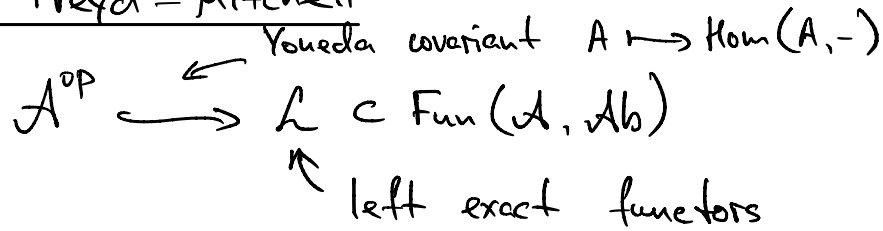


Triangulated & Derived Categories in Geometry & Algebra

Lecture 8

0. Freyd-Mitchell



Wanted to show that \mathcal{L} is abelian, complete, has an injective cogenerator.

We identified \mathcal{L} with "absolutely pure" objects in $\mathcal{M} \leftarrow$ full subcat. of mono functors.

Also a bunch of adjoints to various embeddings.

Defined the class of torsion objects.

If $M \in \mathcal{M} \rightsquigarrow$ an absolutely pure $\mathcal{L}(M)$:

enough to find $0 \rightarrow M \rightarrow L(M) \rightarrow T \rightarrow 0$,
 where $L(M) \in \mathcal{L}$, $T \in \mathcal{T}$.

Constructed an adjoint functor. $M \mapsto L(M)$ is injective.

← but $\mathcal{L} \hookrightarrow \text{Fun}(\mathcal{A}, \mathcal{B})$ is not exact!

Then

- 1) \mathcal{L} is abelian.
- 2) every $L \in \mathcal{L}$ has an injective envelope.
- 3) \mathcal{L} is complete.
- 4) \mathcal{L} has an injective cogenerator.

PF

- 1) $0 \in \mathcal{L}$, L is an additive functor \Rightarrow products & sums.
 From the pure subobject lemma: $\text{Ker}(L_1 \rightarrow L_2) \in \mathcal{L}$.
 $L_1 \rightarrow L_2$ is inj. in $\mathcal{L} \Leftrightarrow L_1 \rightarrow L_2$ is inj. in Fun .

$$0 \rightarrow L_1 \rightarrow L_2 \rightarrow M \rightarrow 0$$

$L_1 \in \mathcal{L} \Leftrightarrow L_1$ is absolutely pure $\Rightarrow M \in \mathcal{M}$.

Can consider $M \hookrightarrow L(M)$.

This is what
 \uparrow
 \mathcal{B}
 the cokernel will be.

Exc Check the properties of an abelian category.

- 2) \mathcal{L} -inj $\Leftrightarrow \text{Fun-inj} \Rightarrow$ an injective envelope formed in Fun is an inj. envelope in \mathcal{L} .
- 3) Products of left exact functors are left exact.
- 4) $\prod_{A \in \mathcal{A}} \mathcal{L}^A$ - left exact & a projective generator.

□

1. Serre & quotient categories

Motivation $F: \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor b/w abelian.

Let $\ker F$ - full subcategory $\{A \in \mathcal{A} \mid F(A) = 0\}$.

$\ker F$ satisfies the following:

$0 \in \ker F$, $X \in \ker F \Rightarrow$ every $Y \hookrightarrow X$ is in $\ker F$
every $X \twoheadrightarrow Z$ is in $\ker F$

$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, $A, C \in \ker F \Rightarrow B \in \ker F$.

Def A Serre subcategory $\mathcal{B} \subseteq \mathcal{A}$ is a full non-empty subcategory such that

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0 \quad A \in \mathcal{B} \Leftrightarrow A' \& A'' \in \mathcal{B}.$$

Thm $\mathcal{B} \subseteq \mathcal{A}$ is Serre $\Leftrightarrow \mathcal{B} = \text{Ker } F$ for some exact $F: \mathcal{A} \rightarrow \mathcal{C}$.

The theorem follows from the following.

Thm Let $\mathcal{B} \subseteq \mathcal{A}$ be Serre. There exists a category \mathcal{A}/\mathcal{B} -abelian & an exact $\mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{B}$ s.t. $\text{Ker } Q = \mathcal{B}$ & it satisfies the UP:
 $\forall F: \mathcal{A} \rightarrow \mathcal{C}$ exact s.t. $F(\mathcal{B}) = 0$
 $\exists! H: \mathcal{A}/\mathcal{B} \rightarrow \mathcal{C}$ s.t. $F = H \circ Q$.

