Triangulated and Derived Categories in Algebra and Geometry

Lecture 21

0. Few words on t-structures

not needed!

$\mathcal{T}$ – triangulated, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ – t-structure

$\rightsquigarrow$ heart $\mathcal{T}^{0} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$

$\cap \mathcal{T}^{\leq p} = \cap \mathcal{T}^{\geq q} = 0$

Prop If the t-structure is non-degenerate, then $\mathcal{T}^{0}$ is abelian.

Pf 1) direct sums are a type of extensions, $\mathcal{T}^{\leq 0}$ & $\mathcal{T}^{\geq 0}$ are closed under extensions $\Rightarrow$ $\mathcal{T}^{0}$ is closed under ext $\Rightarrow$ closed under $\oplus$.

2) $f: X \to Y$ in $\mathcal{T}$, $X, Y \in \mathcal{T}^{0}$ $\rightsquigarrow$ dist Δ

$$U \xrightarrow{u} X \xrightarrow{f} Y \xrightarrow{v} U[1] \qquad U \in \mathcal{T}^{[0,1]} = \mathcal{T}^{\leq 1} \cap \mathcal{T}^{\geq 0}$$

Put $K = \tau_{\leq 0} U$, $C = \tau_{\geq 0}(U[1]) = (\tau_{\geq 1} U)[1]$

$K \to U \to X \qquad\qquad Y \to U[1] \to C$

<u>Claim</u> these are the kernel & the cokernel of f!

For the kernel:

$Z \in \mathcal{T}^0$, $f \circ g = 0$

unique $\longrightarrow$ g'

$\mathrm{Hom}(Z, Y[-1]) = 0$

$$\begin{tikzcd}
 & Z \arrow[dl, dashed, "g'"'] \arrow[d, "g"] & & \\
U \arrow[r, "u"] & X \arrow[r, "f"] & Y \arrow[r, "v"] & U[1]
\end{tikzcd}$$

$Z \in \mathcal{T}^0 \Rightarrow \exists!$

$$\begin{tikzcd}
 & Z \arrow[dl, dashed, "g''"'] \arrow[d, "g'"] \\
K = \tau_{\leq 0} U \arrow[r] & U
\end{tikzcd}$$

Same for the cokernel.

3) Coim = Im

Consider the octahedron diagram:

$$
\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & C[-1] \arrow[r] \arrow[d] & K[1] \arrow[d, phantom, "\wr"] \\
K \arrow[r] & X \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & K[1] \\
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
 & U[1] \arrow[r] & C &
\end{tikzcd}
$$

$X \in \mathcal{T}^0$, $K[1] \in \mathcal{T}^{-1}$

$\Downarrow$

$V \in \mathcal{T}^{[-1,0]}$

$Y \in \mathcal{T}^0$, $C[-1] \in \mathcal{T}^1$

$\Downarrow$

$V \in \mathcal{T}^{[0,1]}$

From the two $\Rightarrow$ $V \in \mathcal{T}^0$.

By definition: $\operatorname{Im} f = \operatorname{Coker}(K \to X) = \tau_{\geq 0} V = V$,

$\operatorname{Coim} f = \operatorname{Ker}(Y \to C) = \tau_{\leq 0} V = V$. $\square$

<u>Warning</u> It is not true in general that, say, if $\mathcal{T} = D^b(\mathcal{A})$, $\mathcal{T}^0$ is the heart of a non-deg. t-structure, then $D^b(\mathcal{A}) \simeq D^b(\mathcal{T}^0)$.

Non-trivial example

$$\mathcal{T} = D(\mathbb{Z}\text{-mod})$$

$$\mathcal{T}^{\leq 0} = \{ X \in D(\mathbb{Z}\text{-mod}) \mid H^t(X) = 0,\ t > 0,\ H^0(X) \text{ is torsion} \},$$

$$\mathcal{T}^{\geq 1} = \{ X \in D(\mathbb{Z}\text{-mod}) \mid H^t(0) = 0,\ t < 0,\ H^0(X) \text{ is torsion free} \}.$$

<u>Exc</u> check that it's a non-deg. t-structure.

For a general t-structure: if $X, Y \in \mathcal{T}^0 \Rightarrow$

$$\mathrm{Hom}(X, Y[1]) \simeq \mathrm{Ext}^1_{\mathcal{T}}(X, Y).$$

Indeed, $f : X \to Y[1] \rightsquigarrow$ dist Δ

$$Y \to Z \to X \to Y[1]$$

check that it gives an element in $\mathrm{Ext}^1_{\mathcal{I}}$.

