t} \hookrightarrow \mathcal{T}$, $\tau_{\ge t}$ is left adjoint to $\mathcal{T}^{\ge t} \hookrightarrow \mathcal{T}$.

<u>Pf</u> Enough to construct $\tau_{\le 0}$ and $\tau_{\ge 1}$, define the rest via shifts. ← after that put

$$\tau_{\le t} = [-t]\circ \tau_{\le 0} \circ [t]$$

$$\tau_{\ge t} = [-t]\circ \tau_{\ge 0} \circ [t]$$

Let $X \in \mathcal{T}$, fix a dist triangle

$$\begin{tikzcd} X_{\le 0} \arrow[r] \arrow[d, equal] & X \arrow[r] & X_{\ge 1} \arrow[r] \arrow[d, equal] & X_{\le 0}[1] \\ \tau_{\le 0} X & & \tau_{\ge 1} X & \end{tikzcd}$$

define

Given $f : X \to Y$

$$\begin{tikzcd}
 & X_{\leq 0} \arrow[r, "u"] \arrow[d, dashed, "g"'] \arrow[dl, bend right, "0"'] & X \arrow[r, "v"] \arrow[d, "f"] & X_{\geq 1} \arrow[r] \arrow[d, dashed, "h"] & X_{\leq 0}[1] \\
Y_{\geq 1}[-1] \arrow[r] & Y_{\leq 0} \arrow[r, "w"] & Y \arrow[r, "z"] & Y_{\geq 1} \arrow[r] & Y_{\leq 0}[1]
\end{tikzcd}$$

$\tau^{\geq 2}$
$\cap$
$\tau^{\geq 1}$

$$z f u : X_{\leq 0} \to Y_{\geq 1} \Rightarrow \exists\, h, g$$

By the lemma long time ago, g is unique!
Same for h.

$\Rightarrow$ $\tau_{\leq 0}$ & $\tau_{\geq 1}$ are indeed functors.

By construction they come with $\tau_{\leq 0} \to \mathrm{id}$ & $\mathrm{id} \to \tau_{\geq 1}$ & $\tau_{\geq 0} \to [1]\,\tau_{\leq 0}$.

Dist Δ $\to$ by construction.

Enough to check that $\tau_{\leq 0}$ is right adjoint to $\mathcal{T}^{\leq 0} \to \mathcal{T}$: $\forall X \in \mathcal{T}$, $\forall Z \in \mathcal{T}^{\leq 0}$

$$\mathrm{Hom}(Z, X) \simeq \mathrm{Hom}(Z, \tau_{\leq 0} X)$$

$$\begin{tikzcd}
 & & Z \arrow[dll, bend right, "\text{only } 0"'] \arrow[dl, dashed] \arrow[d] \arrow[dr, "0 \text{ since } X_{\geq 1} \in \mathcal{T}^{\geq 1}"] & & \\
X_{\geq 1}[-1] \arrow[r] & X_{\leq 0} \arrow[r] & X \arrow[r] & X_{\geq 1} \arrow[r] & X_{\leq 0}[1]
\end{tikzcd}$$

$\Uparrow$

$\mathcal{T}^{\geq 2} \subset \mathcal{T}^{\geq 1}$ $\Rightarrow$ factorization

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr, dashed] & & X \\
 & X_{\leq 0} \arrow[ur] &
\end{tikzcd}$$

is unique! ∎

Same for $\tau_{\geq 1}$.

<u>Lm</u> Let $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ be a strictly full subcategory such that

(1) $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$,

(2) $\mathcal{T}^{\leq 0}$ is closed under extensions, $\leftarrow$ see below

(3) $\mathcal{T}^{\leq 0} \hookrightarrow \mathcal{T}$ has a right adjoint.

Then $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure, where
$\mathcal{T}^{\geq 1} = (\mathcal{T}^{\leq 0})^{\perp}$. $\left(\mathcal{T}^{\geq 0} = (\mathcal{T}^{\leq 0}[1])^{\perp} \right)$.

<u>Def</u> $\mathcal{T}' \subset \mathcal{T}$ – full subcategory is closed under extensions if $\forall$ dist $X \to Y \to Z \to X[1]$ if $X, Z \in \mathcal{T}'$, then $Y \in \mathcal{T}'$.

<u>Problem</u> If $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure, then $\mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 0}$ are closed under extensions.

<u>Pf</u> (Lemma) Define $\mathcal{T}^{\geq 0}$ as $(\mathcal{T}^{\leq 0})^{\perp}[1]$.

Let's check the properties of a t-structure.

(1) trivial (follows from $\mathcal{T}^{\leq -1} \subset \mathcal{T}^{\leq 0}$).

(2) $\mathrm{Hom}(X, Y) = 0$ $\forall X \in \mathcal{T}^{\leq 0}$, $Y \in \mathcal{T}^{\geq 1}$
trivial by the way we defined it.

