Triangulated and Derived Categories in Algebra and Geometry

Lecture 23

1. Derived categories of sheaves

Fix a sheaf of comm. rings $\mathcal{R}$ (in our case A - comm. ring, $\mathcal{R} = A_X \leftarrow$ locally constant sheaf)

Categories of A_X-modules

E.g. $\mathbb{Z}_X$-mod = sheaves of abelian groups

Given $\mathcal{F}, \mathcal{G} \in A_X$-mod $\rightsquigarrow$

$$\mathcal{H}om_{A_X}(\mathcal{F}, \mathcal{G}) : \mathcal{U} \longmapsto \mathrm{Hom}(\mathcal{F}|_{\mathcal{U}}, \mathcal{G}|_{\mathcal{U}})$$

Properties
- sheaf $\leftarrow$ do not need to sheafify
- $\Gamma \circ \mathcal{H}om(-,-) = \mathrm{Hom}(-,-)$
- does not commute with taking stalks

Given $\mathcal{F}, \mathcal{G} \in A_X\text{-mod} \rightsquigarrow$

$$\mathcal{F} \otimes_{A_X} \mathcal{G} : U \longmapsto \mathcal{F}(U) \otimes_{A_X(U)} \mathcal{G}(U) \qquad \text{sheafify!}$$

$\nwarrow$ can put A

<u>Properties</u>

- associative $\longleftarrow$ $\mathcal{F} \otimes (\mathcal{G} \otimes \mathcal{H}) \simeq (\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H}$
- commutative $\longleftarrow$ $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{G} \otimes \mathcal{H}$
- unit $\longleftarrow$ $A_X \otimes \mathcal{F} \simeq \mathcal{F} \otimes A_X \simeq \mathcal{F}$
- commutes with taking stalks

$$(\mathcal{F} \otimes \mathcal{G})_x \simeq \mathcal{F}_x \otimes \mathcal{G}_x$$

Why do we care about stalks? Allows to check that some natural maps are iso (mono/epi).

<u>Prop</u> $\quad - \otimes \mathcal{G} \dashv \mathcal{H}om(\mathcal{G}, -)$

right exact $\nearrow$ $\qquad\qquad$ $\nwarrow$ left exact

Sheaves on different spaces

$f : X \to Y$ continuous map

$f^{-1} : A_Y\text{-mod} \rightleftarrows A_X\text{-mod} : f_*$

Adjunction: $f^{-1} \vdash f_*$ ← left exact

actually, exact

$p : X \to \{pt\} \Rightarrow p_* = \Gamma(X, -)$

$p^{-1} M = M_X$

For both: $X \xrightarrow{f} Y \xrightarrow{g} Z$

$$(g \circ f)_* \simeq g_* \circ f_* \qquad (g \circ f)^{-1} \simeq f^{-1} \circ g^{-1}$$

Put $g : Y \to \{pt\} \rightsquigarrow \Gamma(X, -) \simeq \Gamma(Y, -) \circ f_*$

$\mathcal{H}om$ & $\otimes$ allow to sheafify many relations

Sheafified adjunction: $f: X \to Y$, $\mathcal{F} \in A_X\text{-mod}$, $\mathcal{G} \in A_Y\text{-mod}$

$$\mathcal{H}om(\mathcal{G}, f_* \mathcal{F}) \simeq f_* \mathcal{H}om(f^{-1}\mathcal{G}, \mathcal{F})$$

(What about more general sheaves of rings? Then $\mathcal{G}$ would be $\mathcal{R}$-mod, $\mathcal{F}$ would be $f^{-1}\mathcal{R}$-mod.)

Relation with $\otimes$:

$$f^{-1}\mathcal{F} \otimes f^{-1}\mathcal{G} \simeq f^{-1}(\mathcal{F} \otimes \mathcal{G})$$

one should put $\otimes_{f^{-1}\mathcal{R}}$

From now on assume all the topological spaces are nice. Say, locally compact & Hausdorff. Better - complex algebraic varieties (with complex topology) / smooth manifolds.

