Triangulated & Derived Categories in Algebra & Geometry
Lecture 6

0. Recall what we are doing

fully abelian

Thm $\mathcal{A}$ - small abelian $\Rightarrow$ $\exists$ an exact full embedding into Mod-R for some ring R.

Weaker version: $\mathcal{A} \hookrightarrow$ Mod-R exact embedding.

Last time: thm by Mitchell saying that $\mathcal{A}$ - cocomplete with a projective generator $\Rightarrow$ $\mathcal{A}$ is fully abelian.

Cor $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is fully abelian.

Strategy

$$\begin{array}{ccc} \mathcal{A}^{op} & \hookrightarrow & \mathrm{Fun}(\mathcal{A}, \mathcal{A}b) \\ \cup & & \\ A & \longmapsto & h^A \end{array}, \quad h^A(B) = \mathrm{Hom}_{\mathcal{A}}(A,B).$$

Get $\mathcal{A} \hookrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{A}b)^{op}$.

Could try and show that $\mathrm{Fun}(\mathcal{A}, \mathcal{Ab})^{op}$ is fully abelian by establishing 1) cocomplete 2) has a proj. generator.

1) $\mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$ complete $\Rightarrow$ $\mathrm{Fun}(\mathcal{A}, \mathcal{Ab})^{op}$ cocomplete.

2) Need to show that $\mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$ has an injective cogenerator:

$$I \text{ s.t. } I \text{ is injective } \& \quad f: A \to B,\ f \neq 0$$
$$\exists\ h: B \to I \text{ s.t. } \quad h \circ f \neq 0.$$

Even after this $\mathcal{A} \hookrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{Ab})^{op}$ fully faithful, but not exact: left exact. Needs to be solved.

1. Injective cogenerators

Prop $\mathcal{A}$ - complete with a generator. $\mathcal{A}$ has an injective cogenerator $\Leftrightarrow$ any $A \in \mathcal{A}$ injects into an injective.
(The last property "$\mathcal{A}$ has enough injectives" $\sim$
$\sim$ $\{I \in \mathcal{A} \mid I\text{-inj}\}$ is a cogenerating family.)

Pf $\Rightarrow$ Let C be an injective cogen. $A \in \mathcal{A}$.

$$A \hookrightarrow \prod_{A \to C} C$$ (check that the map is injective)

$\Leftarrow$ G - generator. Put $P = \prod G_i$, G_i runs through the set of quotients of G.

Put $P \hookrightarrow C \leftarrow$ injective. Let's show that C is a cogen.

$f \neq 0$ $\quad \exists\, g : G \to A$ s.t. $f \circ g \neq 0$

$$G \xrightarrow{g} A \xrightarrow{f} B$$

$\Downarrow$

$I = \mathrm{Im}(f \circ g) \neq 0$

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, hook] \arrow[d, hook] \arrow[dd, bend right] & B \arrow[ddl, dashed, "\exists h"] \\
& P \arrow[d, hook] & \\
& C &
\end{tikzcd}$$

Need to see that $h \circ f \neq 0$.

$h \circ f \circ g \neq 0$

$$G \to A \to B \xrightarrow{h} C$$

Conclude $h \circ f \neq 0$. $\square$

We want to apply this to $\mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$. Need to show that it has enough injectives.

2. Grothendieck category

Let I be a linearly ordered set, let $\{A_i\}$ be a collection of subobjects in A, increasing.

If $\mathcal{A}$ is cocomplete, any such chain has a union:

$$\bigcup A_i = \mathrm{Im}\left(\bigoplus A_i \to A\right) \hookrightarrow A.$$

Def A Grothendieck category is an abelian $\mathcal{A}$ with a generator which is cocomplete s.t.

$\forall$ increasing family $A_i \hookrightarrow A$, $\forall B \hookrightarrow A$

$B \cap (\cup A_i) = \cup B \cap A_i$.

Exc Check that $\mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$ is a Grothendieck category.

3. Injective envelopes

<u>Def</u> An extension is just a mono $A \hookrightarrow B$.

An extension is trivial if $\exists$ a section:

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] & B \arrow[l, bend right, "s"'] \end{tikzcd} \qquad s \circ \iota = id_A$$

(In such case $B \simeq A \oplus C$, ι is the inclusion ι_A.)

An extension is essential if $\forall\ C \hookrightarrow B$ $C \cap A \neq 0$.

