

Triangulated and Derived Categories in Algebra and Geometry

Lecture 3

0. Adjoint Functors Revised

Recall (F, G) , $F \dashv G$ $F: \mathcal{A} \rightleftarrows \mathcal{B}: G$ if

$$\mathrm{Hom}_{\mathcal{B}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{A}}(-, G(-))$$

An isom of bifunctors $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \rightarrow \mathrm{Sets}$.

Examples • $\mathrm{Ab} \subset \mathrm{Grp} \leftarrow$ full embedding

Left adjoint:

$$\mathrm{Hom}_{\mathrm{Ab}}(\overset{\uparrow}{?}, A) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, A)$$

quotient by the commutator

• Grothendieck group construction

Consider $\text{AbMon} \supset \text{Ab}$, the full subcategory of abelian groups inside abelian monoids.

Left adjoint functor $M \rightsquigarrow K(A)$, where $K(A)$ is the quotient of

1) consider formal expressions / pairs $(m, n) \in M \times M$, think of (m, n) as a formal difference " $m-n$ ".

It inherits an operation: $(m, n) + (m', n') = (m+m', n+n')$

2) Want " $m-n$ " = " $(m+k) - (n+k)$ "

Consider the equivalence relation

$$(m, n) \sim (m', n') \iff \exists k \in M \text{ s.t.} \\ m+n'+k = m'+n+k.$$

$$"m-n" = "m'-n'" \iff m+n' = m'+n$$

since cancellation might not hold, need a less strong condition

$$\underline{\text{Ex}} \quad K(\mathcal{A}) \simeq \mathbb{Z}$$

Exc Check that K is a functor $\text{AbMon} \rightarrow \text{Ab}$,
it's left adjoint to $\text{Ab} \hookrightarrow \text{AbMon}$.

Unit & Counit

$\text{Hom}_{\mathcal{B}}(F(x), Y) \simeq \text{Hom}_{\mathcal{A}}(X, F(Y))$, plug in $Y = F(x)$
(consider the functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \xrightarrow{\text{Id} \times F} \mathcal{A}^{\text{op}} \times \mathcal{B}$).

$$\text{Hom}_{\mathcal{B}}(F(x), F(x)) \xrightarrow{\eta} \text{Hom}_{\mathcal{A}}(X, (G \circ F)(x)) \quad \text{Id}_{\mathcal{A}} \rightarrow G \circ F$$

$\overset{\text{Id}_{F(x)}}{\uparrow} \quad \xrightarrow{\quad \quad \quad} \quad \xrightarrow{\quad \quad \quad} \quad \xrightarrow{\text{Ex}}$

Similarly, you obtain a nat transformation

$$\gamma: F \circ G \rightarrow \text{Id}_{\mathcal{B}}$$

$$\underline{\text{Lm}} \quad \begin{array}{ccccc} F & \xrightarrow{F \circ \varepsilon} & F \circ G \circ F & \xrightarrow{\gamma \circ F} & F & \leftarrow \text{this composition is the identity} \\ G & \xrightarrow{\varepsilon \circ G} & G \circ F \circ G & \xrightarrow{G \circ \gamma} & G & \leftarrow \text{morphism} \end{array}$$

Exc Prove it!

Hint There is a diagram (functoriality!)

$$\phi: F(x) \rightarrow Y$$

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}}(F(x), F(x)) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{A}}(x, (G \circ F)(x)) \\ \phi \downarrow & & \downarrow G(\phi) \\ \text{Hom}_{\mathcal{B}}(F(x), Y) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{A}}(x, G(Y)) \end{array}$$

← adjunction

1. Monoidal categories

Def A monoidal category \mathcal{M} is a category \mathcal{M} , a bifunctor $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, an object $I \in \mathcal{M}$ called the unit object, and three natural isomorphisms:

$$1) \quad - \otimes (- \otimes -) \xrightarrow{\sim} (- \otimes -) \otimes -,$$

$$2) I \otimes - \xrightarrow{\sim} Id,$$

$$3) - \otimes I \xrightarrow{\sim} Id.$$

Subject to some properties:

$$\begin{array}{ccc}
 a) & A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\sim} (A \otimes B) \otimes (C \otimes D) \xrightarrow{\sim} ((A \otimes B) \otimes C) \otimes D \\
 & \downarrow \cong & \hookrightarrow \\
 & A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\sim} (A \otimes (B \otimes C)) \otimes D \uparrow \cong
 \end{array}$$

$$\begin{array}{ccc}
 b) & & A \otimes (I \otimes B) \xrightarrow{\sim} (A \otimes I) \otimes B \\
 & & \searrow \cong \quad \swarrow \cong \\
 & & A \otimes B
 \end{array}$$

Main examples

- Sets, \otimes is the Cartesian product, $\{x, y\}$.
- Cat, \otimes is the product of cat's, \circ .

- $A\text{-Mod}$, \otimes - tensor product
In particular, $\text{Ab} \cong \mathbb{Z}\text{-Mod}$ and $\text{Vect-}k$.

