Triangulated & Derived Categories in Geometry & Algebra
Lecture 8

0. Freyd – Mitchell

Yoneda covariant $A \mapsto \mathrm{Hom}(A,-)$

$$\mathcal{A}^{op} \longrightarrow \mathcal{L} \subset \mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$$

left exact functors

Wanted to show that $\mathcal{L}$ is abelian, complete, has an injective cogenerator.

We identified $\mathcal{L}$ with "absolutely pure" objects in $\mathcal{M}$ ← full subcat. of mono functors.

<u>Also</u> a bunch of adjoints to various embeddings.

Defined the class of torsion objects.

If $M \in \mathcal{M} \rightsquigarrow$ an absolutely pure $L(M)$:

enough to find $0 \to M \to L(M) \to T \to 0$,
where $L(M) \in \mathcal{L}$, $T \in \mathcal{T}$.

Constructed an adjoint functor. $M \hookrightarrow L(M)$ is injective.

but $\mathcal{L} \hookrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{A}b)$ is not exact!

Thm
1) $\mathcal{L}$ is abelian.
2) every $L \in \mathcal{L}$ has an injective envelope.
3) $\mathcal{L}$ is complete.
4) $\mathcal{L}$ has an injective cogenerator.

Pf 1) $0 \in \mathcal{L}$, L is an additive functor $\Rightarrow$ products & sums.
From the pure subobject lemma: $\mathrm{Ker}(L_1 \to L_2) \in \mathcal{L}$.
$L_1 \to L_2$ is inj. in $\mathcal{L}$ $\iff$ $L_1 \to L_2$ is inj. in Fun.

$0 \to L_1 \to L_2 \to M \to 0$

$L_1 \in \mathcal{L} \iff L_1$ is absolutely pure $\Rightarrow M \in \mathcal{M}$.

Can consider $M \hookrightarrow \underset{\substack{\uparrow \\ B}}{L(M)}$.

This is what the cokernel will be.

Exc Check the properties of an abelian category.

2) $\mathcal{L}$-inj $\Leftrightarrow$ Fun-inj $\Rightarrow$ an injective envelope formed in Fun is an inj. envelope in $\mathcal{L}$.

3) Products of left exact functors are left exact.

4) $\prod_{A \in \mathcal{A}} h^A$ - left exact & a projective generator.

$\square$

1. Serre & quotient categories

Motivation $F: \mathcal{A} \to \mathcal{B}$ be an exact functor b/w abelian.

Put $\ker F$ - full subcategory $\{A \in \mathcal{A} \mid F(A) = 0\}$.

$\ker F$ satisfies the following:

$0 \in \ker F$, $X \in \ker F \Rightarrow$ every $Y \hookrightarrow X$ is in $\ker F$
every $X \twoheadrightarrow Z$ is in $\ker F$

$0 \to A \to B \to C \to 0$, $A, C \in \ker F \Rightarrow B \in \ker F$.

Def A Serre subcategory $\mathcal{B} \subseteq \mathcal{A}$ is a full non-empty subcategory such that

$$0 \to A' \to A \to A'' \to 0 \qquad A \in \mathcal{B} \iff A' \,\&\, A'' \in \mathcal{B}.$$

Thm $\mathcal{B} \subseteq \mathcal{A}$ is Serre $\iff$ $\mathcal{B} = \operatorname{Ker} F$ for some exact $F : \mathcal{A} \to \mathcal{C}$.

The theorem follows from the following.

Thm Let $\mathcal{B} \subseteq \mathcal{A}$ be Serre. There exists a category $\mathcal{A}/\mathcal{B}$ -abelian & an exact $\mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{B}$ s.t. $\operatorname{Ker} Q = \mathcal{B}$ & it satisfies the UP:
$\forall$ $F : \mathcal{A} \to \mathcal{C}$ exact s.t. $F(\mathcal{B}) = 0$
$\exists!$ $H : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ s.t. $F = H \circ Q$.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{C} \\ & \mathcal{A}/\mathcal{B} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

<u>Warning</u> Set-theoretic issues. Insert appropriate words s.a. well-powered (subobjects of every object form a small set).