(Assume $A \hookrightarrow B$, $C \hookrightarrow B$ s.t. $A \wedge C = 0$.

$$A \hookrightarrow B \to B/C$$

<u>Exc</u> Check that $\nearrow$ is an extension.)

<u>Lm</u> $I \in \mathcal{A}$ is injective $\Longleftrightarrow$ I has only trivial extensions.

<u>Pf</u> $\Rightarrow$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "id_I"'] & B \arrow[dl, dashed, "s"] \\ & I & \end{tikzcd}$$

$\swarrow$ UP for injectives

$\Leftarrow$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[d] \arrow[dl, dashed] \\ & I \arrow[r, hook, "\iota"'] & P \arrow[l, dashed, bend right, "s"'] \end{tikzcd}$$

f - mono

P - pushout

$I \hookrightarrow P$ - mono since $A \hookrightarrow B$ $\blacksquare$

Prop If $\mathcal{A}$ is a Grothendieck category, then I - injective $\Leftrightarrow$
$\Leftrightarrow$ I has no proper essential extensions.

Pf $\Rightarrow$ any extension is trivial, trivial extensions are never essential if proper: $I \hookrightarrow I \oplus C$, $I \cap C = 0$.

$\Leftarrow$ $I \hookrightarrow B$

Consider the set $\mathcal{F} = \{ C \overset{\neq 0}{\hookrightarrow} B \mid I \cap C = 0 \}$

since I has no essential ext's, $\mathcal{F} \neq \emptyset$

Ascending chains in $\mathcal{F}$ have upper bounds:

$\{C_i\}$ s.t. $I \cap C_i = 0 \Rightarrow I \cap (\cup C_i) = \cup (I \cap C_i) = 0$

Can apply Zorn lemma, find $C \hookrightarrow B$ which is maximal s.t. $C \cap I = 0$.

$$\begin{tikzcd} I \arrow[r, hook] \arrow[rr, bend right] & B \arrow[r] & B/C \end{tikzcd}$$ $\leftarrow$ essential! (check it)

Thus, $B/C \simeq I$. The original extension splits.
Now use the previous lemma. $\square$

Def An injective envelope of $A \in \mathcal{A}$ is an essential extension $A \hookrightarrow I$ s.t. I is injective.

Want to show that $\mathcal{A}$ - Grothendieck category, then every object has an inj. envelope. In particular, $\mathcal{A}$ has enough injectives.

Observation A composition of essential extensions is essential:

$$\begin{array}{ccccc} & & & & C \\ & & & & \downarrow \\ A & \hookrightarrow & B & \hookrightarrow & B' \end{array} \qquad C \cap B \neq 0 \Rightarrow (C \cap B) \cap B' \neq 0 !$$

Lm If $\mathcal{A}$ - Grothendieck category $A \hookrightarrow E$ - extension, $A \hookrightarrow E_i$ - lin ordered chain of sub extensions, s.t. $A \hookrightarrow E_i$ - essential $\Rightarrow$ $A \hookrightarrow \bigcup E_i$ is essential.

Pf $C \subset \bigcup E_i \subset E$. $C \cap \bigcup E_i = C$, $C \cap \bigcup E_i = \bigcup (E_i \cap C)$, now must exist E_j s.t. $E_j \cap C \neq 0$.

take the intersection with A. Then $C \cap A \supset C \cap E_j \cap A \neq 0$. $\square$

Lm Given a linearly ordered sequence $A \hookrightarrow E_i$, there exists an extension containing this system es subextensions (if $\mathcal{A}$ is Grothendieck).

Pf Put $S = \sum E_i$. Define maps $h_j : S \to S$.

$h_j : E_i \to S$ is given by $E_i \hookrightarrow E_j \to S$ $\quad i \le j$
and by $E_i \to S$ $\quad i \geq j$.

Put h to be the quotient $S \xrightarrow{h} E$ by $\cup \ker h_i$.
Rmk that $\ker h_i$ form an ascending family.
Look at $\mathrm{Im}(E_j \to S) = I$. $I \cap \ker h = I \cap (\cup \ker h_i) =$
$= \cup (I \cap \ker h_i) = 0$.
Conclude that $E_j \hookrightarrow E$. $\square$

Cor Every lin. ordered chain of essential extensions has an upper bound which is an essential extension.

