Triangulated & Derived Categories in Algebra & Geometry
Lecture 5

<u>Goal</u> Prove the Freyd-Mitchell theorem.

If $\mathcal{A}, \mathcal{B}$ are abelian, $F : \mathcal{A} \to \mathcal{B}$ is exact if it transforms SES's in $\mathcal{A}$ into SES's in $\mathcal{B}$.

<u>Def</u> An abelian category $\mathcal{A}$ is <u>fully abelian</u> if for any small full abelian subcategory $\mathcal{B} \subset \mathcal{A}$ the exists a ring R and a fully faithful <u>exact</u> embedding $\mathcal{B} \hookrightarrow \text{Mod-}R$, the cat'y of right R-modules.

<u>Thm</u> (Freyd - Mitchell)
Every $\mathcal{A}$ abelian is fully abelian.

0. Some properties of abelian categories

<u>Def</u> A square

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "q"] \\ B \arrow[r, "p"'] & C \end{tikzcd}$$

is called <u>cartesian</u> if W is the limit of $\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$,

<u>cocartesian</u> if C is the colimit ...

W is called the <u>pullback</u> (C is called the <u>pushout</u>).

<u>Rmk</u> The pushout can be defined as the cokernel of

$$W \xrightarrow{(i,-j)} A \oplus B \longrightarrow C \longrightarrow 0$$

Similarly, the pullback is identified with the kernel

$$0 \longrightarrow W \longrightarrow A \oplus B \xrightarrow{(-p,q)} C.$$

Cor A square
$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "q"] \\ B \arrow[r, "p"'] & C \end{tikzcd}$$
is cartesian $\iff$
$$0 \to W \xrightarrow{(i,j)} A \oplus B \xrightarrow{(-p,q)} C$$
is left exact.

Some properties of (co) cartesian diagrams.

Lm If
$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "j"'] & A \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$
is cartesian, then $\ker i \cong \ker f$.

Pf

$$\begin{tikzcd} 0 \arrow[r] & \ker i \arrow[r, "\eta"] \arrow[d, shift right, "\bar{j}"'] & W \arrow[r, "i"] \arrow[d, "j"] & A \arrow[d, "g"] \\ 0 \arrow[r] & \ker f \arrow[r, "\varepsilon"'] \arrow[u, shift right, "\bar{h}"'] \arrow[ur, "h"] & B \arrow[r, "f"'] & C \end{tikzcd}$$

Since W is the pullback, $\exists\, h : \ker f \to W$ s.t. $j \circ h = \varepsilon$, $i \circ h = 0$. h can be lifted to $\bar{h} : \ker f \to \ker i$.

Wish to prove that $\bar{j} \circ \bar{h} = \mathrm{id}$, enough to show that $\varepsilon \circ (\bar{j} \circ \bar{h}) = \varepsilon$! (Since ε is mono.)

$\varepsilon \circ \bar{j} \circ \bar{h} = j \circ \eta \circ \bar{h} = j \circ h = \varepsilon$!

Similarly show that $\bar{h} \circ \bar{j} = \mathrm{id}$ by showing that

$\eta \circ \bar{h} \circ \bar{j} = \eta$. $\square$

<u>Lm</u> Let
$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$
be cocartesian. If $W \hookrightarrow A$ is injective, then so is $B \hookrightarrow C$.

<u>Pf</u> $W \longrightarrow A \oplus B \to C \to 0$ is right exact.

$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend left, "i"] & A \oplus B \arrow[r] & A \end{tikzcd}$$

i is mono, thus the first map is mono!

(<u>Exc</u> If $X \xrightarrow{f} Y \xrightarrow{g} Z$, $g \circ f$ -mono $\Rightarrow$ f is mono!)

Thus, our square is cartesian as well & $\ker(W \to A) \simeq$
$\simeq \ker(B \to C)$. $\square$

__Exc__ Formulate all the dual statements!

1. Exactness

__Recall__ $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if $g \circ f = 0$ and $\mathrm{Im} f = \ker g$.

__Q__ In what sense equals?

Consider the set of all monomorphisms $A \hookrightarrow B$.
What do we know about commutative diagrams of the form ?

$$\begin{tikzcd}
A \arrow[dr, hook] & \\
& B \\
A' \arrow[ur, hook] &
\end{tikzcd}$$

<u>Claim</u> One can complete it to a commuting triangle in at most one way.

$$\begin{tikzcd} A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dr, hook] & \\ A' \arrow[r, hook, "i'"'] & B \end{tikzcd}$$

i' is mono $\Rightarrow f = g$.

We get a partial order on this set.
A subobject is an isomorphism class $A \hookrightarrow B$.