<u>Construction</u> (due to Serre)

Put $\mathrm{Ob}(\mathcal{A}/\mathcal{B}) = \mathrm{Ob}(\mathcal{A})$.

$$\mathrm{Mor}(X, Y) = \operatorname*{colim}_{\substack{X' \hookrightarrow X,\ X/X' \in \mathcal{B} \\ Y' \hookrightarrow Y,\ Y' \in \mathcal{B}}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$$

Good: 1) morphisms form an abelian group.
2) composition is given by the UP of colim.

Bad: very abstract, impossible to compute anything.

Alternative: localization.

2. Main properties of abelian categories

All kinds of lemmas

Prop (Snake lemma) Given

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

with exact rows

there is an induced exact sequence

$$\begin{tikzcd} \operatorname{Ker} a \arrow[r] & \operatorname{Ker} b \arrow[r] & \operatorname{Ker} c \arrow[dll, "\delta"'] \\ \operatorname{Coker} a \arrow[r] & \operatorname{Coker} b \arrow[r] & \operatorname{Coker} c \end{tikzcd}$$

Very hard to even construct δ without assuming we are in Mod-R.

If in Mod-R:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \operatorname{Ker} a \arrow[d] & \\
 & C \arrow[r] \arrow[d, "c"'] & B \arrow[r, "g"] \arrow[d, "b"] & A \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & B' \arrow[r, "g'"] & A' & \\
 & N & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{pick } y \arrow[r, mapsto, "g"] \arrow[d, mapsto] & x & x \arrow[l, mapsto] \\
z \arrow[r, mapsto] \arrow[d, mapsto] & b(y) \arrow[r, mapsto, "g'"] & 0 & \\
\delta(x) & & &
\end{tikzcd}$$

Need to show that δ is well-defined (preimage of x was a choice).

taking this is where we use that Freyd-Mitchell is fully faithful.

Exc Try to construct δ without appealing to Freyd-Mitchell.

Exc Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in $\mathcal{A}$ - abelian. Construct an exact sequence

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker} gf \to \operatorname{Coker} g \to 0.$$

__Cor__ (5-lemma)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, "d"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

if a & e are epi, d - mono then b is epi

Similarly you get

if b & d are mono, a - epi then c is mono.

3. Serre subcategories

Let $\mathcal{B} \subseteq \mathcal{A}$ be Serre.

__Def__ $f: A \to B$ in $\mathcal{A}$ is

1) $\mathcal{B}$-mono if $\ker f \in \mathcal{B}$,
2) $\mathcal{B}$-epi if $\operatorname{coker} f \in \mathcal{B}$,
3) $\mathcal{B}$-iso if $\mathcal{B}$-epi & $\mathcal{B}$-mono.

__Rmk__ If $F: \mathcal{A} \to \mathcal{C}$ is exact & $\mathcal{B} \subset \operatorname{Ker} F$, then the image of every $\mathcal{B}$-mono (epi/iso) in $\mathcal{C}$ is mon (epi/iso).

__Thought__ Instead of the UP $F(\mathcal{B}) = 0$ let's consider the UP $F(\mathcal{B}\text{-iso}) \subseteq \text{iso}$.

4. Localization

Let $\mathcal{C}$ be a category, Let S be a class of morphisms in $\mathcal{C}$ closed under composition & containing all identify $id_x \in S \quad \forall\, x \in \mathcal{C}$.

Want a universal category $S^{-1}\mathcal{C}$ & a functor $L: \mathcal{C} \to S^{-1}\mathcal{C}$ s.t. $L(S) \subseteq \text{Iso}$ ($L(s)$ is an iso $\forall s \in S$) & $\forall F: \mathcal{C} \to \mathcal{D}$ s.t. $F(S) \subseteq \text{Iso}_{\mathcal{D}}$ $\exists!$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "Q"'] & \mathcal{D} \\ S^{-1}\mathcal{C} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Such a pair $S^{-1}\mathcal{C}$, $\iota : \mathcal{C} \to S^{-1}\mathcal{C}$ is called the localization.
As usual, if exists, then unique up to <u>isom</u> of categories.

(Compare with localization of commutative rings.)

<u>Thm</u> Up to set-theoretic issues localization exists.

