Triangulated and Derived Categories in Algebra and Geometry
Lecture 22

1. Sheaves on a single space

X – top space, A – comm. ring

A_X – modules = sheaves of abelian groups $\mathcal{F}$ s.t. $\forall\, U \subset X$ $\mathcal{F}(U)$ is an A-module, restriction plays well with the action

A_X-mod – abelian category

$U \subset X \rightsquigarrow \mathcal{F}|_U = \iota^{-1}\mathcal{F}$, where $\iota : U \to X$ – open inclusion

__Def__ Given $\mathcal{F}, \mathcal{G} \in A_X$-mod, the sheaf hom is the presheaf

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \longmapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

__Exc__ Check that $\mathcal{H}om$ is a sheaf!

If you look at stalks, you get a map

$$\mathcal{H}om(\mathcal{F},\mathcal{G})_x \longrightarrow \mathrm{Hom}_A(\mathcal{F}_x,\mathcal{G}_x).$$

Warning: in general, neither epi nor mono.

Global sections:

$$\Gamma(X,\mathcal{H}om(\mathcal{F},\mathcal{G})) = \mathrm{Hom}(\mathcal{F}|_X,\mathcal{G}|_X) = \mathrm{Hom}(\mathcal{F},\mathcal{G}).$$

Get a bi-functor

$$\mathcal{H}om(-,-) : A_X\text{-mod}^{op} \times A_X\text{-mod} \longrightarrow A_X\text{-mod}.$$

Exc $\mathcal{H}om$ is left exact in both arguments.

Tensor product

Def Given $\mathcal{F},\mathcal{G} \in A_X$-mod, their tensor product is the sheafification of $U \longmapsto \mathcal{F}(U)\otimes_A \mathcal{G}(U)$.

Exc $(\mathcal{F} \otimes_{A_X} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_A \mathcal{G}_x$.

will drop it in the future

Cor Since exactness of sequences in A_X-mod can be checked on stalks, $\mathcal{F} \otimes -$ is right exact.

For A-modules we have $\otimes \dashv \mathrm{Hom}$.

Lm $\forall\, \mathcal{F}, \mathcal{G}, \mathcal{H} \in A_X$-mod there is a functorial isomorphism

$$\mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

In A_X-mod $\leftarrow$ category of sheaves!

Pf $\mathcal{F} \otimes \mathcal{G}$ is the sheafification. On the right we have a sheaf $\mathcal{H}$ $\Rightarrow$ enough to look at the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes \mathcal{G}(U).$$

Fill in the details (look at sections on both sides). $\square$

Passing to global sections:

$$\mathrm{Hom}(\mathcal{F}\otimes\mathcal{G},\mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F},\mathcal{H}om(\mathcal{G},\mathcal{H}))$$

$-\otimes\mathcal{G}$ is left adjoint to $\mathcal{H}om(\mathcal{G},-)$.

<u>Limits</u> & <u>colimits</u>

Recall that the category of presheaves is the category of functors $Op(X)^{\circ} \longrightarrow A\text{-mod}$

$\Longrightarrow$ presheaves has all limits & colimits.

<u>Exc</u> $\varprojlim$ of a diagram of sheaves is a sheaf.

$\varinjlim$ of a diagram of sheaves needs to be sheafified.

2. Functors b/w different categories

$f : X \to Y$ – cont. map

A_X-mod & A_Y-mod ← two abelian categories

Direct image / push forward

$f_* : A_X\text{-mod} \to A_Y\text{-mod}$

← sheaf

$\mathcal{F} \in A_X\text{-mod} \rightsquigarrow f_*\mathcal{F} : \quad U \mapsto \mathcal{F}(f^{-1}(U))$

Inverse image

$f^{-1} : A_Y\text{-mod} \to A_X\text{-mod}$

← sheafify

$\mathcal{G} \in A_Y\text{-mod} \rightsquigarrow f^{-1}\mathcal{G} : \quad U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$

Rmk If $f(U)$ is open in Y, then $f^{-1}\mathcal{G}(U) = \mathcal{G}(f(U))$

Lm $f^{-1} \dashv f_*$.

Pf Construct the unit & counit morphisms:

$$\mathrm{id} \longrightarrow f_* \circ f^{-1}, \qquad f^{-1} \circ f_* \longrightarrow \mathrm{id}$$

E.g.
$$(f^{-1} \circ f_*)\mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_*\mathcal{F}(V) =$$
$$= \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}(V))$$

If $V \supset f(U)$, then $f^{-1}(V) \supset U \Rightarrow$ there are compatible res: $\mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}(U)$.