Def $f: X \to Y$ is proper if

do not agree in general

a) f is closed,
b) compact fibres

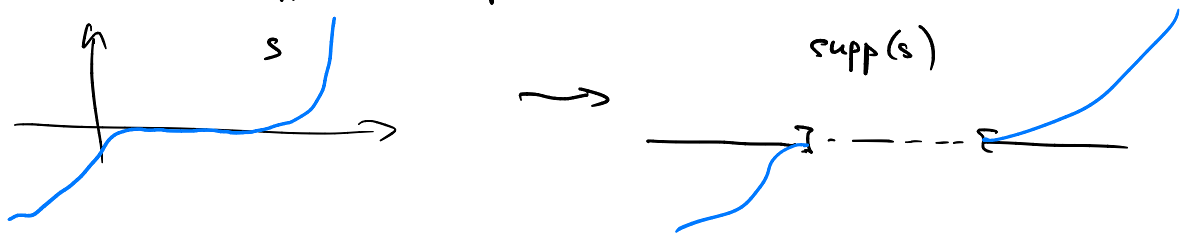
$\iff$ preimage of a compact is compact

Recall: given $s \in \Gamma(\mathcal{U}, \mathcal{F}) \rightsquigarrow \mathrm{supp}(s) \subset \mathcal{U}$

$$\mathrm{supp} = \{ x \in \mathcal{U} \mid s_x \neq 0 \} \longleftarrow \text{closed subset in } \mathcal{U}!$$

If $s_x = 0 \Rightarrow \exists\, V \ni x$ s.t. $s|_V = 0 \Rightarrow \forall y \in V \;\; s_y = 0 \Rightarrow$

$\Rightarrow \mathcal{U} \setminus \mathrm{supp}(s)$ is open.



Direct image with proper supports: given $f : X \to Y$, $\mathcal{F} \in A_X$-mod

$$f_!\mathcal{F} : \mathcal{U} \longmapsto \{ s \in \mathcal{F}(f^{-1}(\mathcal{U})) \mid f : \mathrm{supp}(s) \to \mathcal{U} \text{ is proper} \}$$

$$(f_!\mathcal{F})(\mathcal{U}) \subseteq (f_*\mathcal{F})(\mathcal{U})$$

Since being proper is a local property on the base, $f_!\mathcal{F}$ is a sheaf.

Take $p : X \to \{pt\} \rightsquigarrow$

$p_! = \Gamma_c(X, -)$ ← global sections with compact support

$$\Gamma_c(X, \mathcal{F}) = \{ s \in \Gamma(X, \mathcal{F}) \mid \mathrm{supp}(s) \text{ is compact} \}$$

Need these to deal with, say, duality on non-compact spaces: duality b/w homology & cohomology <u>integration</u> ← need compactness.

<u>Properties</u>

- $X \xrightarrow{f} Y \xrightarrow{g} Z$

  $(g \circ f)_! \simeq g_! \circ f_!$

- $X \xrightarrow{f} Y$ is proper

  $f_! = f_*$

  stalks capture cohomology of fibres

- $(f_! \mathcal{F})_x \xrightarrow{\sim} \Gamma_c(f^{-1}(x), \mathcal{F}|_{f^{-1}(x)})$

  ↑ does not hold for f_*

- $Z \overset{\iota}{\hookrightarrow} X$ - locally closed $\Rightarrow$ $\iota_!$ is exact &

  $\mathcal{F}_Z = \iota_! \circ \iota^{-1} \mathcal{F}$

Very important computational tools:

projection formula & base change.

Work for proper pushforward (if a map is proper $\Rightarrow$ replace with the usual one).

Projection formula

$f: X \to Y$, $\mathcal{G} \in A_X\text{-mod}$, $\mathcal{F} \in A_Y\text{-mod}$, $\mathcal{F}$ - flat $\leftarrow$ $-\otimes\mathcal{F}$ is exact

$$f_! \mathcal{G} \otimes \mathcal{F} \simeq f_!(\mathcal{G} \otimes f^{-1}\mathcal{F})$$

Base change

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Cartesian square: $Y' = Y \times_X X'$

$$g^{-1} \circ f_! \simeq f'_! \circ (g')^{-1}$$

2. Passing to derived categories

Denote $\mathcal{D}^*(X)$ $(* = \emptyset, +, -, b)$ $= \mathcal{D}^*(A_X\text{-mod})$

Recall that A_X-mod is an abelian category which has enough injectives.