In particular,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\ \operatorname{Im} f \arrow[ur, hook] \arrow[r, hook] & \ker g \arrow[u, hook] & \end{tikzcd}$$

$g \circ f \Rightarrow \operatorname{Im} f \leq \ker g$ (as subobjects).

The diagram is exact $\Leftrightarrow \operatorname{Im} f = \ker g$ as subobjects.

Lm The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is left exact if and only if (X, f) is the kernel of g.

Exc Do this carefully using all UP's.

Cor The functor $h^A : \mathcal{A} \to \mathcal{A}b$, $X \mapsto \mathrm{Hom}_{\mathcal{A}}(A, X)$ is left exact.

Pf
$$h \longmapsto 0 = g \circ h$$
$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, X) \longrightarrow \overset{h \,\in}{\mathrm{Hom}_{\mathcal{A}}(A, Y)} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, Z)$$

The first is mono since $X \to Y$ is mono!

X is the kernel, thus the middle term is also exact:

$$\begin{tikzcd}
 & & A \arrow[dll, dashed, "\exists!"'] \arrow[d, "h"] \arrow[dr, "0"] & \\
X & & Y & Z \\
\end{tikzcd}$$
$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

$\square$

Rmk $A \in \mathcal{A}$, $h_A = \mathrm{Hom}_{\mathcal{A}}(-, A) : \mathcal{A}^{op} \to \mathcal{A}b$.

Also <u>left</u> exact!!!

Left exact sequences in $\mathcal{A}^{op}$ = right exact sequences in $\mathcal{A}$.

Def $P \in \mathcal{A}$ is called projective if h^P is exact.
$I \in \mathcal{A}$ is called injective if h_I is exact.

Lm $P \in \mathcal{A}$ is projective $\iff$ $\forall\, B \twoheadrightarrow C$ $\quad h^P(B) \twoheadrightarrow h^P(C)$.

Gives an equivalence with the classical UP definition:

Def $P \in \mathcal{A}$ is projective if $\forall\, B \twoheadrightarrow C$ and $\forall\, P \to C$

$$
\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\
B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

Exc Define and prove the same for injectives.

Ex In Mod-R projective objects are direct summands of free modules.

Warning Injective objects are usually horrible **but** exist more often...

Lm Direct sums of projective objects are projective!

Pf $P = \bigoplus_{i \in I} P_i$ $\qquad 0 \to A \to B \to C \to 0$

$$h^P(0 \to A \to B \to C \to 0) =$$

$$= \quad 0 \longrightarrow \prod \mathrm{Hom}(P_i, A) \longrightarrow \prod \mathrm{Hom}(P_i, B) \to \prod \mathrm{Hom}(P_i, C) \to 0$$

Obviously exact! $\blacksquare$

Lm Direct products of injectives are injective.

2. Generators

Def A set of objects $\{G_i\}$ is called a family of generators

if $\forall f, g : X \to Y$, $f \neq g$ $\exists h : G_i \to X$ s.t. $f \circ h \neq g \circ h$.
An object $G \in \mathcal{C}$ is a generator if $\{G\}$ is a generating family.

Ex $\mathcal{A} = \text{Mod-}R$, then R, the rank 1 free module is a generator.

If $\mathcal{A}$ is abelian, a projective generator is easy to detect.

Lm $P \in \mathcal{A}$ – projective, generator $\iff$ $h^P(A) \neq 0$ $\forall A \neq 0$.

Pf $\Rightarrow$ $A \neq 0$, assume that $\operatorname{Hom}(P, A) = 0$.

Then $\operatorname{Hom}(P, -)$ does not distinguish

$$A \xrightarrow{0} A, \quad A \xrightarrow{id} A.$$

Know that $0 = id \iff A = 0$.

$\Leftarrow$ P - projective. Consider $B \overset{f}{\underset{g}{\rightrightarrows}} C$. Enough to find $P \xrightarrow{h} B$ s.t. $(f-g)\circ h \neq 0$.

In other words, in $\mathcal{A}$ - abelian G is a generator $\Leftrightarrow$ $\forall\ A \xrightarrow{f} B$, $f \neq 0$ $\exists\ h : G \to A$ s.t. $f \circ h \neq 0$.

$$0 \to \ker f \longrightarrow B \xrightarrow{f} C \qquad f \neq 0 \Rightarrow \ker f \neq B.$$

Assume any $P \to B$ composed with $f = 0$!

