Triangulated and Derived Categories in Geometry and Algebra
Lecture 4

0. Adjoint functors & limits

We define limits of diagrams $\mathcal{D} \in \mathrm{Fun}(I, \mathcal{C})$.
Inside $\mathrm{Fun}(I, \mathcal{C})$ there are constant functors:
$\forall\, X \in \mathcal{C}$
$\mathrm{Const}(X) : I \to \mathcal{C}$ takes any $i \in I$ to X, all morphisms to id_X.
Get a functor $\mathcal{C} \xrightarrow{\mathrm{Const}} \mathrm{Fun}(I, \mathcal{C})$, fully faithful (if $I \neq \emptyset$). check

Rmk $\mathcal{D}$ has a colimit $\iff$ the constant diagram functor of X, $\mathrm{Const}(X)$, is initial in the category of arrows $\mathcal{D} \to \mathrm{Const}(Y)$.

Rmk Assume all $\mathcal{D} \in \mathrm{Fun}(I, \mathcal{C})$ have a colimit. Then there is an isomorphism of functors

$$\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(D, \mathrm{Const}(-)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(\varinjlim D, -)$$

In other words, this functor is representable.

Follows from Problem 8 that $\varinjlim$ is left adjoint to Const. Similarly, $\mathrm{Const} \dashv \varprojlim$.

<u>Prop</u> Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be adjoint, $F \dashv G$. Assume $D : I \to \mathcal{A}$ has a colimit. Then $F(\varinjlim D)$ is the colimit of $F \circ D : I \to \mathcal{B}$. Similarly, G preserves limits: given $D : I \to \mathcal{B}$, if $\varprojlim D$ exists, then $G(\varprojlim D)$ is $\varprojlim G \circ D$.

<u>Proof</u> $\forall Y \in \mathcal{B}$ $\quad \mathrm{Hom}_{\mathcal{B}}(F(\varinjlim X_i), Y) \simeq \mathrm{Hom}_{\mathcal{A}}(\varinjlim X_i, G(Y)) \simeq$

$$\simeq \varprojlim \mathrm{Hom}_{\mathcal{A}}(X_i, G(Y)) \simeq \varprojlim \mathrm{Hom}_{\mathcal{B}}(F(X_i), Y)$$

functor which $\varinjlim F \circ D$ should represent $\quad \square$

<u>Exc</u> Redo using the UP.

Immediate example. Consider a commutative ring A, then $\forall\, M \in A\text{-Mod}$ $\quad - \otimes_A M \dashv \mathrm{Hom}_A(M, -)$.

Conclude that $- \otimes_A M$ commutes with direct sums!

<u>Rmk</u> The proposition gives you a way to prove that a functor does not admit a left/right adjoint.

1. Additive cat's & functors

<u>Def</u> $\mathcal{A}, \mathcal{B}$ - additive, then $F: \mathcal{A} \to \mathcal{B}$ is additive if $\forall\, X, Y \in \mathcal{A}$ $\quad F: \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$ is a group homomorphism $\Longleftrightarrow$ F preserves finite products (coproducts).

<u>Recall</u> $\forall\, f, g : X \to Y$

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[rrr, bend right, "f+g"'] & X \oplus Y \arrow[r, "f \times g"] & X \oplus Y \arrow[r, "\nabla"] & Y \end{tikzcd}$$

<u>Observation</u> $\forall X, Y \in \mathcal{A}$ - additive $\mathrm{Hom}_{\mathcal{A}}(Y, X)$ is an abelian group!

$$\begin{tikzcd}
h_{_} : \mathcal{A} \arrow[rr, hook] \arrow[dr] & & \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Sets}) \\
& \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{Ab}) \arrow[ur] &
\end{tikzcd}$$

One should think of h_X as functors $\mathcal{A}^{op} \to \mathcal{Ab}$.
Same for $h^X : \mathcal{A} \to \mathcal{Ab}$.

<u>Lm</u> Assume $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ are adjoint b/w additive. Then both F & G are additive!