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
 Q \searrow & & \nearrow \exists! \\
 & \mathcal{A}/\mathcal{B} &
 \end{array}$$

Warning Set-theoretic issues. Insert appropriate words
s.a. well-powered (subobjects of every object
form a small set).

Construction (due to Serre)

$$\text{Put } \text{Ob}(A/B) = \text{Ob}(A).$$

$$\text{Mor}(X, Y) = \text{colim} \text{Hom}_A(X', Y/Y')$$

$X' \hookrightarrow X, X' \in B$
 $Y' \hookrightarrow Y, Y' \in B$

Good: 1) morphisms form an abelian group.
2) composition is given by the UP of colim.

Bad: very abstract, impossible to compute anything.

Alternative: localization.

2. Main properties of abelian categories

All kinds of lemmas

Prop (Snake lemma) Given

with exact rows

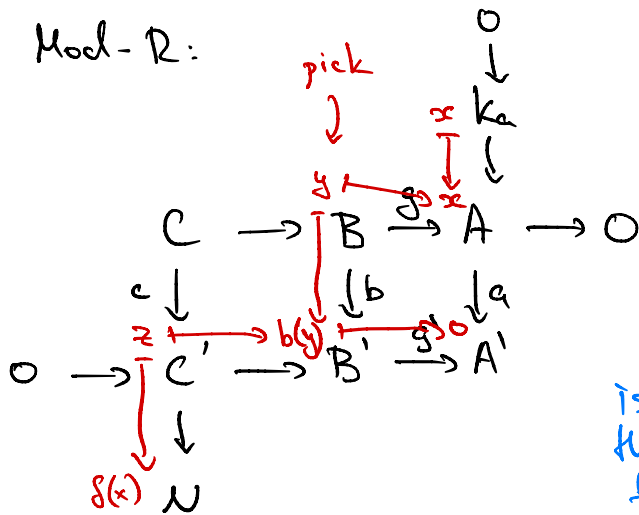
$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \rightarrow & 0 \\ & a \downarrow & & b \downarrow & & c \downarrow & \\ 0 & \rightarrow & A' & \rightarrow & B' & \rightarrow & C' \end{array}$$

there is an induced exact sequence

$$\begin{array}{c} \ker a \rightarrow \ker b \rightarrow \ker c \\ \text{\scriptsize } \delta \\ \hookrightarrow \text{Coker } a \rightarrow \text{Coker } b \rightarrow \text{Coker } c \end{array}$$

Very hard to even construct δ without assuming we are in $\text{Mod-}R$.

If in Mod-R:



Need to show that \mathcal{S} is well-defined (preimage of x was a choice).

is where we use that Freyd-Mitchell is fully faithful.

Exc Try to construct \mathcal{S} without appealing to Freyd-Mitchell.

Exc Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in \mathcal{A} -abelian. Construct an exact sequence

$$0 \rightarrow \ker f \rightarrow \ker gf \rightarrow \ker g \rightarrow \operatorname{coker} f \rightarrow \operatorname{coker} gf \rightarrow \operatorname{coker} g \rightarrow 0.$$

Cor (5-lemma)

$$\begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & C & \rightarrow & D \\ a \downarrow & & \downarrow b & & \downarrow c & & \downarrow d \\ A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' \end{array}$$

Similarly you get

if a, c, d are epi, d -mono
then b is epi

if b, c, d are mono, a -epi
then c is mono.

3. Serre subcategories

Let $\mathcal{B} \subseteq \mathcal{A}$ be Serre.

Def $f: A \rightarrow B$ in \mathcal{A} is

- 1) \mathcal{B} -mono if $\ker f \in \mathcal{B}$,
- 2) \mathcal{B} -epi if $\operatorname{coker} f \in \mathcal{B}$,
- 3) \mathcal{B} -iso if \mathcal{B} -epi & \mathcal{B} -mono.

Prop If $F: \mathcal{A} \rightarrow \mathcal{C}$ is exact & $B \in \text{Ker } F$,
 then the image of every B -mono (epi/iso) in \mathcal{A}
 is mono (epi/iso).

Thought Instead of the WP $F(B) = 0$ let's
 consider the WP $F(B\text{-iso}) \subseteq \text{iso}$.

4. Localization

Let \mathcal{C} be a category, let S be a class of morphisms
 in \mathcal{C} closed under composition & containing all identity
 $\text{id}_x \in S \quad \forall x \in \mathcal{C}$.

Want a universal category $S^{-1}\mathcal{C}$ & a functor $L: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$
 s.t. $L(S) \subseteq \text{Iso}$ ($L(s)$ is an iso $\forall s \in S$) & $\forall F: \mathcal{C} \rightarrow \mathcal{D}$
 s.t. $F(S) \subseteq \text{Iso}$ $\exists!$

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow Q & & \uparrow \exists! \\ S^{-1}\mathcal{C} & & \end{array}$$

Such a pair $S^{-1}\mathcal{C}$, $L: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ is called the localization.
As usual, if exists, then unique up to isom of categories.