Gives you the counit. ... ∎

Lm If $f: X \to Y$ is cont's, $\mathcal{F} \in A_Y$-mod, $\mathcal{G} \in A_X$-mod, then

$$\mathcal{H}om(\mathcal{F}, f_*\mathcal{G}) \simeq f_* \mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G}).$$

in A_Y-mod

Rmk Apply global sections:

$$\Gamma(Y, \mathcal{H}om(\mathcal{F}, f_* \mathcal{G})) = \mathrm{Hom}(\mathcal{F}, f_* \mathcal{G})$$

$$\Gamma(Y, f_* \mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G})) = \Gamma(X, \mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G})) = \mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G})$$

Sheafified version of the adjunction!

Pf

$$\Gamma(U, f_* \mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G})) = \Gamma(f^{-1}(U), \mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G})) =$$

$$\begin{array}{c} f^{-1}(U) \\ \downarrow f \\ U \end{array}$$

$$= \mathrm{Hom}(f^{-1}\mathcal{F}|_{f^{-1}(U)}, \mathcal{G}|_{f^{-1}(U)}) =$$

$$= \mathrm{Hom}(\mathcal{F}|_U, f_* \mathcal{G}|_U) = \Gamma(U, \mathcal{H}om(\mathcal{F}, f_* \mathcal{G})) \qquad \square$$

Exe Check that for $f: X \to Y$ cont's, $\mathcal{F}, \mathcal{G} \in A_Y\text{-mod}$,

$$f^{-1}\mathcal{F} \otimes f^{-1}\mathcal{G} \simeq f^{-1}(\mathcal{F} \otimes \mathcal{G}).$$

Warning Not the case for f_*. Is there a nat map?

3. Support

Let $\mathcal{F}$ be a sheaf of A_X-modules. $x \in X$

$\mathcal{F} \longrightarrow \mathcal{F}_x$

Take a section $s \in \Gamma(U, \mathcal{F})$. If $s_x \in \mathcal{F}_x$ is equal to zero, it means that $\exists\ U \supset V \ni x$ s.t.
$s|_V = 0 \implies s_y = 0$ for all $y \in V$.

In other words, the set of $x \in U$ st. $s_x = 0$ is open.

Def $\mathrm{Supp}(s) = \{x \mid s_x \neq 0\}$ – support of s.

$\mathrm{Supp}(s)$ is closed.

Goal: define various sheaves associated to a locally closed $Z \hookrightarrow X$.

intersection of an open & a closed

Case $Z \hookrightarrow X$ is closed

$j : Z \hookrightarrow X$

Put $\mathcal{F}_Z = j_* j^{-1} \mathcal{F}$. Sheaf on X!

$$\mathcal{F}_Z(U) = j_* j^{-1} \mathcal{F}(U) = j^{-1}\mathcal{F}(U \cap Z) = \varinjlim_{V \supset U \cap Z} \mathcal{F}(V)$$

Adjunction unit:

$$\mathcal{F} \longrightarrow \mathcal{F}_Z$$

Exc Check that

1) $$(\mathcal{F}_Z)_x = \begin{cases} \mathcal{F}_x, & x \in Z, \\ 0, & x \notin Z. \end{cases}$$

2) $\mathcal{F}_x \longrightarrow (\mathcal{F}_Z)_x$ is iso for $x \in Z$.

Rmk $\mathcal{F} \longrightarrow \mathcal{F}_Z$ is surjective (look at the stalks).

$\mathcal{G} = \ker(\mathcal{F} \to \mathcal{F}_Z)$ — subsheaf in $\mathcal{F}$ satisfying

$$\mathcal{G}_x = \begin{cases} 0, & x \in Z, \\ \mathcal{F}_x, & x \notin Z. \end{cases}$$

<u>Case $U \hookrightarrow X$ open</u>

defined since $X \setminus U$ is closed

Put $\mathcal{F}_U = \ker(\mathcal{F} \to \mathcal{F}_{X \setminus U})$

Comes with $\mathcal{F}_U \hookrightarrow \mathcal{F}$.