Can define $\mathrm{Ext}^i(X, Y) = \mathrm{Hom}(X, Y[i])$.

In general, different from $\mathrm{Ext}^i_{\mathcal{Y}}(X,Y)$.

1. <u>Cohomology w/r to a t-structure</u>

$\mathcal{T}$ - Δ category, $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ - t-structure.

<u>Lm</u> For $a \leq b$ $\exists$ canonical isomorphisms

$$\tau_{\leq a} \circ \tau_{\leq b} \simeq \tau_{\leq b} \circ \tau_{\leq a} \simeq \tau_{\leq a} \quad \text{and}$$

$$\tau_{\geq b} \circ \tau_{\geq a} \simeq \tau_{\geq a} \circ \tau_{\geq b} \simeq \tau_{\geq b}.$$

<u>Pf</u> $$\mathcal{T}^{\leq a} \hookrightarrow \mathcal{T}^{\leq b} \hookrightarrow \mathcal{T}$$

<u>Lm</u> For $a \leq b$ $\exists$ canonical isomorphism

$$\tau_{\leq b} \circ \tau_{\geq a} \simeq \tau_{\geq a} \circ \tau_{\leq b}.$$

<u>Pf</u> Rmk that $\tau_{\leq a-1} \tau_{\leq b} \simeq \tau_{\leq a-1}$ (previous lemma).

$\tau_{\leq a-1} X \to \tau_{\leq b} X \to X \rightsquigarrow$ octahedral diagram

$$
\begin{tikzcd}
\tau_{\le a-1} X \arrow[r] \arrow[d, equal] & \tau_{\le b} X \arrow[r] \arrow[d] & \tau_{\ge a}\tau_{\le b} X \arrow[r] \arrow[d] & \tau_{\le a-1} X[1] \arrow[d, equal] \\
\tau_{\le a-1} X \arrow[r] & X \arrow[r] \arrow[d] & \tau_{\ge a} X \arrow[r] \arrow[d] & \tau_{\le a-1} X[1] \\
 & \tau_{\ge b+1} X \arrow[r, equal] \arrow[d] & \tau_{\ge b+1} X \arrow[d] & \\
 & \tau_{\le b} X[1] \arrow[r] & \tau_{\ge a}\tau_{\le b} X[1] &
\end{tikzcd}
$$

From the Ist row we get $\tau_{\ge a}\tau_{\le b} X \in \mathcal{T}^{\le b}$ $\Rightarrow$

$\Rightarrow$ IInd column is a Δ associated to $(\mathcal{T}^{\le b}, \mathcal{T}^{\ge b+1})$. $\square$

<u>Def</u> Put

$$H^a(X) = (\tau_{\le a}\tau_{\ge a} X)[a] \in \mathcal{T}^{0}.$$

Cohomology w/r to the t-structure.

Prop Let $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ be a t-structure. Then

1) $\forall X \in \mathcal{T}$ $H^{p+q}(X) \simeq H^p(X[q])$,
2) $X \to Y \to Z \to X[1]$ - dist $\rightsquigarrow$

$$\ldots \to H^p(X) \to H^p(Y) \to H^p(Z) \to H^{p+1}(X) \to \ldots \quad \text{LES.}$$

2. Sheaves & the 6-functor formalism

Instead of sheaves of abelian groups, one usually deals with modules over a sheaf of rings.

Def A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space, $\mathcal{O}_X$ - sheaf of commutative rings: $U \rightsquigarrow \mathcal{O}_X(U) \in \mathrm{Comm}$, restrictions are homomorphisms of rings.

The category of $\mathcal{O}_X$-modules: an object is a sheaf of abelian groups on X + $\forall U \subset X \rightsquigarrow \mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$: $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$, these should commute with restriction: $\forall\, V \subset U$

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d, "\mathrm{res} \times \mathrm{res}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(U) \arrow[d, "\mathrm{res}"] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V). \end{tikzcd}$$

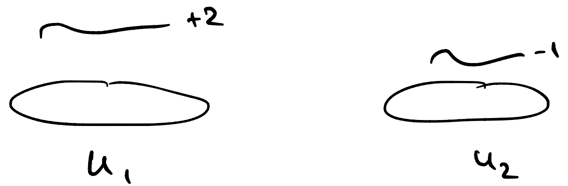
<u>Exc</u> Define morphisms of $\mathcal{O}_X$-modules & check that $\mathcal{O}_X$-mod forms an abelian category.