2. Enriched categories

Def Given a monoidal category \mathcal{M} , an enriched category \mathcal{C} (enriched in \mathcal{M}) consists of a set of objects $\text{Ob } \mathcal{C}$, $\forall x, y \in \text{Ob } \mathcal{C}$ an object $\text{Hom}_{\mathcal{C}}(x, y) \in \mathcal{M}$, $\forall x$ a morphism $\text{id}_x : I \rightarrow \text{Hom}_{\mathcal{C}}(x, x)$, $\forall x, y, z \in \mathcal{C}$

$0 : \text{Hom}_{\mathcal{C}}(y, z) \otimes \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$, such that...

Rule Associativity of 0 is embedded in associativity of \otimes .

Rule If you want some set of morphisms $x \rightarrow y$, look at arrows $I \rightarrow \text{Hom}_{\mathcal{C}}(x, y)$ in \mathcal{M} .

- Ex
- Categories are enriched categories over $(\text{Sets}, \times, \text{4x4})$.
 - 2-categories are enriched over $(\text{Cat}, \times, \cdot)$.
 - Preadditive categories are enriched categories over (Ab, \otimes) .

3. Preadditive categories

Def A preadditive category is a category \mathcal{A} s.t. every $\text{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group, the composition is bilinear

$$\text{Hom}_{\mathcal{A}}(Y, Z) \times \text{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{A}}(X, Z)$$

$$\begin{array}{ccc} & & \nearrow \\ \searrow & & \\ \text{Hom}(Y, Z) \otimes_{\mathbb{Z}} \text{Hom}_{\mathcal{A}}(X, Y) & & \end{array} .$$

Ln Let \mathcal{A} be preadditive. The following are equivalent for an object $X \in \mathcal{A}$.

- (1) X is initial,
- (2) X is final,
- (3) $\text{id}_X = 0$ in $\mathcal{A}(X, X)$.

Def An object which is both initial and final is called a zero object, denoted by 0 .

Pf (of the lemma)

(1), (2) \Rightarrow (3) is trivial since in both cases

$\text{Hom}(X, X)$ consists of a single element.

Assume $\text{id}_X = 0$ in $\text{Hom}(X, X)$

$\forall Y$ consider

$$\text{Hom}(X, X) \times \text{Hom}(Y, X) \longrightarrow \text{Hom}(Y, X)$$

then $\forall f \in \text{Hom}(Y, X)$

$$f = \text{id}_X \circ f = 0 \circ f = 0 \quad \leftarrow \text{bilinearity}$$

(χ is a bilinear form, then $\chi(0, v) = \chi(0+0, v) = 2\chi(0, v) \Rightarrow \Rightarrow \chi(0, v) = 0.$)

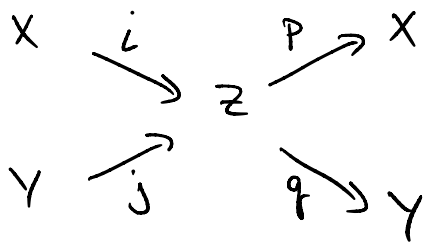
Thus, $f=0 \Rightarrow \text{Hom}(Y, X) = \{0\} \Rightarrow X$ is final.

Similarly, (3) \Rightarrow (1). □

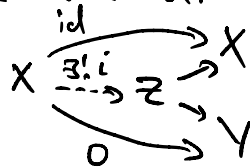
Cor If a preadditive category has an initial / final object, then it has a zero object.

Lim Let \mathcal{A} be preadditive, $X, Y \in \mathcal{A}$. If the product $X \times Y$ exists, so does the coproduct. Moreover, $X \sqcup Y \simeq X \times Y$.

Proof Let $Z = X \times Y$. Let's show that Z is a coproduct.

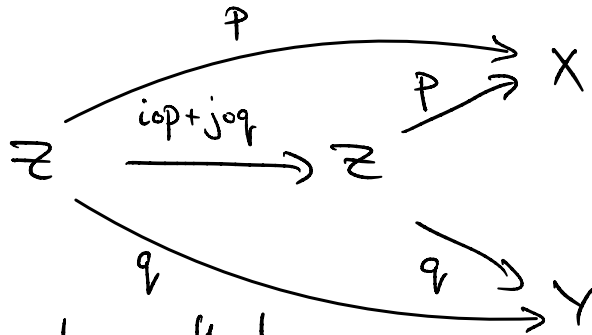


Let's define i and j by the the UP of Z



These maps satisfy: $p \circ i = \text{id}_X$, $q \circ i = 0$,
 $q \circ j = \text{id}_Y$, $p \circ j = 0$.

Claim $i \circ p + j \circ q = \text{id}_Z$

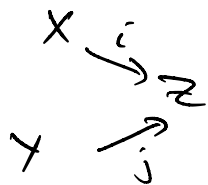


Enough to show that the compositions with $p \neq q$ are $p \neq q$ again.

$$p \circ (i \circ p + j \circ q) = \underbrace{(p \circ i) \circ p}_{\text{id}_X} + \underbrace{(p \circ j) \circ q}_0 = \text{id}_X \circ p + 0 \circ q = p.$$

Same for q .