<u>Case Z is locally closed</u>

$Z = U \cap A$, U - open, A - closed

Put $\mathcal{F}_Z = (\mathcal{F}_U)_A$

<u>Properties</u>

1) $$(\mathcal{F}_Z)_x = \begin{cases} \mathcal{F}_x, & x \in Z, \\ 0, & x \notin Z. \end{cases}$$

2) $\mathcal{F} \longmapsto \mathcal{F}_Z$ is an exact functor (look at stalks).

3) Z, Z' - locally closed

$(\mathcal{F}_Z)_{Z'} \simeq \mathcal{F}_{Z \cap Z'}$.

4) $Z' \subset Z$ - closed

$$0 \to \mathcal{F}_{Z \setminus Z'} \to \mathcal{F}_Z \to \mathcal{F}_{Z'} \to 0$$

(Generalization of $0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_{X \setminus U} \to 0$.)

5) Z_1 & Z_2 - closed

$$0 \to \mathcal{F}_{Z_1 \cup Z_2} \to \mathcal{F}_{Z_1} \oplus \mathcal{F}_{Z_2} \to \mathcal{F}_{Z_1 \cap Z_2} \to 0.$$

6) U_1 & U_2 - open

$$0 \to \mathcal{F}_{U_1 \cap U_2} \to \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2} \to \mathcal{F}_{U_1 \cup U_2} \to 0.$$

Another construction: sheaf of sections supported on Z.

Let Z be a closed subset in U.

$$\Gamma_Z(U, \mathcal{F}) = \operatorname{Ker}\left(\mathcal{F}(U) \to \mathcal{F}(U \setminus Z)\right)$$

Sections vanishing outside Z = supported on Z.

If Z is contained in $V^{\subset U} \Rightarrow Z$ is closed in $V \subset U$ ← open.

Remark: $\Gamma_Z(U, \mathcal{F}) \longrightarrow \Gamma_Z(V, \mathcal{F})$ - isomorphism!

For Z locally closed, define

$$\Gamma_Z(X, \mathcal{F}) = \Gamma_Z(U, \mathcal{F}), \quad U\text{-open s.t. } Z \hookrightarrow U \text{ is closed.}$$

can alternatively define this as $\varinjlim$

<u>Def</u> The sheaf of sections of $\mathcal{F}$ supported at Z is given by $\Gamma_Z(\mathcal{F}) : U \longmapsto \Gamma_{Z \cap U}(U, \mathcal{F})$.

<u>Exc</u> $\Gamma_Z(\mathcal{F})$ is a sheaf!

Properties

shef version of $\Gamma_Z(X,-)$

1) $\Gamma_Z(X,-) = \Gamma(X,-)\circ\Gamma_Z(-)$.

2) U - open $\Rightarrow$

$$\Gamma_U(\mathcal{F}) = \iota_*\circ\iota^{-1}\mathcal{F}.$$

(If $Z\hookrightarrow X$ - closed, then $\mathcal{F}_Z = j_*\circ j^{-1}\mathcal{F}$.)

Exc Let $Z\hookrightarrow X$ be locally closed. Then $\forall\,\mathcal{F}\in A_X$-mod

- $\mathcal{F}_Z = (A_X)_Z\otimes_{A_X}\mathcal{F}$,
- $\mathcal{H}om((A_X)_Z,\mathcal{F})\simeq\Gamma_Z(\mathcal{F})$.

Lm $\forall\,\mathcal{F}_1,\mathcal{F}_2\in A_X$-mod & $Z\hookrightarrow X$ locally closed

$$\mathcal{H}om((\mathcal{F}_1)_Z,\mathcal{F}_2)\simeq\mathcal{H}om(\mathcal{F}_1,\Gamma_Z(\mathcal{F}_2))\simeq\Gamma_Z\mathcal{H}om(\mathcal{F}_1,\mathcal{F}_2).$$

previous exc + the sheafified adj.

4. Injective and flat sheaves

Def An injective sheaf is an injective object in A_X-mod.

Lm If $\mathcal{F}$ is injective, then for any $U \subset X$ open $\mathcal{F}|_U$ is injective.

Pf For any $\mathcal{G} \in A_U$-mod

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}|_U) = \mathrm{Hom}((\iota_* \mathcal{G})|_U, \mathcal{F}|_U) =$$

$$= \mathrm{Hom}((\iota_* \mathcal{G})_U, \mathcal{F})$$

The functor $\mathcal{G} \longmapsto (\iota_* \mathcal{G})_U$ is exact (what is it?).