<u>Pf</u> $F \dashv G \Rightarrow F$ preserves colimits $\Rightarrow F(0) = 0$,
$F(X \oplus Y) = F(X) \oplus F(Y)$. Same for G since
G preserves limits. $\square$

Rmk If $\mathcal{A}$ is additive, then $\mathrm{Fun}(I, \mathcal{A})$ is also additive. $0: F(i) = 0 \ \forall i$

$$(F \oplus G)(i) = F(i) \oplus G(i)$$

2. Complexes & homotopy

Def If $\mathcal{A}$ - additive, the category of differential objects is $\mathrm{Diff}(\mathcal{A}) = \mathrm{Fun}(\mathbb{Z}, \mathcal{A})$.

Objects:

Morphisms:

$$\begin{tikzcd} \cdots \arrow[r] & X^i \arrow[r, "f^i"] \arrow[d, "\varphi^i"'] \arrow[dr, phantom, "\circlearrowleft"] & X^{i+1} \arrow[r, "f^{i+1}"] \arrow[d, "\varphi^{i+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & X^{i+2} \arrow[r] \arrow[d, "\varphi^{i+2}"'] & \cdots \\ \cdots \arrow[r] & Y^i \arrow[r, "g^i"'] & Y^{i+1} \arrow[r, "g^{i+1}"'] & Y^{i+2} \arrow[r] & \cdots \end{tikzcd}$$

Versions: $\mathrm{Diff}^b(\mathcal{A})$ — full subcategory $X^i = 0, |i| \gg 0$.

$\mathrm{Diff}^-(\mathcal{A})$ — " — $X^i = 0, i \gg 0$.

$\mathrm{Diff}^+(\mathcal{A})$ — " — $X^i = 0, i \ll 0$.

As a functor category, it's additive!

Rmk $\mathcal{A} \hookrightarrow \mathrm{Diff}^b(\mathcal{A})$ fully faithfully

$$X \longmapsto \ldots \to 0 \to 0 \to \underset{\substack{\uparrow \\ \text{in degree } 0}}{X} \to 0 \to \ldots$$

Def $C(\mathcal{A}) \subset \mathrm{Diff}(\mathcal{A})$ – full subcategory of complexes $= (X^i, d^i)$ s.t. $\forall i \; d^{i+1} \circ d^i = 0$.

Similarly define $C^b(\mathcal{A})$, $C^-(\mathcal{A})$, $C^+(\mathcal{A})$.

Homotopy of morphisms

Def Consider $f: X^\bullet \to Y^\bullet$ in $\mathcal{C}(\mathcal{A})$. Say that $f \sim 0$ homotopic to 0 if $\exists h^i : X^i \to Y^{i-1}$ s.t. $f^i = h^{i+1} \circ d^i + d^{i-1} \circ h^i$.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{i-1} \arrow[r] & X^i \arrow[r, "d^i"] \arrow[dl, "h^i"'] \arrow[d, "f^i"] & X^{i+1} \arrow[r] \arrow[dl, "h^{i+1}"] & \cdots \\
\cdots \arrow[r] & Y^{i-1} \arrow[r, "d^{i-1}"'] & Y^i \arrow[r] & Y^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

- $f \sim g$ (f is homotopic to g) if $f - g \sim 0$
- $X \in C(\mathcal{A})$ is homotopic to 0 if $\mathrm{id}_X \sim 0$
- $X, Y \in C(\mathcal{A})$ are homotopy equivalent if
  $\exists\ f: X \to Y,\ g: Y \to X$ s.t. $g \circ f \sim \mathrm{id}_X$, $f \circ g \sim \mathrm{id}_Y$.

Rmk Zero morphisms are ~ 0, if $f \sim 0$, $g \sim 0$, $f, g: X \to Y$, then $-f$ & $f+g$ are ~ 0.

Lm Consider $X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet$. Then if f or g ~ 0, then $g \circ f \sim 0$.
(Morphisms homotopic to 0 form some kind of ideal!)

Pf Assume $f \sim 0$

$$\begin{tikzcd}
\arrow[r] & X^{i-1} \arrow[r] \arrow[d, "f"] & X^i \arrow[r] \arrow[d, "f"] \arrow[dl, "h"'] & X^{i+1} \arrow[r] \arrow[d, "f"] \arrow[dl, "h"'] & {} \\
\arrow[r] & Y^{i-1} \arrow[r] \arrow[d, "g"] & Y^i \arrow[r] \arrow[d, "g"] & Y^{i+1} \arrow[r] \arrow[d, "g"] & 0 \\
\arrow[r] & Z^{i-1} \arrow[r] & Z^i \arrow[r] & Z^{i+1} \arrow[r] & 0
\end{tikzcd}$$

if h is a homotopy, then $g \circ h$ is a homotopy!

