Triangulated and Derived Categories in Algebra and Geometry
Lecture 3

0. Adjoint Functors Revised

Recall (F, G), $F \dashv G$ $\quad F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ if

$$\mathrm{Hom}_{\mathcal{B}}(F(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(-, G(-))$$

An isom of bifunctors $\mathcal{A}^{op} \times \mathcal{B} \to \mathrm{Sets}$.

Examples · $\mathrm{Ab} \subset \mathrm{Grp}$ ← full embedding

Left adjoint :

$$\mathrm{Hom}_{\mathrm{Ab}}(\,?\,, A) \simeq \mathrm{Hom}_{\mathrm{Grp}}(G, A)$$

quotient by the commutator

· Grothendieck group construction

Consider $\mathrm{AbMon} \supset \mathrm{Ab}$, the full subcategory of abelian groups inside abelian monoids.

Left adjoint functor $M \rightsquigarrow K(A)$, where $K(A)$ is the quotient of

1) consider formal expressions / pairs $(m,n) \in M \times M$, think of (m,n) as a formal difference "$m-n$".

   It inherits an operation : $(m,n) + (m',n') = (m+m', n+n')$

2) Want "$m-n$" = "$(m+k)-(n+k)$"

   Consider the equivalence relation

$$(m,n) \sim (m',n') \iff \exists\, k \in M \text{ s.t. } m+n'+k = m'+n+k.$$

"$m-n$" = "$m'-n'$" $\iff$ $m+n' = m'+n$

since cancellation might not hold, need a less strong condition

Ex $K(\mathbb{N}) \simeq \mathbb{Z}$

Exc Check that K is a functor $\mathrm{AbMon} \to \mathrm{Ab}$, it's left adjoint to $\mathrm{Ab} \hookrightarrow \mathrm{AbMon}$.

Unit & Counit

$\mathrm{Hom}_{\mathcal{B}}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, F(Y))$, plug in $Y = F(X)$

(consider the functor $\mathcal{A}^{op} \times \mathcal{A} \xrightarrow{\mathrm{Id} \times F} \mathcal{A}^{op} \times \mathcal{B}$).

$$\mathrm{Id}_{F(X)} \longmapsto \varepsilon_X$$

$$\mathrm{Hom}_{\mathcal{B}}(F(X), F(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X, (G \circ F)(X)) \qquad \mathrm{Id}_{\mathcal{A}} \to G \circ F$$

Similarly, you obtain a nat transformation $\eta : F \circ G \to \mathrm{Id}_{\mathcal{B}}$.

Lm $F \xrightarrow{F\varepsilon} F \circ G \circ F \xrightarrow{\eta F} F$ ⟵ this composition is the identity morphism

$G \xrightarrow{\varepsilon G} G \circ F \circ G \xrightarrow{G\eta} G$ ⟵

__Exc__ Prove it!

__Hint__ There is a diagram (functoriality!)

$\phi : F(X) \to Y$

adjunction

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(F(X), F(X)) \arrow[r, "\sim"] \arrow[d, "\phi"'] & \mathrm{Hom}_{\mathcal{A}}(X, (G \circ F)(X)) \arrow[d, "G(\phi)"] \\
\mathrm{Hom}_{\mathcal{B}}(F(X), Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{A}}(X, G(Y))
\end{tikzcd}$$

1. Monoidal categories

__Def__ A monoidal category $\mathcal{M}$ is a category $\mathcal{M}$, a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an object $I \in \mathcal{M}$ called the unit object, and three natural isomorphisms:

1) $- \otimes (- \otimes -) \xrightarrow{\sim} (- \otimes -) \otimes -$,

2) $I \otimes - \xrightarrow{\sim} \mathrm{Id}$,
3) $- \otimes I \xrightarrow{\sim} \mathrm{Id}$.

Subject to some properties:

a)
$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "\sim"] \arrow[d, "\wr"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\sim"] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\wr"']
\end{tikzcd}$$

b)
$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\sim"] \arrow[dr, "\wr"'] & & (A \otimes I) \otimes B \arrow[dl, "\wr"] \\
& A \otimes B &
\end{tikzcd}$$

<u>Main examples</u>
- Sets, $\otimes$ is the Cartesian product, $\{*\}$.
- Cat, $\otimes$ is the product of cat's, $\bullet \circlearrowleft 1$.

- A-Mod, $\otimes$ – tensor product
  In particular, $Ab \simeq \mathbb{Z}$-Mod and Vect-k.