<u>Pf</u> Construction: put $Ob(S^{-1}\mathcal{A}) = Ob(\mathcal{A})$

Morphisms $X \to Y$

1) Consider the "set" of diagrams of the form

$$\begin{tikzcd}
 & Z_0 & & Z_1 & & \cdots & \\
X \arrow[ur, "f_0"] & & Y_0 \arrow[ul, "s_0"'] \arrow[ur, "f_1"] & & Y_1 \arrow[ul, "s_1"'] \arrow[ur] & & Y_n = Y \arrow[ul, "s_n"']
\end{tikzcd}$$

Where $s_i \in S$.
Think of such a chain as of
$$(s_n)^{-1} \circ \dots \circ (s_1)^{-1} \circ f_1 \circ (s_0)^{-1} \circ f_0 .$$

2) Composition is obvious: concatenation

$\Rightarrow$ associative

$$X \to \cdot \leftarrow \cdots \to \cdot \leftarrow Y \to \cdot \leftarrow \cdots \to \cdot \leftarrow Z$$

3) Put an equivalence relation

a) You can insert / remove

$$X \xrightarrow{s} Y \xleftarrow{s} X \sim X \qquad (s)^{-1} \circ s = \mathrm{id}_X$$

b)

$$Z \xleftarrow{s} Y \xrightarrow{\mathrm{id}} Y \xleftarrow{t} W \sim Z \xleftarrow{s \circ t} W \qquad \forall s, t \in S$$

c)

$$X \xrightarrow{f} Z \xleftarrow{\mathrm{id}} Z \xrightarrow{g} Y \sim X \xrightarrow{g \circ f} Y \qquad \forall f, g.$$

Generated by a, b, c.

<u>Exc</u> Persuade yourself that this thing with

the functor $\mathcal{C} \to S^{-1}\mathcal{C}$

$$X \longmapsto X$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \end{tikzcd} \longmapsto \begin{tikzcd} & Y & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "id_Y"'] \end{tikzcd}$$

satisfies the UP of localization. $\square$

<u>Problem</u> Can't compute anything + massive set-theoretic issues.

5. Calculus of fractions

<u>Def</u> A class of morphisms S in $\mathcal{C}$ is a <u>left localization system</u> if

1) $id_X \in S \quad \forall X \in \mathcal{C}$ & S closed under composition,

2)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ Z \arrow[r, dashed, "g"'] & W \end{tikzcd}$$

$s \in S \Rightarrow \exists$ a completed diag. with $t \in S$

$(g \circ s = t \circ f \rightsquigarrow t^{-1} g = f \circ s^{-1})$

3) if $X \xrightarrow{s} Y \overset{f}{\underset{g}{\rightrightarrows}} Z$ $\quad s \in S$ equalizes f & g: $fs = gs$

$\Rightarrow \exists \; Y \overset{f}{\underset{g}{\rightrightarrows}} Z \xrightarrow{t} W \quad t \in S$, t coeq. f & g.

<u>Prop</u> Consider diagrams of the form

$$\begin{tikzcd} & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd} , \qquad s \in S.$$

Will think of it as of $s^{-1} \circ f$.

If S is a left localization system, then the following is an equivalence relation:

$X \to Y' \leftarrow Y \;\sim\; X \to Y'' \leftarrow Y$ if

$$\begin{tikzcd} & Y' & \\ X \arrow[ur] \arrow[r] \arrow[dr] & Y''' \arrow[u] \arrow[d] & Y \arrow[ul, "s'"'] \arrow[l, "s'''"'] \arrow[dl, "s''"] \\ & Y'' & \end{tikzcd}$$

$s', s''', s'' \in S$.

Define composition:

$$\begin{tikzcd}[column sep=small] & & & W & & \\ & Y' \arrow[urr, dashed, "f'"] & & & Z' \arrow[ul, dashed, "s'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & & Z \arrow[ul, "t"'] \end{tikzcd}$$

$\leftarrow$ in S

the completion exists!

$s' \circ t \in S$

$$\rightsquigarrow \quad \begin{tikzcd} & W & \\ X \arrow[ur, "f' \circ f"] & & Z \arrow[ul, "s' \circ t"'] \end{tikzcd}$$

Prop This defines an associative composition law on these fractions $/\sim$.
Thus, one gets a category with $\mathrm{Ob} = \mathrm{Ob}\,\mathcal{C}$.
Morphisms: fractions$/\sim$.

Consider the functor $\mathcal{C} \to S^{-1}\mathcal{C}$,

$$X \to X, \quad X \xrightarrow{f} Y \longmapsto \begin{tikzcd} & Y & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "id"'] \end{tikzcd}$$

Exc Check the UP.

Rmk One could dually define right localization systems $\rightsquigarrow$ localization with morphisms given by equiv. classes of $X \xleftarrow{s} X' \xrightarrow{f} Y$, $s \in S$.