Enough injectives $\rightsquigarrow$ derived functors for all left exact

$$
\begin{aligned}
R\Gamma(X,-) &: \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(\mathcal{A}b) \\
R\Gamma_Z(X,-) &: \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(\mathcal{A}b) \\
R\Gamma_Z(-) &: \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(X) \\
R\mathcal{H}om(-,-) &: \mathcal{D}^-(X)^\circ \times \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(X) \\
R\Gamma_c(X,-) &: \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(\mathcal{A}b) \\
Rf_*(-) &: \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(Y) \\
Rf_!(-) &: \mathcal{D}^+(X) \longrightarrow \mathcal{D}^+(Y)
\end{aligned}
$$

$Z \hookrightarrow X$ loc. closed

$f : X \to Y$

Since f^{-1}, $(-)_Z$ are exact, get

$f^{-1}(-) : \quad D^*(Y) \longrightarrow D^*(X)$

$(-)_Z : \quad D^*(X) \longrightarrow D^*(X)$

Tensor product: checked that there are enough flat sheaves. Assume that A has finite flat dimension: any A-module has a resolution by flat modules of length at most N. ($A = \mathbb{Z} \Rightarrow$ any module has a resolution of length ≤ 2 by projectives $\Rightarrow$ by flat.)

$- \overset{L}{\otimes} - : \quad D^*(X) \times D^*(X) \longrightarrow D^*(X).$

Some properties / definitions:

- cohomology of $\mathcal{F} \in A_X$-mod

$$H^i(X, \mathcal{F}) = H^i(R\Gamma(X, \mathcal{F})) \quad \text{(same for } H^i_c(X, \mathcal{F})\text{)}$$

- $R^i f_*(\mathcal{F}) = H^i(Rf_* \mathcal{F})$ is the sheafification of

$$U \longmapsto H^i(f^{-1}(U), \mathcal{F})$$

Want to compose these (derived) functors. Need adapted classes of objects. This is why we needed flabby sheaves.

$\mathcal{F}$ - flabby $\Rightarrow$ $f_* \mathcal{F}$ is flabby,
$\mathcal{F}$ - injective $\Rightarrow$ $\mathcal{F}$ is flabby

$$\Rightarrow \quad R(g \circ f)_* \simeq Rg_* \circ Rf_*$$

Similarly, $\quad R\mathrm{Hom} \simeq R\Gamma \circ R\mathcal{H}om$

What about proper direct image?

<u>Def</u> $\mathcal{F}$ - c-soft if $\forall\ K \subset X$ - compact
$$\Gamma(X, \mathcal{F}) \twoheadrightarrow \Gamma(K, \mathcal{F}).$$

either you give some definition, better put $\varinjlim\limits_{U \supset K} \Gamma(U, \mathcal{F})$.

Immediate corollary: flabby sheaves a c-soft.
c-soft - adapted class for proper support.

$$R(g\circ f)_! \simeq Rg_! \circ Rf_!$$

<u>Adjunctions</u> $\quad - \overset{L}{\otimes} \mathcal{G} \dashv R\mathcal{H}om(\mathcal{G}, -)$

More general: shefified

$$R\mathcal{H}om(\mathcal{F} \overset{L}{\otimes} \mathcal{G}, \mathcal{H}) \simeq R\mathcal{H}om(\mathcal{F}, R\mathcal{H}om(\mathcal{G}, \mathcal{H}))$$

<u>Also</u>

- $f^{-1} \dashv f_*$
- $R\mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}) \simeq R\mathrm{Hom}(f^{-1}\mathcal{F}, \mathcal{G})$ $\quad \mathcal{F} \in D^-, \ \mathcal{G} \in D^+$
- $R\mathcal{H}om(\mathcal{F}, f_*\mathcal{G}) \simeq Rf_* R\mathcal{H}om(f^{-1}\mathcal{F}, \mathcal{G})$ $\quad$ — " —
- $f^{-1}\mathcal{F} \overset{L}{\otimes} f^{-1}\mathcal{G} \simeq f^{-1}(\mathcal{F} \overset{L}{\otimes} \mathcal{G})$
- projection formula
- base change