2. Enriched categories

<u>Def</u> Given a monoidal category $\mathcal{M}$, an <u>enriched category</u> $\mathcal{C}$ (enriched in $\mathcal{M}$) consists of a set of objects $Ob\,\mathcal{C}$, $\forall\, X, Y \in Ob\,\mathcal{C}$ an <u>object</u> $Hom_{\mathcal{C}}(X,Y) \in \mathcal{M}$, $\forall\, X$ a morphism $id_X : I \longrightarrow Hom_{\mathcal{C}}(X,X)$, $\forall\, X, Y, Z \in \mathcal{C}$

$\circ : Hom_{\mathcal{C}}(Y,Z) \otimes Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{C}}(X,Z)$, such that...

<u>Rmk</u> Associativity of $\circ$ is embedded in associativity of $\otimes$.

<u>Rmk</u> If you want some set of morphisms $X \to Y$, look at arrows $I \to Hom_{\mathcal{C}}(X,Y)$ in $\mathcal{M}$.

Ex
- Categories are enriched categories over $(\mathrm{Sets}, \times, \{*\})$.
- 2-categories are enriched over $(\mathrm{Cat}, \times, \bullet)$.
- Preadditive categories are enriched categories over $(\mathrm{Ab}, \otimes)$.

3. Preadditive categories

Def A preadditive category is a category $\mathcal{A}$ s.t. every $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group, the composition is bilinear

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(Y,Z) \times \mathrm{Hom}_{\mathcal{A}}(X,Y) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\mathcal{A}}(X,Z) \\
& \mathrm{Hom}(Y,Z) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X,Y) \arrow[ur] &
\end{tikzcd}$$

.

Lm Let $\mathcal{A}$ be preadditive. The following are equivalent for an object $X \in \mathcal{A}$.

(1) X is initial,
(2) X is final,
(3) $\mathrm{id}_X = 0$ in $\mathcal{A}(X,X)$.

Def An object which is both initial and final is called a zero object, denoted by 0.

Pf (of the lemma)

(1), (2) $\Rightarrow$ (3) is trivial since in both cases $\mathrm{Hom}(X,X)$ consists of a single element.

Assume $\mathrm{id}_X = 0$ in $\mathrm{Hom}(X,X)$

$\forall\, Y$ consider

$$\mathrm{Hom}(X,X) \times \mathrm{Hom}(Y,X) \longrightarrow \mathrm{Hom}(Y,X)$$

then $\forall\, f \in \mathrm{Hom}(Y,X)$

$$f = \mathrm{id}_X \circ f = 0 \circ f = 0$$

bilinearity

(χ is a bilinear form, then $\chi(0,v) = \chi(0+0,v) = 2\chi(0,v) \Rightarrow$
$\Rightarrow \chi(0,v) = 0$.)

Thus, $f=0 \Rightarrow \mathrm{Hom}(Y,X) = \{0\} \Rightarrow X$ is final.

Similarly, $(3) \Rightarrow (1)$. $\square$

Cor If a preadditive category has an initial / final object, then it has a zero object.

Lm Let $\mathcal{A}$ be preadditive, $X, Y \in \mathcal{A}$. If the product $X \times Y$ exists, so does the coproduct. Moreover, $X \sqcup Y \simeq X \times Y$.

Proof Let $Z = X \times Y$. Let's show that Z is a coproduct.

$$\begin{tikzcd}
X \arrow[dr, "i"] & & X \\
& Z \arrow[ur, "p"] \arrow[dr, "q"'] & \\
Y \arrow[ur, "j"'] & & Y
\end{tikzcd}$$

Let's define i and j by the the UP of Z

$$\begin{tikzcd}
& & X \\
X \arrow[urr, bend left, "\mathrm{id}"] \arrow[r, dashed, "\exists! i"] \arrow[drr, bend right, "0"'] & Z \arrow[ur] \arrow[dr] & \\
& & Y
\end{tikzcd}$$

These maps satisfy: $p \circ i = \mathrm{id}_X$, $q \circ i = 0$,
$q \circ j = \mathrm{id}_Y$, $p \circ j = 0$.

<u>Claim</u> $i \circ p + j \circ q = \mathrm{id}_Z$

$$
\begin{tikzcd}
Z \arrow[r, "i\circ p + j \circ q"] \arrow[rr, bend left, "p"] \arrow[rrd, bend right, "q"'] & Z \arrow[r, "p"] \arrow[rd, "q"'] & X \\
& & Y
\end{tikzcd}
$$

Enough to show that the compositions with p & q are p & q again.

$$p \circ (i \circ p + j \circ q) = \underbrace{(p \circ i)}_{\mathrm{id}_X} \circ p + \underbrace{(p \circ j)}_{0} \circ q = \mathrm{id}_X \circ p + 0 \circ q = p.$$

Same for q.

Check that
$$
\begin{tikzcd}
X \arrow[rd, "i"] & \\
& Z \\
Y \arrow[ru, "j"'] &
\end{tikzcd}
$$
satisfies the UP of $X \sqcup Y$. $\square$

Rmk 1) We could have done the same proof for coproduct $\Rightarrow$ product!