$\square$

Cor There is a well-defined category, the homotopy category of $\mathcal{A}$, s.t.

quotient groups

$$\mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\, C(\mathcal{A}), \quad \mathrm{Hom}_{K(\mathcal{A})}(X,Y) = \mathrm{Hom}_{C(\mathcal{A})}(X,Y)/\sim .$$

Similarly are defined $K^b(\mathcal{A})$, $K^-(\mathcal{A})$, $K^+(\mathcal{A})$.

3. Abelian categories

Recall that the kernel of $X \xrightarrow{f} Y$ in $\mathcal{A}$-additive is the equalizer $\mathrm{Eq}(f,0)$. (If exists!)

$$\mathrm{Ker} f \to X \to Y$$

Cokernel — $\mathrm{Coeq}(f,0)$, coimage — cokernel of the kernel, image — kernel of the cokernel.

Lm If $\mathrm{Ker}(f)$ exists, then it's a monomorphism. $\mathrm{Ker} f \to X$

If $\mathrm{Coker}(f)$ exists, then it's an epimorphism. $Y \to \mathrm{Coker} f$

Pf Enough to check the first (opposite cat. argument).

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & \operatorname{Ker} f \arrow[r, "i"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Assume $\quad i \circ g = i \circ h$. Then $i \circ (g-h) = 0$
Thus, $\quad f \circ i \circ (g-h) = 0$, Also $\quad f \circ i \circ 0 = 0$!
The UP of the kernel implies that $g - h = 0$. $\quad \square$

Construction

Assume $\operatorname{Ker} f$, $\operatorname{Coker} f$, $\operatorname{Im} f$, $\operatorname{Coim} f$ exist. Let's construct a canonical arrow $\operatorname{Coim} f \to \operatorname{Im} f$.

$$\begin{tikzcd} \operatorname{Ker} f \arrow[r, "i"] \arrow[rr, bend left, "0"] & X \arrow[r, "f"] \arrow[d, "q"'] & Y \arrow[r, "p"] & \operatorname{Coker} f \\ & \operatorname{Coim} f \arrow[ur, dashed, "\exists!\, h"] \arrow[r, dashed, "\exists!"] & \operatorname{Im} f \arrow[u, "j"'] & \end{tikzcd}$$

Consider $p \circ h$.
Claim $\quad p \circ h = 0$
$\Rightarrow \exists \operatorname{Coim} f \to \operatorname{Im} f$

Rmk $\quad p \circ f = 0$, q is an epimorphism and $p \circ h \circ q = p \circ f = 0 \Rightarrow$
$\Rightarrow p \circ h = 0$!

<u>Example</u> Fil-k – category of $\mathbb{Z}$-filtered vector spaces $/\mathbb{k}$.

Objects : $\ldots \supseteq V^i \supseteq V^{i+1} \supseteq \ldots \supseteq \ldots$

Morphisms: $\varphi : \cup V^i \to \cup W^i$ s.t. $\varphi(V^i) \subseteq W^i$.

<u>Exc</u> Check that it is additive!

Consider V : $V^i = \begin{cases} k, & i<0 \\ 0, & i \geq 0 \end{cases}$, W : $W^i = \begin{cases} k, & i \leq 0 \\ 0, & i > 0 \end{cases}$

There is an obvious morphism $f : V \to W$.

$$\begin{tikzcd}
\vdots & \vdots \\
0 \arrow[r, hook, "0"] & 0 \\
0 \arrow[u, phantom, "\cap" rotate=0] \arrow[r, hook, "0"] & k \arrow[u, phantom, "\cap"] \\
k \arrow[u, phantom, "\cap"] \arrow[r, hook, "1"] & k \arrow[u, phantom, "\cap"] \\
k \arrow[u, phantom, "\cap"] \arrow[r, hook, "1"] & k \arrow[u, phantom, "\cap"] \\
\vdots \arrow[u, phantom, "\cap"] & \vdots \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

<u>Check</u> :
1. $\ker f = 0$
2. $\operatorname{Coker} f = 0$
3. $\operatorname{Coim} f = V$
4. $\operatorname{Im} f = W$
5. f is not an isom.