Construction $\mathcal{A}$ - Grothendieck category. If $A \in \mathcal{A}$ is injective, put $E(A) = A$. Otherwise, choose an essential proper extension $A \to E(A)$.

Extend to all ordinals. If $\gamma = \alpha+1$, put $E^{\gamma}(A) = E(E^{\alpha}(A))$. Otherwise, $\{E^{\alpha}(A) \mid \alpha < \gamma\}$, using the previous corollary choose an upper bound $E^{\gamma}(A)$.

If these sequences stabilize, we get essential extensions which can not be further extended $\Rightarrow$ injective envelopes.

Thm $\mathcal{A}$ - Grothendieck category $\Rightarrow$ every object has an injective envelope.

Look at the generator G. Put $R = \mathrm{End}_{\mathcal{A}}(G)$. Recall that we get a functor to Mod-R. Namely,
$F = h^G$, $F(A) = \mathrm{Hom}_{\mathcal{A}}(G, A)$.

$F: \mathcal{A} \to \text{Mod-}R$.

Lm $A \overset{\iota}{\hookrightarrow} E$ is an essential extension $\Rightarrow$ $F(A) \to F(E)$ is an essential extension.

Pf

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, phantom, "\parallel"] & F(E) \arrow[d, phantom, "\parallel"] \\ \mathrm{Hom}(G,A) \arrow[r, "\iota\circ"] & \mathrm{Hom}(G,E) \end{tikzcd}$$

ι is mono $\Rightarrow$ $F(A) \hookrightarrow F(E)$

Given $0 \neq M \subset \mathrm{Hom}(G,E) = F(E)$ need to show that $M \cap \mathrm{Im}(F(A)) \neq 0$

$M \neq 0 \Rightarrow \exists f \in M,\ f \neq 0,\ f: G \to E.$

$$\begin{tikzcd} G \arrow[r] & P \arrow[r, hook] \arrow[d] \arrow[dr] & G \arrow[d, "f"] \\ 0 \arrow[r] & A \arrow[r, hook] & E \end{tikzcd}$$

$A \cap \mathrm{Im} f \neq 0$

The morphism $P \to E$ is non-trivial. $\exists$ $G \to P$ s.t. $G \to P \to E$ is non-trivial

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend left, "x"] & P \arrow[r] & G \arrow[r] & E \end{tikzcd}$$ is non-trivial

The latter is fx in $F(E)$, lies in the image of $F(A)$. $\square$

Pf (Theorem) Pick $A \in \mathcal{A}$, consider an injective extension

$$F(A) \xhookrightarrow{\varepsilon} Q \quad \text{in } \mathrm{Mod}\text{-}R.$$

Here we use the standard fact from homological algebra: Mod-R has enough injectives.

Q - injective, then for any $A \hookrightarrow E$ essential

$$\begin{tikzcd} 0 \arrow[r] & F(A) \arrow[r, hook] \arrow[d, hook, "\varepsilon"'] & F(E) \arrow[dl, dashed, "h"] \\ & Q & \end{tikzcd}$$

factorization $F(E)$ becomes a subextension in Q.

$\ker h \cap F(A) \subset \ker \varepsilon = 0$ Since $F(A) \hookrightarrow F(E)$ is essential, we conclude that $\ker h = 0$.

To every essential extension $\rightsquigarrow$ a subset in Q!

Take any ordinal γ larger than the # of subsets in Q. Conclude that $E^{\gamma'}$ stabilizes for $\gamma' \geq \gamma$. $\square$

Cor $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is Grothendieck, thus has injective envelopes.

4. Weak embedding theorem

Lm If $E \in \mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is injective, then it is right exact.

Pf Given $A' \to A \to A'' \to 0$ in $\mathcal{A}$, we get a sequence

$0 \to h^{A''} \to h^{A} \to h^{A'}$ in $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$. Apply $\mathrm{Hom}(-, E)$

$E(A') \to E(A) \to E(A'') \to 0 \leftarrow$ exact on the right since E is injective. ∎

Def $E \in \mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is mono if it preserves monomorphisms.

Lm $E \hookrightarrow F$ is an essential extension in $\mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$, then E - mono $\Rightarrow F$ is mono. $\leftarrow$ we will prove it next time

Cor Every small abelian category can be exactly embedded into $\mathcal{A}b$!

Problem Prove it! (Take an inj. envelope $\bigoplus h^A$, use the lemme.)