Triangulated & Derived Categories in Geometry & Algebra
Lecture 7

<u>Goal</u> Finish the proof of Freyd-Mitchell.

Last time we showed that if $\mathcal{A}$ is small abelian, then $\mathcal{A}$ embeds exactly into $\mathcal{Ab}$: $\exists$ a faithful exact functor $E: \mathcal{A} \longrightarrow \mathcal{Ab}$.

How did we do that?

<u>Def</u> $F \in \mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$ is <u>mono</u> if F preserves monomorphisms.

<u>Observation</u> $E \in \mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$ is injective $\Rightarrow$ E is right exact.

<u>Lm</u> If $F \to E$ - essential extension. Then F - mono $\Rightarrow$ E mono.

<u>Cor</u> An injective envelope ($F \hookrightarrow E$ s.t. E - injective, $F \hookrightarrow E$ is essential) of a mono F is an exact functor.

Cor Take $F = \bigoplus_{A \in \mathcal{A}} h^A$. Let E be its injective envelope.

Then $E : \mathcal{A} \longrightarrow \mathcal{A}b$ is a (faithful) exact embedding.

Pf (Ess. extension Lemma)

$0 \to F \to E$ - essential extension. Assume that E is not mono. Then $\exists\ A \xhookrightarrow{f} B$ s.t. $E(A) \not\hookrightarrow E(B)$. For instance, $\exists\ 0 \neq x \in E(A)$ s.t. $E(f)(x) = 0$.

By Yoneda (covariant version) $E(A) \simeq \mathrm{Hom}(h^A, E)$.

$$x \longmapsto \eta$$

Look at $\mathrm{Im}\,\eta \subset E$. Denote $\mathrm{Im}\,\eta = M$.

$$M(c) = \{ y \in E(c) \mid \exists\, g : A \to c \text{ s.t. } E(g)(x) = y \}.$$

$$M(c) \subseteq E(c).$$

Exc Check that $M(c)$ is a subgroup, M - functor w/r to restrictions of maps coming from E.

M – subfunctor "generated" by x.

<u>Claim</u> $M \cap F$ in E is 0. Since $M \neq 0$, we get a contr'n.

Assume $M \cap F \neq 0$. $\exists\, C$ s.t. $M(C) \cap F(C) \neq 0$. Let $0 \neq y \in M(C) \cap F(C)$. $y = E(g)(x)$ for some $g : A \to C$

$$\begin{tikzcd}
x \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & 0 \\
& A \arrow[r, hook, "f"] \arrow[d, "g"] & B \arrow[d] & \\
& C \arrow[r, hook] & D \arrow[ul, phantom, "\lrcorner"] & \\
y \arrow[rrr, mapsto] & & & \neq 0
\end{tikzcd}$$

$\leftarrow$ pushout diagram.

Follow the x along the square!

$\square$

1. Proof of the strong embedding theorem

<u>Know</u> $\mathcal{B}$ – abelian, complete, has a projective generator, every object has an injective hull (can be embedded in an injective) $\Rightarrow$ $\mathcal{B}$ has an inj. cogenerator.

For instance, $\mathcal{B}^{op}$ then has a projective generator!
By Mitchell's thm can be fully exactly embedded into Mod-R for some big R.

<u>Strategy</u> $\mathcal{A}^{op} \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathrm{Ab})$. Rmk that h^A is a left exact functor for all A. Let's look at the category of left exact functors $\mathcal{L} \subset \mathrm{Fun}(\mathcal{A}, \mathrm{Ab})$ (full subcategory).

<u>Claim</u> $\mathcal{L}$ satisfies all the properties we need.
$\mathcal{L}$ - abelian $\longleftarrow$ hard, every object has an injective envelope, $\mathcal{L}$ - complete, $\mathcal{L}$ has an injective cogenerator.

<u>Thm</u> $h : \mathcal{A}^{op} \longrightarrow \mathcal{L}$ is an exact full embedding.

<u>Pf</u> Yoneda $\Rightarrow$ full embedding. Why exact?
For exactness it's enough to check that

for an injective cogenerator E and any $0 \to A'' \to A \to A' \to 0$

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}(h^{A'}, E) & \to & \mathrm{Hom}(h^{A}, E) & \to & \mathrm{Hom}(h^{A''}, E) & \to 0 \\ & \shortparallel & & \shortparallel & & \shortparallel & \\ & E(A') & & E(A) & & E(A'') & \end{array}$$

iff E - exact functor.

will follow from the lemma $\square$

2. Left exact functors form an abelian category

Def $\mathcal{M} \subset \mathrm{Fun}(\mathcal{A}, Ab)$ - full subcategory of mono functors.