Main examples:

0) X is some kind of manifold, $\mathcal{O}_X$ - sheaf of "nice" functions.

1) A - commutative ring $\rightsquigarrow$ A_X - sheafification of the constant presheaf $U \mapsto A$.

$$A_X(U) = \{ U \to A \text{ cont, where } A \text{ has discrete top.} \}$$



want to glue these

$\mathbb{Z}_X$ - mod = sheaves of abelian groups.

From now on, fix A, deal with A_X-modules.

The abelian category structure on A_X-mod is induced by that of $AbSh(X)$.

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \quad \text{SES} \iff$$
$$0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0 \quad \text{is SE} \quad \forall x \in X.$$

3. <u>Restrictions to subsets</u>

$U \subset X$, $\mathcal{F} \in A_X$-mod $\rightsquigarrow$ $\mathcal{F}|_U \in A_U$-mod

$$\mathcal{F}|_U(V) = \mathcal{F}(V) \quad \forall\, V \subset U$$

Recall that $\forall\ f: X \to Y$, $\mathcal{F} \in A_Y\text{-mod}$ we defined

$$f^{-1}\mathcal{F}(U) = \varinjlim_{V \supset f(U)} \mathcal{F}(V) \qquad (\text{presheaf} \rightsquigarrow \text{sheafify}).$$

In the case $\iota: U \to X$:

$$\mathcal{F}|_U = \iota^{-1}\mathcal{F}.$$

We had another operation: $\mathcal{G} \in A_X\text{-mod}$

$$f_*\mathcal{G}(V) = \mathcal{G}(f^{-1}(V)) \qquad (\text{sheaf})$$

We know that $f^{-1} \dashv f_*$.

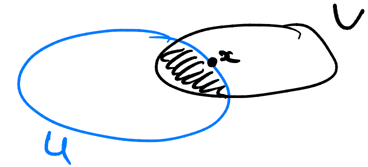
For $\iota: U \to X$

$$\iota_*\mathcal{G}(V) = \mathcal{G}(\iota^{-1}(V)) = \mathcal{G}(U \cap V)$$

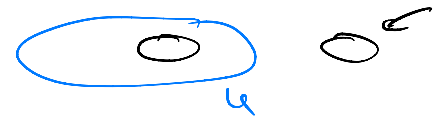
- $|_U$ is left adjoint to ι_*.

It turns out, there is a left adjoint to $-|_U$, extension by zero.

$$\iota_! : A_U\text{-mod} \longrightarrow A_X\text{-mod}$$



$$\iota_! \mathcal{G}(V) = \begin{cases} 0, & V \not\subseteq U, \\ \mathcal{G}(V), & V \subseteq U. \end{cases} \quad \longleftarrow \text{sheafify}$$



$$\iota_! \mathcal{G} \subset \iota_* \mathcal{G} \quad - \text{ subsheaf.}$$

<u>Exc</u> Check that for $\iota : U \hookrightarrow X$ $\iota_!$ is left adjoint to $\iota^{-1} = -|_U$.

<u>Properties</u>: • $(\iota_! \mathcal{G})_x = \begin{cases} \mathcal{G}_x, & x \in U \\ 0, & x \notin U \end{cases}$

• $V \xhookrightarrow{j} U \xhookrightarrow{i} X$

$$\iota_! \circ j_! \simeq (\iota \circ j)_!$$

Observation: $\mathrm{Hom}(A_X, \mathcal{F}) \simeq \mathcal{F}(X) = \Gamma(X, \mathcal{F})$.

Indeed, for every $U \subset X$: $A \longrightarrow \mathcal{F}(U)$ - morphism of A-modules. Any such morphism is given by the image of 1!