(Compare with localization of commutative rings.)

Thm Up to set-theoretic issues localization exists.

Pf Construction: put $\text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C})$

Morphisms $X \rightarrow Y$

1) Consider the "set" of diagrams of the form

$$X \xrightarrow{f_0} Z_0 \xleftarrow{s_0} Y_0 \xrightarrow{f_1} Z_1 \xleftarrow{s_1} Y_1 \rightarrow \dots \xleftarrow{s_n} Y_n = Y$$

Where $s_i \in S$.

Think of such a chain as of
 $(s_n)^{-1} \circ \dots \circ (s_1)^{-1} \circ f_1 \circ (s_0)^{-1} \circ f_0$.

2) Composition is obvious:
concatenation

\Rightarrow associative

$$x \xrightarrow{\alpha} \dots \xrightarrow{\beta} y \xrightarrow{\gamma} \dots \xrightarrow{\delta} z$$

3) Pnf an equivalence relation
a) You can insert / remove

$$x \xrightarrow{s} y \xrightarrow{s} x \sim x \quad (s)^{-1}os = id_x$$

$$b) \quad z \xrightarrow{s} y \xrightarrow{id} y \xrightarrow{t} w \sim z \xrightarrow{st} w \quad \forall s, t \in S$$

$$c) \quad x \xrightarrow{f} z \xrightarrow{id} z \xrightarrow{g} y \sim x \xrightarrow{g \circ f} y \quad \forall f, g.$$

Generated by a, b, c.

Exe Persuade yourself that this thing with

the functor $\mathcal{C} \rightarrow \mathcal{S}^{-1}\mathcal{C}$

$$X \mapsto X$$

$$X \xrightarrow{f} Y \mapsto X \xrightarrow{f} Y \xleftarrow{\text{id}_Y} Y$$

satisfies the UP of localization. \square

Problem Can't compute anything + massive set-theoretic issues.

5. Calculus of fractions

Def A class of morphisms S in \mathcal{C} is a left localization system if

1) $\text{id}_X \in S \quad \forall X \in \mathcal{C}$ & S closed under composition,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ s \downarrow & & \downarrow t \\ Z & \xrightarrow{g} & W \end{array}$$

$s \in S \Rightarrow \exists$ a completed diag.
with $t \in S$

$$(g \circ s = t \circ f \rightsquigarrow t' \circ g = f \circ s')$$

$$3) \text{ if } X \xrightarrow{s} Y \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Z \stackrel{s \in S}{\text{equalizes } f \neq g} : fs = gs$$

$$\Rightarrow \exists Y \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Z \xrightarrow{t} W \quad t \in S, \quad t \text{ coeq. } f \neq g.$$

Prop

Consider diagrams of the form

$$X \xrightarrow{f} Y' \begin{matrix} \xleftarrow{s} \\ \xleftarrow{r} \end{matrix} Y$$

$s \in S.$

Will think of it as
of s -b.f.

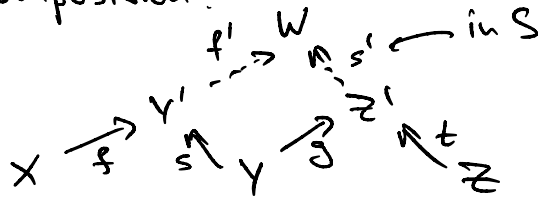
If S is a left localization system, then
the following is an equivalence relation:

$$X \rightarrow Y' \leftarrow Y \sim X \rightarrow Y'' \leftarrow Y \text{ if}$$

$$\begin{array}{c}
 \begin{matrix} X & \nearrow & Y' \\ & \rightarrow & Y'' \\ & \searrow & Y'' \\ & & \downarrow \\ & & Y'' \\ & & \downarrow \\ & & Y'' \\ & & \downarrow \\ & & Y'' \end{matrix}
 \end{array}$$

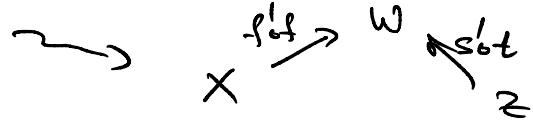
$s', s'', s''' \in S.$

Define composition:



the completion exists!

$$s't \in S$$



Prop This defines an associative composition law on these fractions $/_R$. Thus, one gets a category with $\text{Ob} = \text{Ob } \mathcal{C}$. Morphisms: fractions $/_R$.