Check that



satisfies the UP of $X \sqcup Y$.

□

Remark 1) We could have done the same proof for coproduct \Rightarrow product!

2) What we actually proved is TFAE

(1) $X \times Y$ exists

(2) $X \sqcup Y$ exists

(3) \exists Z $\begin{array}{ccc} X & \xrightarrow{L} & Z \\ Y & \xrightarrow{j} & Z \end{array}$ $\begin{array}{ccc} & & \xrightarrow{p} X \\ & & \xrightarrow{q} Y \end{array}$ $\begin{array}{l} p \circ i = 1, \quad p \circ j = 0 \\ q \circ i = 0, \quad q \circ j = 1 \\ i \circ p + j \circ q = 1 \end{array}$

Def Let \mathcal{A}, \mathcal{B} be preadditive. A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is additive if the induced maps on Hom's are homomorphisms of abelian groups: $\forall f, g: X \rightarrow Y$
 $F(f+g) = F(f) + F(g)$.

Cor An additive functor preserves products, coproducts and zero objects.

Pf $\cdot Z$ is zero $\Leftrightarrow \text{id}_Z = 0$. If $\text{id}_Z = 0$, then
 $F(\text{id}_Z) = \text{id}_{F(Z)} = F(0) = 0$.

\cdot for products use this diagram:
$$\begin{array}{ccccc} X & \rightarrow & Z & \rightarrow & X \\ Y & \rightarrow & & \rightarrow & Y \end{array} \quad \square$$

Def An additive category is a preadditive category in which all finite products exist (including the empty = final obj).

Notation In an additive category $X \times Y$ is denoted by $X \oplus Y$ and is called the direct sum.

Main examples

- Ab - abelian groups
- A - ring, $A\text{-Mod}$ and $\text{Mod-}A$
left modules right modules
- look at free or f.g. free A -modules

- Let \mathcal{A} be an additive category, the \mathcal{A}^{op} is additive!
- Given a category I & an additive category \mathcal{A} , the category of functors $\text{Fun}(I, \mathcal{A})$ is additive!

Abelian groups of natural transformations (pointwise addition).

Direct sums are object-wise: $F, G: I \rightarrow \mathcal{A}$,

$$(F \oplus G)(x) = F(x) \oplus G(x).$$

Exc Work out the details.

- X - topological space $\rightsquigarrow \text{Op}(X)$
 objects are $U \subset X$ - open, $U \subseteq V \iff U \subseteq V$
 $\text{PSh}_{\mathcal{A}}(X) = \text{Fun}(\text{Op}(X)^{\text{op}}, \mathcal{A})$
 presheaves on X with values in \mathcal{A} .

Def Let \mathcal{A} be additive, $X \in \mathcal{A}$. The diagonal
 $\Delta: X \rightarrow X \oplus X$ is induced by $X \begin{array}{c} \xrightarrow{1} X \\ \searrow 1 \\ X \end{array}$.

$\nabla: X \oplus X \rightarrow X$ is induced by $\begin{array}{ccc} X & \xrightarrow{f} & X \\ X & \xrightarrow{g} & X \end{array}$.

\swarrow codiagonal

Exc Show that the following commutes $\forall f, g: X \rightarrow Y$

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta_X} & X \oplus X & \xrightarrow{f+g} & Y \oplus Y & \xrightarrow{\nabla_Y} & Y \\ & & & \searrow^{f+g} & & & \nearrow \\ & & & & & & \end{array}$$

Exc Using \nearrow show that a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is additive \Leftrightarrow it preserves products:
 $F(X \oplus Y) \cong F(X) \oplus F(Y)$.

Exc Redefine additive categories as categories with zero objects and direct sums ($\Pi = \cup$). Reconstruct the abelian group structure on Hom spaces, show bilinearity.

E.g. $\begin{array}{ccc} X & \xrightarrow{0} & Y \\ & \searrow & \nearrow \\ & & 0 \end{array}$

Def Let A be additive, $f: X \rightarrow Y$.

The kernel (if exists) of f is the equalizer $\text{Eq}(0, f)$.

The cokernel (if exists) of f is the coequalizer $\text{Coeq}(0, f)$.

The coimage is the cokernel of the kernel.

The image is the kernel of the cokernel.

Exc Show that $\ker(f)$ is what you are more used to:

$$\begin{array}{ccc} \ker(f) & \xrightarrow{\varepsilon} & X \xrightarrow{f} Y & f \circ \varepsilon = 0 \\ \exists! h \uparrow & \nearrow g & & \forall g \text{ st. } f \circ g = 0 \\ \mathbb{Z} & & & \exists! h \text{ st. } \varepsilon \circ h = g. \end{array}$$

Next time Abelian categories.

Before Try and construct $\text{Coim}(f) \rightarrow \text{Im}(f)$
if everything exists.