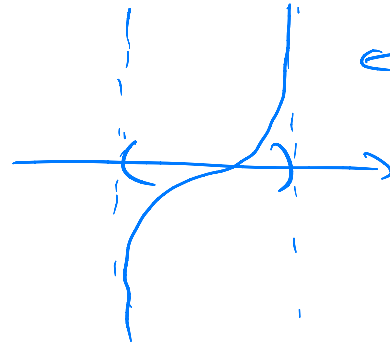
$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[ddd, bend right] & s \arrow[ddd, bend left] \\
A \arrow[r] \arrow[d, "U"] & \mathcal{F}(X) \arrow[d, "\mathrm{res}^X_U"] \\
A \arrow[r] & \mathcal{F}(U) \\
1 & s|_U
\end{tikzcd}$$

Thus, $\mathrm{Hom}(\iota_! A_U, \mathcal{F}) \simeq \mathrm{Hom}(A_U, \mathcal{F}|_U) \simeq \mathcal{F}(U)$.

adjunction

Lm Let $\mathcal{I}$ be an injective sheaf of A_X-modules. Then $\forall\, V \subset U$ $\mathrm{res}^U_V : \mathcal{I}(U) \longrightarrow \mathcal{I}(V)$ is surjective!

A bit strange since



$\leftarrow$ does not extend to a cont. function on a larger interval.

<u>Pf</u> Let $\mathcal{I}$ be injective. Means that $\forall\ 0 \to \mathcal{G} \to \mathcal{F}$

$\mathrm{Hom}(\mathcal{F}, \mathcal{I}) \to \mathrm{Hom}(\mathcal{G}, \mathcal{I})$ is surjective.

$V \subset U \subset X$ – open $\qquad V \overset{j}{\hookrightarrow} U \overset{\iota}{\hookrightarrow} X$

Consider $(\iota \circ j)_! A_V \hookrightarrow \iota_! A_U$ (indeed, look at the stalks: left $= A$ $\forall x \in V$, 0 – otherwise, right $= A$ $\forall x \in U$, 0 – otherw).

$$\begin{array}{ccc} \mathrm{Hom}(\iota_! A_U, \mathcal{I}) & \twoheadrightarrow & \mathrm{Hom}((\iota \circ j)_! A_V, \mathcal{I}) \\ \wr\| & & \wr\| \\ \mathcal{I}(U) & \xrightarrow{\text{res}} & \mathcal{I}(V) \end{array}$$

$\square$

Def A sheaf $\mathcal{F}$ is called flabby (flasque) if $\forall\, V \subset U$
$res^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Exc 1) $U \subset X$, $\mathcal{F}$ is flabby $\Rightarrow$ $\mathcal{F}|_U$ is flabby.
2) $f: X \to Y$, $\mathcal{F}$ is flabby $\Rightarrow$ $f_*\mathcal{F}$ is flabby.

Lm Let $0 \to \mathcal{F}' \to \mathcal{F} \xrightarrow{\alpha} \mathcal{F}'' \to 0$ be an exact sequence.
If $\mathcal{F}'$ is flabby, then the sequence

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

here should be $H^1(X, \mathcal{F}')$

is exact! (In general, only left exact.)

Pf Enough to check that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is surjective.
Let $s'' \in \mathcal{F}''(X)$. Consider the set of pairs
$\{(U, s) \mid s \in \mathcal{F}(U) \text{ s.t. } \alpha(s)|_U = s''|_U\}$.
This set has a natural partial order and is non-empty!

$\forall x \in X \quad \exists \ U_x \ni x$ in X s.t. $s''|_{U_x}$ lifts to some $s \in \mathcal{F}(U_x)$ (surjectivity on stalks).

Given an open chain, there is an upper bound:

$$(U_i, s_i) \le (U_{i+1}, s_{i+1}) \le \dots \quad \rightsquigarrow (\cup U_i, s)$$

$U_i \subset U_{i+1}, \quad s_{i+1}|_{U_i} = s_i$

exists by the sheaf condition

By Zorn lemma $\rightsquigarrow$ maximal (U, s). If $U = X$, we are done. Assume $U \neq X$, pick $x \in X \setminus U$.

Locally $\exists \ V \ni x, \ t \in \mathcal{F}(V)$ s.t. $\alpha(t)|_V = s''$.

Both s & t map to the same element in $\mathcal{F}''(U \cap V)$.

$\Rightarrow (s-t)|_{U \cap V}$ is in the subsheaf $\mathcal{F}' \Rightarrow$

$\Rightarrow \exists$ a section $s' \in \Gamma(X, \mathcal{F}')$ s.t. $s'|_{U \cap V} = s - t$.

Replace t by $t + s'$. Then $t \in \mathcal{F}(V)$, $s \in \mathcal{F}(U)$

$t|_{U \cap V} = s|_{U \cap V}$ $\Rightarrow$ glue $\Rightarrow$ we can enlarge U! $\square$