Triangulated and Derived Categories in Algebra and Geometry
Lecture 24

1. Poincaré - Verdier duality

Deal with reasonable topological spaces (later - complex algebraic var's)

$f : X \to Y$

$D^*(X)$ - derived category of k_X-modules, where k - nice enough ring of coeff's (comm. ring of finite global dim = = any module has a universally bounded proj. resolution)
$k = \mathbb{C}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}_p \ldots$

exact

Derived functors: $- \overset{L}{\otimes} -$, $R\mathcal{H}om(-,-)$, Rf_*, $Rf_!$, f^{-1}

__Rmk__ $f_!$ is left exact $\Rightarrow$ no chance
$f_! : k_X\text{-mod} \to k_Y\text{-mod}$ in general has a right adjoint

It turns out, $Rf_!$ has a right adjoint

$$f^! : D^+(k_Y\text{-mod}) \longrightarrow D^+(k_X\text{-mod})$$

Properties:

- $\mathrm{Hom}(Rf_! \mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, f^! \mathcal{G})$
- $f: X \to \{pt\}$, X - n-dimensional oriented mfld over $\mathbb{R}$

  $f^! k \simeq k_X[n]$
- $X \xrightarrow{f} Y \xrightarrow{g} Z$

  $(g \circ f)^! \simeq f^! \circ g^!$
- Base change:

  if the maps are nice enough

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f^! \circ Rg_* \simeq Rg'_* \circ (f')^!$$

- Again, for nice morphisms

$$Rf_* R\mathcal{H}om(\mathcal{F}, f^! \mathcal{G}) \simeq R\mathcal{H}om(Rf_! \mathcal{F}, \mathcal{G}) \quad \ldots$$

2. 6 functor formalism

Stick to complex algebraic varieties for simplicity
Homological algebra that you can do (in derived cat's)
using these 6 functors:

$\overset{L}{\otimes}$, $R\mathcal{H}om$, f^{-1}, Rf_*, $f^!$, $Rf_!$

Properties: a) adjunctions

- $\overset{L}{\otimes} \vdash R\mathcal{H}om$
- $f^{-1} \vdash Rf_*$
- $Rf_! \vdash f^{-1}$

b) f - proper $f_* = f_!$

c) fundamental distinguished triangles

$Z \overset{\iota}{\hookrightarrow} X$ - closed, $\mathcal{U} \overset{j}{\hookrightarrow} X$ - complement

- $\iota_! \iota^! \to id \to j_* j^{-1} \xrightarrow{+1}$
- $j_! j^! \to id \to \iota_* \iota^{-1} \to$

d) duality: given X, $p: X \to \{pt\}$

<u>Def</u> The dualizing complex

$$\omega_X = p^! k.$$

Define $\mathbb{D}_X = R\mathcal{H}om(-, \omega_X) \leftarrow$ dualization functor.

- $X = \{pt\} \Rightarrow \mathbb{D}_{\{pt\}}$ is the usual duality of complexes
- $\mathbb{D}^2 = id$
- $\mathbb{D} f^{-1} \simeq f^! \mathbb{D}$ $\qquad \mathbb{D} Rf_! \simeq Rf_* \mathbb{D}$

  $\mathbb{D} f^! \simeq f^{-1} \mathbb{D}$ $\qquad \mathbb{D} Rf_* \simeq Rf_! \mathbb{D}$

real dim

<u>Ex</u>: X – smooth oriented mfld $\Rightarrow \omega_X = k[d]$

$$H^i(X, \mathbb{C})^* = H^{-i} \mathbb{D} Rp_* k_X \simeq H^{-i}(Rp_! \mathbb{D} k_X) \simeq$$

$$\simeq H^{-i}(Rp_! R\mathcal{H}om(k_X, \omega_X)) \simeq H^{-i}(Rp_! R\mathcal{H}om(k_X, k_X[d])) \simeq$$

$$\simeq H^{2d-i}(Rp_! k_X) \simeq H_c^{2d-i}(X, k) = H_c^{2d-i}(X, \mathbb{C}).$$

Thing to remember: $f^!$ allows two things

1) easier computations,
2) gives you duality!