Consider the functor $\mathcal{C} \rightarrow \mathcal{S}^{-1}\mathcal{C}$,

$$X \rightarrow X, \quad X \xrightarrow{f} Y \mapsto \begin{array}{ccc} & Y & \\ f \nearrow & & \text{id} \searrow \\ X & & Y \end{array}$$

Exc Check the UP.

Remark One could dually define right localization systems \leadsto localization with morphisms

given by equiv. classes of $X \xleftarrow{s} X' \xrightarrow{f} Y$, $s \in S$.

If S is a localization system (both left & right),
by the UP the two localizations are isomorphic categories.

6. Quotients as localizations

Lim $\mathcal{B} \subseteq \mathcal{A}$ is Serre $\Leftrightarrow \mathcal{B}$ is non-empty
and $\forall A' \rightarrow A \rightarrow A''$ if $A', A'' \in \mathcal{B} \Rightarrow A \in \mathcal{B}$.

Pf \Leftarrow Only need to show that sub(quotients) of
objects in \mathcal{B} are in \mathcal{B} .

1) $\mathcal{B} \neq \emptyset \Rightarrow \exists B \in \mathcal{B}$.

$$B \rightarrow 0 \rightarrow B \Rightarrow 0 \in \mathcal{B}$$

2) $0 \rightarrow A' \rightarrow A \Rightarrow A' \in \mathcal{B}$ if $A \in \mathcal{B}$.

3) $A \rightarrow A'' \rightarrow 0$ same!

$$\Rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \quad A', A'' \in \mathcal{B}$$

$$0 \rightarrow \ker f \rightarrow A' \rightarrow \text{Im } f = \ker g \rightarrow 0 \quad A' \in \mathcal{B} \Rightarrow \ker g \in \mathcal{B}$$

$$0 \rightarrow \text{Im } g \rightarrow A'' \rightarrow \text{Coker } g \rightarrow 0 \quad A'' \in \mathcal{B} \Rightarrow \text{Im } g \in \mathcal{B}$$

$$0 \rightarrow \ker g \rightarrow A \rightarrow \text{Im } g \rightarrow 0 \Rightarrow A \in \mathcal{B}. \quad \square$$

Prop If $\mathcal{B} \subset \mathcal{A}$ is Serre, the class of \mathcal{B} -iso is a localization system.

Pf Let's check the "complete the square", the rest - exc.

$$\begin{array}{ccc}
 \ker s & \twoheadrightarrow & \ker t \xleftarrow{\text{epi}} \\
 \downarrow & & \downarrow \\
 A & \twoheadrightarrow & C \\
 s \downarrow & & \downarrow t \\
 B & \twoheadrightarrow & C \cup B \\
 \downarrow & & \downarrow \\
 \text{Coker } s & \xrightarrow{\sim} & \text{Coker } t \Rightarrow \text{Coker } t \in \mathcal{B}
 \end{array}$$

$s \in \mathcal{B}\text{-iso} \Leftrightarrow \ker s \in \mathcal{B}$
 $\text{Coker } s \in \mathcal{B}$

thus, $\ker t \in \mathcal{B}$
 as a quotient
 of $\ker s \in \mathcal{B}$.

\square

We can now define A/B as $S^{-1}A$, where $S = B - \text{iso}$.

4. What are derived categories

Let \mathcal{A} - abelian. Recall that we defined $\mathcal{C}(\mathcal{A})$ - the cat of complexes: objects

$$\dots \rightarrow X^i \xrightarrow{d^i} X^{i+1} \rightarrow \dots \quad \text{st. } d^{i+1} \circ d^i = 0 \quad \forall i \in \mathbb{Z}.$$

Morphisms:

$$\begin{array}{ccccccc} \dots & \rightarrow & X^i & \rightarrow & X^{i+1} & \rightarrow & \dots \\ & & \downarrow f^i & & \downarrow f^{i+1} & & \\ \dots & \rightarrow & Y^i & \rightarrow & Y^{i+1} & \rightarrow & \dots \end{array} \quad d^i \circ f^i = f^{i+1} \circ d^i$$

Def The nth cohomology of $X^\bullet \in \mathcal{C}(\mathcal{A})$ is

$$H^i(X^\bullet) = \text{Ker } d^i / \text{Im } d^{i-1}.$$

Observation H^i is a functor $\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}$ for all i .

Def $f: X^\bullet \rightarrow Y^\bullet$ in $\mathcal{C}(\mathcal{A})$ is a quasi-isomorphism
if $H^i(f): H^i(X^\bullet) \xrightarrow{\sim} H^i(Y^\bullet) \quad \forall i \in \mathbb{Z}$.

Def The derived category $\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{C}(\mathcal{A})$,
where S is the class of quasi-isom's.

Next week begin to study such things.