2) What we actually proved is TFAE

(1) $X \times Y$ exists

(2) $X \sqcup Y$ exists

(3) $\exists$
$$\begin{tikzcd} X \arrow[dr, "i"] & & X \\ & Z \arrow[ur, "p"] \arrow[dr, "q"'] & \\ Y \arrow[ur, "j"'] & & Y \end{tikzcd}$$
$p \circ i = 1, \quad p \circ j = 0$
$q \circ i = 0 \quad q \circ j = 1$
$i \circ p + j \circ q = 1$

Def Let $\mathcal{A}, \mathcal{B}$ be preadditive. A functor $F : \mathcal{A} \to \mathcal{B}$ is additive if the induced maps on Hom's are homomorphisms of abelian groups: $\forall f, g : X \to Y$
$F(f+g) = F(f) + F(g)$.

Cor An additive functor preserves products, coproducts and zero objects.

Pf • Z is zero $\Leftrightarrow$ $id_Z = 0$. If $id_Z = 0$, then
$F(id_Z) = id_{F(Z)} = F(0) = 0$.

• for products use this diagram:
$$\begin{tikzcd} X \arrow[dr] & & X \\ & Z \arrow[ur] \arrow[dr] & \\ Y \arrow[ur] & & Y \end{tikzcd}$$
$\square$

Def An additive category is a preadditive category in which all finite products exist (including the empty = final obj).

Notation In an additive category $X \times Y$ is denoted by $X \oplus Y$ and is called the direct sum.

Main examples

• $\mathcal{A}b$ – abelian groups
• A – ring, A-Mod and Mod-A

left modules — right modules

• look at free or f.g. free A-modules

- Let $\mathcal{A}$ be an additive category, the $\mathcal{A}^{op}$ is additive!
- Given a category I & an additive category $\mathcal{A}$, the category of functors $\mathrm{Fun}(I, \mathcal{A})$ is additive!

  Abelian groups of natural transformations (pointwise addition).
  Direct sums are object-wise: $F, G : I \to \mathcal{A}$,
  $(F \oplus G)(x) = F(x) \oplus G(x)$.

<u>Exc</u> Work out the details.

- X – topological space $\rightsquigarrow$ $\mathrm{Op}(X)$
  objects are $U \subset X$ – open, $U \leq V \iff U \subseteq V$
  $PSh_{\mathcal{A}}(X) = \mathrm{Fun}(\mathrm{Op}(X)^{op}, \mathcal{A})$
  presheaves on X with values in $\mathcal{A}$.

<u>Def</u> Let $\mathcal{A}$ be additive, $X \in \mathcal{A}$. The <u>diagonal</u>

$\Delta : X \to X \oplus X$ is induced by

$$\begin{tikzcd} & X \\ X \arrow[ur, "1"] \arrow[dr, "1"'] & \\ & X \end{tikzcd}.$$

$\nabla : X \oplus X \to X$ is induced by
$$\begin{tikzcd}[row sep=small] X \arrow[dr, "1"] & \\ & X \\ X \arrow[ur, "1"'] & \end{tikzcd}.$$

codiagonal

<u>Exc</u> Show that the following commutes $\forall f, g : X \to Y$

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[rrr, bend right, "f+g"'] & X \oplus X \arrow[r, "f \times g"] & Y \oplus Y \arrow[r, "\nabla_Y"] & Y \end{tikzcd}.$$

<u>Exc</u> Using show that a functor $F : \mathcal{A} \to \mathcal{B}$ is additive $\iff$ it preserves products:
$$F(X \oplus Y) \simeq F(X) \oplus F(Y).$$

<u>Exc</u> Redefine additive categories as categories with zero objects and direct sums $(\Pi = \sqcup)$.
Reconstruct the abelian group structure on Hom spaces, show bilinearity.

E.g.
$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr] & & Y \\ & 0 \arrow[ur] & \end{tikzcd}.$$

Def Let $\mathcal{A}$ be additive, $f: X \to Y$.

The kernel (if exists) of f is the equalizer $Eq(0,f)$.
The cokernel (if exists) of f is the coequalizer $Coeq(0,f)$.
The coimage is the cokernel of the kernel.
The image is the kernel of the cokernel.

Exc Show that $Ker(f)$ is what you are more used to:

$$\begin{tikzcd} Ker(f) \arrow[r, "\varepsilon"] & X \arrow[r, "f"] & Y \\ Z \arrow[u, dashed, "\exists! h"] \arrow[ur, "g"'] & & \end{tikzcd}$$

$f \circ \varepsilon = 0$

$\forall g$ s.t. $f \circ g = 0$
$\exists!\, h$ s.t. $\varepsilon \circ h = g$.

Next time Abelian categories.

Before Try and construct $Coim(f) \to Im(f)$ if everything exists.