3. Back to basics

Why sheaves of modules at all? Which sheaves?

Observation: $\underbrace{H^i(X, k)}_{\text{your fav. cohomology theory}} \simeq H^i(X, \underbrace{k_X}_{\text{locally constant sheaf}}) = H^i(R\Gamma(X, k_X))$

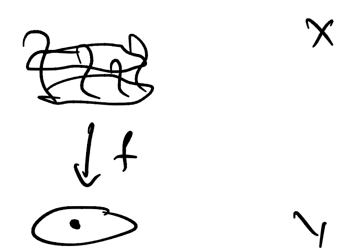
Let's consider a reasonable map

$$f : X \to Y$$

namely, a fibration:

$\forall y \in Y \ \exists\, U \ni y$ s.t.

$$\begin{tikzcd} f^{-1}(U) \arrow[dr, "f"] \arrow[d, phantom, "\cong" rotate=-90] & \\ U \times F \arrow[r, "p_U"'] & U \end{tikzcd}$$



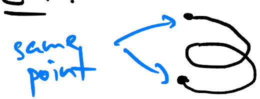
The cohomology of the fibers is "constant"?
The best you can say about

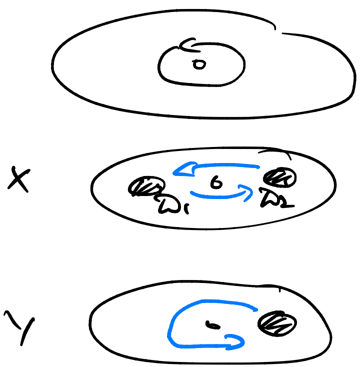
$$\mathcal{U} \longmapsto H^i(f^{-1}(\mathcal{U}), k) \qquad\qquad R^i f_* k_X$$

is locally isomorphic to a sheaf of the form $M_{\mathcal{U}}$ for some k-module M.

Ex: $X = \mathbb{C}^*$, $Y = \mathbb{C}^*$, $k = \mathbb{C}$, $f: X \to Y$

$$z \longmapsto z^2$$





There is no global isomorphism b/w $f_* k_X$ and $(k \oplus k)_Y$.

If $\mathcal{U} \subset Y$ is a small disk,

then $f_* k(\mathcal{U}) \simeq k(D_1) \oplus k(D_2)$

If I move $\mathcal{U}$ around the origin back to itself $\rightsquigarrow$ monodromy.

$$k(D_1) \oplus k(D_2) \xrightarrow{\sim} k(D_1) \oplus k(D_2)$$

$$\begin{matrix} \| & & \| \\ f_* k(\mathcal{U}) & & f_* k(\mathcal{U}) \end{matrix}$$

Any path in Y $\rightsquigarrow$ automorphism of the stalk!

Def A local system on X is a sheaf of k_X-mod that locally is isomorphic to $M_\mathcal{U}$, M - f.g. k-module: $\forall\, x \in X$ $\exists\, \mathcal{U} \ni x$ s.t. $\mathcal{L}|_\mathcal{U} \simeq M_\mathcal{U}$. $M_\mathcal{U} = p^* M$, $p: M \to \{pt\}$

Exc $\mathcal{L}oc(X)$ form an abelian subcategory in k_X-mod.

Thm If X is connected, then there is an equivalence

$$\mathcal{L}oc_k(X) \simeq \mathrm{Rep}_k(\pi_1(X, x)).$$

If you want to be fancy and get rid of those extra conditions (connected, base point)

$$\mathrm{Loc}_k(X) \simeq \mathrm{Fun}(\underbrace{\pi_1(X)}_{\text{fundamental groupoid}}, \; k\text{-mod})$$

4. Constructible sheaves

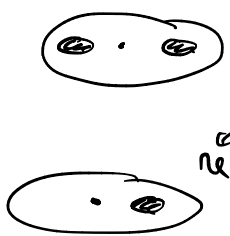
Observation: if $f : X \to Y$ is smooth (say, a topologically locally trivial fibration of varieties), then $R^i f_* k_X$ — a local system.