Properties
1) $\mathcal{M}$ is closed under subobjects.
2) $\mathcal{M}$ is closed under products.
3) $\mathcal{M}$ is closed under essential extensions.

Objects of $\mathcal{M}$ will be called mono.

Think of $\mathcal{M}$ as of torsion-free modules.

Denote $\mathcal{B} = \mathrm{Fun}(\mathcal{A}, \mathrm{Ab})$. $\mathcal{M} \subseteq \mathcal{B}$.

<u>Lm</u> $F \in \mathcal{B}$. Then F has a maximal quotient object $F \to M(F) \in \mathcal{M}$.

<u>Pf</u> Consider $Q = \{ F \twoheadrightarrow B \mid B \in \mathcal{M} \}$.

Put $M(F) = \mathrm{Im}\left(F \longrightarrow \prod_{B \in Q} B \right)$.

If $F \twoheadrightarrow B'$, $B' \in \mathcal{M}$.

$M(F) \hookrightarrow \prod_{B \in Q} B \xrightarrow{\pi} B'$. Check that the comp. is surj!

<u>Lm</u> $F \in \mathcal{B}$, $M \in \mathcal{M}$, $f : F \to M$, then f factors uniquely through $M(F)$.

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[dr] & M \\ & M(F) \arrow[u] \\ \mathcal{B} & \mathcal{M} \end{tikzcd}$$

Pf Immediate since the coimage of f is a subobject in $\mathcal{M}$. $F \to \mathrm{Coim} \hookrightarrow \mathcal{M}$. $\square$

Cor M is an additive functor!

$M : \mathcal{B} \to \mathcal{M}$, satisfies $\forall B \in \mathcal{B}$, $\forall M \in \mathcal{M}$

$$\mathrm{Hom}_{\mathcal{B}}(B, M) \simeq \mathrm{Hom}_{\mathcal{M}}(M(B), M)$$

$\mathcal{M}$ is a full subcategory.

M – left adjoint to $\iota : \mathcal{M} \hookrightarrow \mathcal{B}$.

Def $T \in \mathcal{B}$ is called <u>torsion</u> if $\mathrm{Hom}(T, M) = 0$ for all $M \in \mathcal{M}$.

(Think torsion modules if $\mathcal{M}$ = torsion-free modules.)

Lm If $F \in \mathcal{B}$, then $\mathrm{Ker}(F \to M(F))$ is a maximal torsion subobject.

<u>Pf</u> If $T \hookrightarrow F$ is torsion, then $\mathrm{Im}(T \hookrightarrow F \twoheadrightarrow M(F))$ is 0 since $T \to M(F)$ must be zero.
$T \hookrightarrow \ker(F \to M(F))$. Enough to show that the latter is torsion.

in $\mathcal{M}$ $\longrightarrow$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "f"'] \arrow[dr, dashed] & F \arrow[r] \arrow[d, dashed] & M(F) \arrow[r] \arrow[dl, dashed] & 0 \\
0 \arrow[r, hook] & M \arrow[r, "\iota"] & \Sigma & &
\end{tikzcd}$$

Let $K \xrightarrow{f} M$, $M \to \Sigma$ – inj. envelope.

All commutes, $\iota \circ f = 0$. ι – injective $\Rightarrow f = 0$. $\square$

<u>So far</u> $\mathcal{M}$ – mono objects $\rightsquigarrow$ $\mathcal{T}$ – torsion objects.

Any $F \in \mathcal{B}$ can be decomposed as

$$0 \to \underset{\mathcal{T}}{\underset{\rotatebox{90}{$\in$}}{T(F)}} \to F \to \underset{\mathcal{M}}{\underset{\rotatebox{90}{$\in$}}{M(F)}} \to 0 \qquad \mathrm{Hom}(\mathcal{T}, \mathcal{M}) = 0.$$

How to find left exact functors among mono functors?

<u>Lm</u> $\mathcal{M}$ is closed under extensions: $0 \to M_1 \to F \to M_2 \to 0$, $M_1, M_2 \in \mathcal{M} \Rightarrow F \in \mathcal{M}$.

<u>Pf</u>

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[dl, dashed] & M_2 \arrow[r] & 0 \\ & E & & & \end{tikzcd}$$

$E \in \mathcal{M}$

inj. envelope

since E injective

the induced map $F \longrightarrow E \oplus M_2$ is injective $\Rightarrow$ $\Rightarrow F \in \mathcal{M}$ as a subobject $\square$

$\mathcal{M}$ is an additive subcategory closed under extensions & taking subobjects.

<u>Problem</u> quotients of $M_1 \longrightarrow M_2$ in $\mathcal{M}$ might not be in $\mathcal{M}$. $\left(0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \notin \text{torsion-free.}\right)$

<u>Def</u> $M' \hookrightarrow M$ in $\mathcal{M}$ is <u>pure</u> if $M/M' \in \mathcal{M}$.