Def An abelian category $\mathcal{A}$ is an additive category which has all kernels & cokernels and the natural $\mathrm{Coim} f \to \mathrm{Im} f$ are isomorphisms.

Main example

- R - ring with 1 (associative, might not be commutative)

R-Mod — left
Mod-R — right
R-modules are abelian categories.

- Ab, Ab^f
- $\mathcal{A}$ - abelian $\Rightarrow$ $\mathcal{A}^{op}$ - abelian
- $\mathcal{A}$ - abelian $\Rightarrow$ $\mathrm{Fun}(I, \mathcal{A})$ - abelian category

Exc Check this!

- $C(\mathcal{A})$, $C^b(\mathcal{A})$, $C^-(\mathcal{A})$, $C^+(\mathcal{A})$ are abelian if $\mathcal{A}$ is.

Def $\mathcal{A}$ - abelian, $f: X \to Y$

- f is <u>injective</u> if $\operatorname{Ker} f = 0$,
- f is <u>surjective</u> if $\operatorname{Coker} f = 0$.

Lm f is injective $\iff$ f is a monomorphism,
f is surjective $\iff$ f is an epimorphism.

Cor Any $f: X \to Y$ in an abelian category decomposes as $f = p \circ \iota$, ι - injective, p - surjective

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & \operatorname{Coim} f \simeq \operatorname{Im} \arrow[ur, "\iota"'] & \end{tikzcd}$$

Lm $\mathcal{A}$ - abelian $\Rightarrow$ all finite limits & colimits exist in $\mathcal{A}$.

Pf 1) Finite products exist.
2) $\operatorname{Eq}(f,g) = \ker(f-g)$. check this
3) Colimits by duality. $\square$

Def A complex $X^\bullet \in C(\mathcal{A})$, $\mathcal{A}$-abelian is exact if $\forall i$ $\operatorname{Im} d^{i-1} \simeq \operatorname{Ker} d^i$.

$$\begin{tikzcd}
X^{i-1} \arrow[r, "d^{i-1}"] \arrow[dr, "p"'] & X^i \arrow[r, "id"] & X^i \arrow[r, "d^i"] & X^{i+1} \\
& \operatorname{Im} d^{i-1} \arrow[u, "\varphi"] \arrow[r, dashed, "j"] \arrow[urr, bend right, "h"] & \operatorname{Ker} d^i \arrow[u, "\psi"'] \arrow[ur] &
\end{tikzcd}$$

$h = d^i \circ \varphi$. Know that $h \circ p = d^i \circ \varphi \circ p = d^i \circ d^{i-1} = 0$.
Since p is an epimorphism, $d^i \circ \varphi = h = 0$.
Thus, $\exists !\, j$ s.t. everything commutes.

Exc Check that $\operatorname{Im} d^{i-1} \longrightarrow \operatorname{Ker} d^i$ is injective.

Def Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ s.t. $g \circ f = 0$, say that it's a left exact sequence if $\operatorname{Ker} g \simeq \operatorname{Im} f$, and f – injective. Right exact if $\operatorname{Ker} g \simeq \operatorname{Im} f$,

g is surjective. <u>Exact</u> if both right + left.

<u>Notation</u>

$0 \to X \to Y \to Z$ left exact

$X \to Y \to Z \to 0$ right exact

$0 \to X \to Y \to Z \to 0$ exact (short exact sequence = SES)

<u>Def</u> An additive functor is <u>left exact</u> if it maps left exact sequences to left exact:

$$0 \to X \to Y \to Z \quad \rightsquigarrow \quad 0 \to F(X) \to F(Y) \to F(Z)$$

<u>Right exact</u>: preserves right exactness.

<u>Exact</u>: preserves exactness.

<u>Thm</u> (Freyd-Mitchell embedding theorem)

Let $\mathcal{A}$ be abelian. Then there exists a ring R and a fully faithful exact embedding $\mathcal{A} \hookrightarrow R\text{-Mod}$.

Allows us to think of $\mathcal{A}$ as of a full subcategory of a category of modules closed under $\oplus$, Ker, Coker.

<u>Problem 2</u> Let $\mathcal{A}, \mathcal{B}$ be abelian. Show that $F: \mathcal{A} \to \mathcal{B}$ is left (right) exact $\Leftrightarrow$ F preserves finite limits (colimits).