Thus, all morphisms $P \to B$ factor throng $\ker f$.

$$0 \to \ker f \to B \to \operatorname{Coim} f = \overset{\neq 0}{\operatorname{Im} f} \to 0$$

But

$$0 \to \operatorname{Hom}(P, \ker f) \xrightarrow{\text{iso}} \operatorname{Hom}(P, B) \to \operatorname{Hom}(P, \operatorname{Im} f) \to 0$$

$$\Rightarrow \operatorname{Hom}(P, \operatorname{Im} f) = 0$$

But $\operatorname{Im} f \neq 0$! $\square$.

Def $\mathcal{A}$ is complete if it has all possible products,
$\mathcal{A}$ is cocomplete if it has all possible coproducts.

Ex Modules are complete & cocomplete.
Finitely generated modules are not!

Prop Let $\mathcal{A}$ be abelian, then $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is complete, cocomplete & has a projective generator.

Pf $\mathcal{A}b$ is complete & cocomplete: know how to form

$\bigoplus A_i$, $\prod B_j$.

Thus, as we discussed, $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is complete and cocomplete: limits & colimits are defined point-wise.

Lm $A \in \mathcal{A}$, then h^A is a projective object in $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$.

Pf Need to prove that if $E, F: \mathcal{A} \to \mathcal{A}b$, $\eta: E \twoheadrightarrow F$, $\alpha: h^A \to F$, then $\exists$ a lift.

In other words, $\mathrm{Hom}(h^A, E) \twoheadrightarrow \mathrm{Hom}(h^A, F)$.

$$\begin{array}{ccc} \mathrm{Hom}(h^A, E) & \twoheadrightarrow & \mathrm{Hom}(h^A, F) \\ \wr\!\| & & \wr\!\| \\ E(A) & \twoheadrightarrow & F(A) \end{array}$$

$\square$

Put $P = \bigoplus\limits_{A \in \mathcal{A}} h^A$. It is projective! Enough to show that $\forall\ E \in \mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ $\mathrm{Hom}(P, E) \neq 0$ if $E \neq 0$.

But $\mathrm{Hom}(P, E) = \prod\limits_{A \in \mathcal{A}} E(A)$. $\square$

3. Mitchell's theorem

Thm A cocomplete abelian category with a projective generator is fully abelian.

Pf $\mathcal{A}' \subseteq \mathcal{A}$ - small full exact subcategory, $\bar{P}$ - projective generator in A.

Put $P = \sum_{\bar{P} \to A'} \bar{P}$ $\leftarrow$ runs over all $P \to A'$, $A' \in \mathcal{A}'$.

Now $\forall\, A \in \mathcal{A}'$ $\exists$ a surjection $P \to A \to 0$!
(Defined on $P_{\alpha: \bar{P} \to A}$ as α & 0 otherwise.)

Put $R = \mathrm{End}_{\mathcal{A}}(P) = \mathrm{Hom}(P, P)$.

Remark that $\forall\, A \in \mathcal{A}$ $\mathrm{Hom}(P, A)$ is a right R-module: precompose $P \to A$ with an endo of P.

We get a functor $h^P : \mathcal{A}' \longrightarrow \text{Mod-}R$.
P - projective $\Rightarrow$ h^P is exact.
P - generator $\Rightarrow$ h^P is faithful.

Remains to verify fullness.

Denote $F = h^P$. Given $\bar{f} : F(A) \to F(B)$, want to find f s.t. $F(f) = \bar{f}$. Let $0 \to K \to P \to A \to 0$

$P \to B \to 0$

$$\begin{tikzcd} 0 \arrow[r] & F(K) \arrow[r] & R \arrow[r] \arrow[d, "r"'] \arrow[dr, dashed] & F(A) \arrow[r] \arrow[d, "\bar{f}"] & 0 \\ & & R \arrow[r] & F(B) \arrow[r] & 0 \end{tikzcd}$$

$R = F(P)$

Any morphism of R-modules $R \to R$ is given by mult. by some element (on the left).

The corresponding element $r \in R = \mathrm{Hom}(P, P)$.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d, "r"'] \arrow[dr, dashed] & A \arrow[r] \arrow[d, "f"] & 0 \\ & & P \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

The composition $K \to P \to B$ is zero since

$F(K) \to R \to F(B)$ is zero & F is faithful.

Remains to check that $F(f) = \bar{f}$. They are equal since $R \to F(A)$ is epi & the pre composition with it is the same for $F(f)$ & $\bar{f}$. $\square$

<u>Cor</u> $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is fully abelian!

Strategy of the proof of Freyd-Mitchell:

embed $\mathcal{A} \hookrightarrow \mathcal{B}$, $\mathcal{B}$ is cocomplete & has a projective generator.

Attempt: $\mathcal{A} \hookrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{A}b)$ ← complete/cocomplete + proj. generator.

Problem: Yoneda embedding is not exact!

<u>Next time</u>: solve this problem, quotient categories.