M is absolutely pure if whenever $M \hookrightarrow M''$ in $\mathcal{M}$, then $M''/M \in \mathcal{M}$.

Want to show that absolutely pure = left-exact among all mono functors.

Lm A pure subobject of an absolutely pure one is absolutely pure.

Pf A – absolutely pure, $P \hookrightarrow A$ – pure, $P \hookrightarrow M$.
Need to show that $M/P \in \mathcal{M}$.

$A/P \in \mathcal{M}$ ← since P – pure subobject

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, hook] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & A/P \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & M \arrow[r, hook] \arrow[d] \arrow[ur, phantom, "\lrcorner"] & N \arrow[r] \arrow[d] & A/P \arrow[r] & 0 \\
 & M/P \arrow[r, "\sim"] & N/A & &
\end{tikzcd}$$

injective as a pushout of an inj.

iso as kernels in a pushout square

$M/P \simeq N/A \in \mathcal{M}$!

∎

If $F \hookrightarrow E$ is an injective envelope, $F \in \mathcal{M}$, then $E \in \mathcal{M}$. Also E is absolutely pure!

$E \in \mathcal{M}$, E - injective, $E \hookrightarrow N$, the embedding splits!

$N \simeq E \oplus M$, M is in $\mathcal{M}$ since $M \hookrightarrow N$. $N/E \simeq M \in \mathcal{M}$.

<u>Prop</u> $M \in \mathcal{M} \subset \mathrm{Fun}(\mathcal{A}, \mathcal{Ab})$ is absolutely pure iff M is left exact.

<u>Pf</u> $M \hookrightarrow E$ - inj. envelope. E - absolutely pure, E - left exact (last lecture). Enough to show that a pure subfunctor of a left exact functor is left exact.

$0 \to M \to E \to F \to 0$, E - left exact, F - mono

Pick $0 \to A' \to A \to A''$ in $\mathcal{A}$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & M(A') \arrow[r] \arrow[d] & M(A) \arrow[r] \arrow[d] & M(A'') \arrow[d] \\
0 \arrow[r] & E(A') \arrow[r] \arrow[d] & E(A) \arrow[r] \arrow[d] & E(A'') \\
 & F(A') \arrow[r] \arrow[d] & F(A) \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}
$$

<u>Claim</u> (Problem)

$F(A') \to F(A)$ is mono $\Leftrightarrow$ the top row is left exact.

$\square$

$\mathcal{L} \subset \mathcal{M}$ – full subcategory of absolutely pure objects.

Want: for any $M \in \mathcal{M} \rightsquigarrow L(M) \in \mathcal{L}$ s.t.

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & L \\ & L(M) \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

In other words, a left adjoint to $\iota : \mathcal{L} \hookrightarrow \mathcal{M}$.

$$\mathrm{Hom}_{\mathcal{L}}(L(M), L) \simeq \mathrm{Hom}_{\mathcal{M}}(M, L).$$

Lm Given $0 \to M \to L \to T \to 0$, s.t. $M \in \mathcal{M}$, $L \in \mathcal{L}$, T - torsion, $L \simeq L(M)$ (if $L(M)$ exists).

Pf

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] \arrow[dl, dashed] \arrow[dr, "0"] & T \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & L' \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

since E is injective

$L' \to E$ - inj. envelope.

$F \in \mathcal{M}$ since L' is absolutely pure

Exc show uniqueness $L \to L'$. ∎

Thm $L : \mathcal{M} \to \mathcal{L}$ exists; moreover, $M \to L(M)$ is injective.

<u>Pf</u> Enough to show that $L(M)$ exists for every M. (Problem 1 says that adj. functor is defined by representability object-wise.)

$0 \to M \to E$, E - injective envelope

thus, $E \in \mathcal{L}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & M(F) \arrow[r, "\sim"] \arrow[d] & M(F) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

our decomposition for $(\mathcal{T}, M)$

cokernel in $\mathcal{B}$

N - pure subobject of an absolutely pure $\Rightarrow N \in \mathcal{L}$.

$T \in \mathcal{T} \Rightarrow$ By the previous lemma $N = L(M)$. $\square$

Tomorrow Explain how this gives us an abelian category structure.

Problem Let the following be a diagram in an abelian category $\mathcal{A}$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[d] \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \\
 & C' \arrow[r] \arrow[d] & C & \\
 & 0 & &
\end{tikzcd}$$

the columns are exact, so is the middle row

Then the top row is exact $\iff$

$C' \to C$ is mono.

Warning Can only use the axioms of abelian cat's, no picking elements in modules.