If S is a localization system (both left & right), by the UP the two localizations are isomorphic categories.

6. Quotients as localizations

<u>Lm</u> $\mathcal{B} \subseteq \mathcal{A}$ is Serre $\iff$ $\mathcal{B}$ is non-empty and $\forall$ $A' \to A \to A''$ if $A', A'' \in \mathcal{B} \Rightarrow A \in \mathcal{B}$.

<u>Pf</u> $\Leftarrow$ Only need to show that sub/quotients of objects in $\mathcal{B}$ are in $\mathcal{B}$.

1) $\mathcal{B} \neq \emptyset \Rightarrow \exists\ B \in \mathcal{B}$.
   $B \to 0 \to B \Rightarrow 0 \in \mathcal{B}$
2) $0 \to A' \to A \Rightarrow A' \in \mathcal{B}$ if $A \in \mathcal{B}$.
3) $A \to A'' \to 0$ same!

$$\Rightarrow \quad A' \xrightarrow{f} A \xrightarrow{g} A'' \qquad A', A'' \in \mathcal{B}$$

$$0 \to \ker f \to A' \to \operatorname{Im} f = \ker g \to 0 \qquad A' \in \mathcal{B} \Rightarrow \ker g \in \mathcal{B}$$

$$0 \to \operatorname{Im} g \to A'' \to \operatorname{Coker} g \to 0 \qquad A'' \in \mathcal{B} \Rightarrow \operatorname{Im} g \in \mathcal{B}$$

$$0 \to \ker g \to A \to \operatorname{Im} g \to 0 \qquad \Rightarrow A \in \mathcal{B}. \qquad \square$$

<u>Prop</u> If $\mathcal{B} \subset \mathcal{A}$ is Serre, the class of $\mathcal{B}$-iso is a localization system.

<u>Pf</u> Let's check the "complete the square", the rest - <u>exc</u>.

$$\begin{tikzcd}
\ker s \arrow[r, two heads] \arrow[d] & \ker t \arrow[d] \\
A \arrow[r] \arrow[d, "s"'] & C \arrow[d, "t"] \\
B \arrow[r] \arrow[d] & C \sqcup B \arrow[d] \\
\operatorname{Coker} s \arrow[r, "\sim"] & \operatorname{Coker} t
\end{tikzcd} \Rightarrow \operatorname{Coker} t \in \mathcal{B} \qquad \square$$

$\leftarrow$ epi $\leftarrow$ exercise

thus, $\ker t \in \mathcal{B}$ as a quotient of $\ker s \in \mathcal{B}$.

$s \in \mathcal{B}\text{-iso} \Leftrightarrow \ker s \in \mathcal{B}$, $\operatorname{Coker} s \in \mathcal{B}$

We can now define $\mathcal{A}/\mathcal{B}$ as $S^{-1}\mathcal{A}$, where $S = \mathcal{B}$-iso.

7. What are derived categories

Let $\mathcal{A}$ - abelian. Recall that we defined $C(\mathcal{A})$ - the catg of complexes: objects

$$\cdots \xrightarrow{d^i} X^i \xrightarrow{d^{i+1}} X^{i+1} \longrightarrow \cdots \qquad \text{s.t. } d^{i+1} \circ d^i = 0 \quad \forall i \in \mathbb{Z}.$$

Morphisms:

$$\begin{array}{ccccccc} \cdots \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \\ & f^i \downarrow & & \downarrow f^{i+1} & & \\ \cdots \longrightarrow & Y^i & \longrightarrow & Y^{i+1} & \longrightarrow & \cdots \end{array} \qquad d^i \circ f^i = f^{i+1} \circ d^i$$

<u>Def</u> The <u>i-th cohomology</u> of $X^\bullet \in C(\mathcal{A})$ is

$$H^i(X^\bullet) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}.$$

<u>Observation</u> H^i is a functor $C(\mathcal{A}) \to \mathcal{A}$ for all i.

Def $f: X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ is a quasi-isomorphism

if $H^i(f) : H^i(X^\bullet) \xrightarrow{\sim} H^i(Y^\bullet)$ $\forall\, i \in \mathbb{Z}$.

Def The derived category $D(\mathcal{A}) = S^{-1}C(\mathcal{A})$,
where S is the class of quasi-isom's.

Next week begin to study such things.