(3) dist triangles. $\iota : \mathcal{T}^{\leq 0} \longrightarrow \mathcal{T}$, $\iota^!$ - right adjoint.

Consider any dist Δ:

$$\iota\iota^! X \longrightarrow X \longrightarrow X' \longrightarrow \iota\iota^! X[1]$$

Want to check that $X' \in \mathcal{T}^{\geq 1}$.

$X' \in \mathcal{T}^{\geq 1} \iff \mathrm{Hom}(Z, X') = 0 \quad \forall\, Z \in \mathcal{T}^{\leq 0}$.

$$\begin{tikzcd}
\mathcal{T}^{\leq 0} \ni \iota\iota^! X \arrow[r, "u'"] \arrow[d, equal] & Z' \arrow[r] \arrow[d, "g"] \arrow[dl, "g'"'] & Z \in \mathcal{T}^{\leq 0} \arrow[r, "wf"] \arrow[d, "f"] & \iota\iota^! X[1] \arrow[d, equal] \\
\iota\iota^! X \arrow[r, "u"'] & X \arrow[r, "v"'] & X' \arrow[r, "w"'] & \iota\iota^! X[1]
\end{tikzcd}$$

g - completion to a morphism of dist Δ's

$\iota\iota^! X$ & $Z \in \mathcal{T}^{\leq 0} \Rightarrow Z' \in \mathcal{T}^{\leq 0}$ (closed under ext's)

$Z' \xrightarrow{g} X$ factors uniquely through $\iota\iota^! X \to X$

$g' \circ u' = \mathrm{id}$! $\Rightarrow$ the triangle above splits $\Rightarrow$

$\Rightarrow Z \xrightarrow{wf} \iota\iota^! X[1]$ is 0 (exc/problem some lectures ago) $\Rightarrow$ $f : Z \to X'$ lifts to a morphism $Z \to X$ $\overset{Z \in \mathcal{T}^{\le 0}}{\Longrightarrow}$ factors through $Z \to u^! X$ $\Rightarrow$

$$\begin{tikzcd}
 & & & Z \arrow[dlll] \arrow[d, "f"] & \\
u^! X \arrow[r] \arrow[rrr, bend right, "0"'] & & X \arrow[r] & X' \arrow[r] & u^! X[1]
\end{tikzcd} \quad \Rightarrow f = 0 !$$

$\square$

<u>Prop</u> $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ – non-degenerate t-structure $\Rightarrow$ $\Rightarrow$ $\mathcal{T}^{0} = \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$ is abelian!

<u>Pf</u> For additivity – enough to check that it's closed under $\oplus$. Both $\mathcal{T}^{\le 0}$ & $\mathcal{T}^{\ge 0}$ are closed under extensions. $X, Y \in \mathcal{T}^0$ $\Rightarrow$

$$X \to X \oplus Y \to Y \to X[1] \text{ is dist!}$$

Existence of kernels & cokernels:

$$f : U \to V \quad - \text{ morphism in } \mathcal{T}^0$$

$$U \xrightarrow{u} X \xrightarrow{f} Y \xrightarrow{v} U[1]$$

Rmk: $U \in \mathcal{T}^{[0,1]}$, where $\mathcal{T}^{[a,b]} = \mathcal{T}^{\leq b} \cap \mathcal{T}^{\geq a}$

(think cohomology only in term $a, a+1, \dots, b$).

$\mathcal{T}^{[a,b]}$ always closed under extensions

$$Y[-1] \to U \to X \to Y$$

$$Y[-1] \in \mathcal{T}^1 \subset \mathcal{T}^{[0,1]}, \qquad X \in \mathcal{T}^0 \subset \mathcal{T}^{[0,1]}$$

Put $K = \tau_{\leq 0} U$, $\quad C = \tau_{\geq 0}(U[1]) = (\tau_{\geq 1} U)[1]$

$$\begin{tikzcd} K = \tau_{\leq 0} U \arrow[r] \arrow[rr, bend right] & U \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[rr, bend right] & U[1] \arrow[r] & \tau_{\geq 0}(U[1]) = C \end{tikzcd}$$

Claim these are the kernel & the cokernel of $f: X \to Y$.

Check the UP: $Z \xrightarrow{g} X$ s.t. $Z \xrightarrow{g} X \xrightarrow{f} Y$ is 0, $Z \in \mathcal{T}^0$

$$
\begin{tikzcd}
 & & Z \in \mathcal{T}^{\le 0} \arrow[dll, dashed, "\exists!"'] \arrow[dl, dashed, "\exists!"'] \arrow[d, "g"] \arrow[dr, "0"] & & \\
\tau_{\le 0} U \arrow[r] & U \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & U[1]
\end{tikzcd}
$$

Same thing for the cokernel.

Coimage = Image – next time...

Also next time: derived category of constructible sheaves, 6 functor formalism.