However, let's look at

$$f : \mathbb{C} \to \mathbb{C}.$$
$$z \mapsto z^2$$

What is $f_* k_X$?

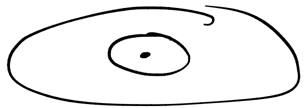
X



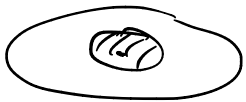
Y

small disc U

$$f_* k_X(U) \simeq k \oplus k$$



$$f_* k_x(V) \simeq k \quad \text{if } \overset{0}{\overset{\cup}{x}} \in V$$



V ← small disk

Relation: $\mathcal{U} \subset V$ are small discs, $\overset{0}{\overset{\cup}{x}} \in V$, $x \notin \mathcal{U}$

$$\mathrm{res}^V_{\mathcal{U}} : k \longrightarrow k \oplus k$$

maps k to the monodromy invariants!

$f_* k_x$ is not a local system! How to fix it?

Enlarge our category.

Def A stratification of X is a decomposition

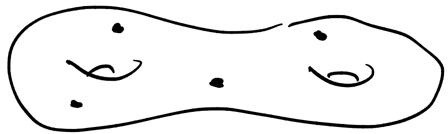
strata

$X = \bigsqcup X_\lambda$ s.t. X_λ is locally closed & $\forall \lambda$

$\overline{X_\lambda}$ is a union of other strata.

Example: C - curve (2-dim surface)

$C = \mathcal{U} \sqcup \{p_1\} \sqcup \{p_2\} \sqcup \ldots \sqcup \{p_k\}$



Def A constructible sheaf is a sheaf of k_X-modules such that $\exists$ a stratification: it's restriction to every stratum is a local system.

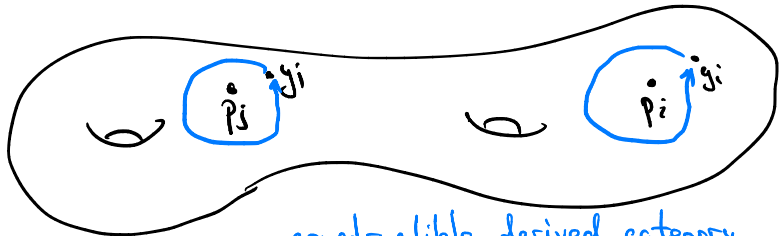
$f_* k$ — constructible for $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$.

Example $C = \mathcal{U} \sqcup \{p_1\} \sqcup \ldots \sqcup \{p_k\}$, $\mathcal{F}$ is constructible w/r to such a stratification.

$\mathcal{F}$ is fully defined by 1) $\mathcal{F}|_{\mathcal{U}} = \mathcal{L} \leftarrow$ local system

2) $\forall i$ $M_i \in k\text{-mod}$ — stalk at p_i

3) $\forall i$ $M_i \to \mathcal{L}_{y_i}^{inv} \leftarrow$ monodromy invariants



y_i is a point close to p_i

constructible derived category

<u>Def</u> $D^b_c(X)$ is the full triangulated subcategory in $D^b(X)$ of those complexes whose cohomology (coh. of complexes) is constructible.

<u>Comment</u> We do not specify the stratification. You can always refine stratifications.

<u>Thm</u> (meta-theorem)

The 6 functor formalism works for constructible derived categories.

It turns out, $D^b_c(X)$ has a very interesting t-structure.

5. Perverse sheaves

Under some conditions t-structures can be glued
Recollement (BBD)

these will correspond to $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$

Given: Δ categories D_Z, D, D_U

& $D_Z \xrightarrow{i_*} D \xrightarrow{j^{-1}} D_U$

From now on, $j^{-1} = j^*$...

s.t. 1) i_* has both a left and right adjoint i^{-1}, $i^!$ (think/put $i_! = i_*$)

j^{-1} has left and right adjoints $j_!$ and j_* (think/put $j^! = j^*$)