<u>Example</u> Let X be a smooth manifold. Denote by $\mathcal{E}_X^p$ - sheaf of smooth p-forms (differential) $/\mathbb{C}$

Poincaré lemma: $\quad 0 \to \mathbb{C}_X \to \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \to \dots \to \mathcal{E}_X^n \to 0$ $\quad n = \dim X$

exact!

You get an isom in $D^b(X)$

$$\mathbb{C}_X \simeq \left(0 \to \mathcal{E}^0_X \to \dots \to \mathcal{E}^n_X \to 0\right)$$

All the $\mathcal{E}^p_X$ are c-soft $\Rightarrow$ you can apply $\Gamma_c(X,-)$ or $\Gamma(X,-)$ to the resolution term-wise to compute $R\Gamma(X,-)$ / $R\Gamma_c(X,-)$. Conclude:

$$H^i(X, \mathbb{C}_X) \simeq H^i\left(0 \to \mathcal{E}^0_X(X) \to \mathcal{E}^1_X(X) \to \dots \to \mathcal{E}^n_X(X) \to 0\right)$$

de Rham cohomology

if you believe that this is isom to $H^i(X,\mathbb{C})$

$\Rightarrow$ get a comparison theorem

Compositions of derived functors $\rightsquigarrow$ spectral sequences

Assume $F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{C}$ are left exact,

$R(G \circ F) \simeq RG \circ RF \longleftarrow$ we had conditions for this
injectives are sent to G-adapted.

Passing to cohomology and picking resolutions $\rightsquigarrow$
$\rightsquigarrow$ spectral sequence of a double complex

$$E_2^{pq} = R^pG \circ R^qF(x) \Rightarrow R^{p+q}(G \circ F)(x).$$

Grothendieck spectral sequence

Typical example:

$$X \xrightarrow{f} Y \to \{pt\} \qquad \Gamma(X,-) \simeq \Gamma(Y,-) \circ f_*$$

Get: $\quad H^p(Y, R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$

Assume that locally $\forall\, y \in Y$ $\exists\, \mathcal{U} \ni y$ s.t.
$$\begin{array}{ccc} f^{-1}(\mathcal{U}) & \longrightarrow & \mathcal{U} \\ \cong & \nearrow_{p} & \\ \mathcal{U} \times F & & \end{array}$$

Then locally $R^qf_*\mathbb{C} \simeq H^q(F, \mathbb{C})$
$\rightsquigarrow$ recover the usual Serre spectral sequence!

3. <u>Duality</u>

$f : X \to Y$ - map of nice spaces
$f_!$ - left exact functor, but not exact in general
$\Rightarrow f_!$ can not have a right adjoint

It turns out, there is a right adjoint

$$Rf_! : D^+(X) \rightleftarrows D^+(Y) : f^!$$

$$\mathrm{Hom}(Rf_! \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, f^! \mathcal{G})$$

For $\mathcal{F} \in D^+(X)$, $\mathcal{G} \in D^+(Y)$.

Assume that X is a smooth manifold (oriented) of dim n.
Assume that Y is a point.
Put $\mathcal{F} = \mathbb{C}_X$, $\mathcal{G} = \mathbb{C}$.

$$Rf_! \mathcal{F} = R\Gamma_c(X, \mathbb{C})$$

We will see that $f^! \mathcal{G} \simeq \mathbb{C}_X[n]$.

$$\mathrm{Hom}(R\Gamma_c(X, \mathbb{C})[n], \mathbb{C}) \simeq R\Gamma(X, \mathbb{C}) = R\mathrm{Hom}(\mathbb{C}_X, \mathbb{C}_X)$$

complex of vector spaces!

$$(H_c^{n-i}(X, \mathbb{C}))^* \simeq H^i(X, \mathbb{C})$$

classical Poincaré duality!