$\Rightarrow$ $\mathrm{Hom}(-, \mathcal{F}|_U) \simeq \mathrm{Hom}((\iota_* -)_U, \mathcal{F})$ is exact! $\square$

Cor If $\mathcal{F}$ is injective, then $\mathcal{H}om(-, \mathcal{F})$ is exact.

Lm $f: X \to Y$, $\mathcal{F} \in A_X$-mod – injective $\Rightarrow f_* \mathcal{F}$ is injective.

Pf $\mathrm{Hom}(-, f_* \mathcal{F}) \simeq \mathrm{Hom}(\underbrace{f^{-1}(-)}_{\text{exact.}}, \mathcal{F})$ $\square$

Allows to demystify the construction of injective hulls.

<u>Prop</u> A_X-mod has enough injectives.

<u>Pf</u> Consider the space $\underline{X} \leftarrow X$ with discrete topology.

$f : \underline{X} \longrightarrow X$, $\mathcal{F} \in A_X$-mod

$f^{-1}\mathcal{F}$ has an injective hull:

injective

for every $x \in X$ choose $0 \to \mathcal{F}_x \to I_x$

Put $\mathcal{I} = \prod_{x \in X} I_x$. $f^{-1}\mathcal{F} \hookrightarrow \mathcal{I}$, $0 \to \mathcal{F} \hookrightarrow f_* \mathcal{I}$

injective by the previous $\square$

<u>Problem</u> In general A_X-mod does not have enough projectives. $\otimes$ is right exact, how do we compute its left derived?

Need enough objects s.t. $\mathcal{F} \otimes -$ is exact.

Def $\mathcal{F} \in A_X$-mod is flat if $\mathcal{F} \otimes -$ is exact.

Lm $\mathcal{F}$ is flat $\iff$ $\forall x \in X$ $\mathcal{F}_x$ is a flat A-module.
(In A-mod $\mathcal{F}_x \otimes_A -$ is exact. E.g. $\mathcal{F}_x$ is free.)

Pf Look at stalks.

Prop There are enough flat A_X-modules: $\forall \mathcal{F} \in A_X$-mod
$\exists\, \mathcal{P} \twoheadrightarrow \mathcal{F}$, $\mathcal{P}$ – flat.

Pf For every (U, s), $U \subset X$ – open, $s \in \mathcal{F}(U)$ consider the sheaf $(A_X)_U$. There is a natural map

$$(A_X)_U \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F}$$

$$\begin{array}{ccc} \Gamma(U, (A_X)_U) & \longrightarrow & \Gamma(U, \mathcal{F}_U) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1 & \longmapsto & s \end{array}$$

Put $\mathcal{P} = \bigoplus_{(U,s)} (A_X)_U \twoheadrightarrow \mathcal{F}$. $\square$

<u>Lm</u> If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact and $\mathcal{F}$ & $\mathcal{F}''$ are flat $\Rightarrow$ $\mathcal{F}'$ is flat.

<u>Conclusion</u> One can compute left derived functors of $\otimes$ using flat resolutions.

We have enough injectives $\Rightarrow$ can compute various right derived functors.

Alternatively - use flabby resolutions.

<u>Lm</u> If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact and $\mathcal{F}'$ & $\mathcal{F}$ are flabby $\Rightarrow$ $\mathcal{F}''$ is flabby.

Why is flabby nice?

<u>Lm</u> If $\mathcal{F}$ is flabby on X, Z - locally closed, then $\Gamma_Z(\mathcal{F})$ is flabby.

5. Proper pushforward

From now on, all top. spaces are considered to be very nice. Locally compact for instance. Think about nice manifolds.

Def $f: X \to Y$ is proper if f is closed (maps closed subsets to closed subsets) and its fibers are relatively Hausdorff and compact.

two points in a fiber have disj. neib's in X

Def The proper pushforward of $\mathcal{F} \in A_X$-mod is a subsheaf in $f_* \mathcal{F}$:

($Z \hookrightarrow X$ is locally closed, $i_! i^* \mathcal{F} = \mathcal{F}_Z$)

$$f_! \mathcal{F} : \; U \longmapsto \{ s \in \mathcal{F}(f^{-1}(U)) \mid \operatorname{supp}(s) \to U \text{ is proper} \}.$$

Next time
- Discuss all the functors in the derived setting.
- Poincaré - Verdier duality.

Lec 24 $\to$
- Perverse sheaves (non-trivial t-structures).