2) $j^* i_* = 0 \Rightarrow i^* j_! = 0$ & $i^! j_*$

3) dist. triangles

$j_! j^! \to \mathrm{id} \to i_* i^* \xrightarrow{+1}$

$i_! i^! \to \mathrm{id} \to j_* j^* \xrightarrow{+1}$

the map - adj units/counits

4) $\iota_*, j_!, j_*$ are fully faithful

<u>Thm</u> (BBD) Given t-structures on $\mathcal{D}_Z$ & $\mathcal{D}_U$,

The full subcategories

$$\mathcal{D}^{\leq 0} = \{ X \in \mathcal{D} \mid \iota^* X \in \mathcal{D}_Z^{\leq 0} \ \& \ j^* X \in \mathcal{D}_U^{\leq 0} \}$$

$$\mathcal{D}^{\geq 0} = \{ X \in \mathcal{D} \mid \iota^! X \in \mathcal{D}_Z^{\geq 0} \ \& \ j^! X \in \mathcal{D}_U^{\geq 0} \}$$

define a t-structure on $\mathcal{D}$.

<u>Example</u> $X = \mathbb{P}^1_{\mathbb{C}} \simeq S^2$

$S^2 = \mathbb{C} \cup \{p\} \leftarrow \Lambda$ (our stratification)

$$D^b_{l.c.}(pt) \hookrightarrow D^b_{\Lambda}(X) \longrightarrow D^b_{l.c.}(\mathbb{C})$$

constructible w/r to Λ

$\mathbb{C}$ is contractible

$\Rightarrow$ locally constant

$=$ constant!

6 functor formalism implies that we can use recollement

What if we glue

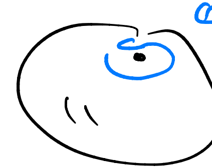
$$\mathrm{Loc}(p) \quad \text{with} \quad \mathrm{Loc}(\mathbb{C})[d] \ ?$$

<u>Answer</u>:

- $d=0 \Rightarrow$ standard t-structure

  an object in the heart is equivalent to (V_0, V_1) - k-modules and

  $$V_0 \longrightarrow V_1$$



the loop is trivial $\Rightarrow$ $\Rightarrow$ no taking invariants

- $d=2$

  the object in the heart is given by

  $$V_0 \longleftarrow V_1$$

- $d<0$ or $d>0$

  the heart is generated by two simple objects!

• $d=1$ (example of perverse sheaves)
object $\rightsquigarrow$

$$V_0 \underset{f}{\overset{e}{\rightleftarrows}} V_1 \qquad ef=0.$$

Perverse sheaves in general. Assume Λ-stratification is nice enough (Whitney).

$$X = \bigsqcup_{\lambda\in\Lambda} X_\lambda, \qquad \iota_\lambda : X_\lambda \hookrightarrow X$$

dimension of X_λ

$$D_\lambda^{\leq 0} = \{ X \in D^b_{lc}(X_\lambda) \mid H^i(X) = 0 \text{ for } i > -d_\lambda \}$$
$$D_\lambda^{\geq 0} = \{ X \in D^b_{lc}(X_\lambda) \mid H^i(X) = 0 \text{ for } i < -d_\lambda \}$$

Heart — $Loc(X_\lambda)[d_\lambda]$

$${}^pD^{\leq 0} = \{ X \in D^b_\Lambda(X) \mid \iota_\lambda^* X \in D_\lambda^{\leq 0} \quad \forall \lambda \in \Lambda \}$$
$${}^pD^{\geq 0} = \{ X \in D^b_\Lambda(X) \mid \iota_\lambda^! X \in D_\lambda^{\geq} \quad \forall \lambda \in \Lambda \}$$

The heart is called the category of perverse sheaves w/r to Δ.

In general — pass to $\varinjlim$ over all stratifications.

1) We got a category $\mathrm{Perv}(X)$ – new abelian category associated to X! In particular, an invariant

2) Riemann–Hilbert correspondence:

$$D^b_h(\mathcal{D}_X\text{-mod}) \xrightarrow{\sim} D^b_c(X)$$

$\mathcal{D}$-modules, regular holonomic cohom (differential equations) | constructible sheaves (topology)

standard t-structure $\longleftrightarrow$ perverse t-structure!

regular hol. modules $\simeq$ perverse sheaves