Triangulated
Lecture 9

0. Recap

Left / right localization systems in categories.
Left system:
1) $id_X \in S$ $\forall X \in \mathcal{C}$, S closed under comp
2)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ Z \arrow[r, dashed, "g"'] & W \end{tikzcd}$$
$s, t \in S$
3) $X \xrightarrow{s} Y \overset{f}{\underset{g}{\rightrightarrows}} Z \overset{t}{\dashrightarrow} W$

Dually – right localization system.

Define $S^{-1}\mathcal{C}$ (also denoted $\mathcal{C}[S^{-1}]$):
$Ob = Ob\,\mathcal{C}$

$$\mathrm{Hom}(X,Y) = \left\{ \begin{tikzcd}[column sep=small] & Y' & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \end{tikzcd} \;\middle|\; s \in S \right\} \Big/ \sim$$

Equivalence relation:

in the previous lecture there is a mistake!
the vertical arrows were reversed

$$\begin{tikzcd}
 & Y_1 \arrow[d] & \\
X \arrow[ur, "f"] \arrow[r] \arrow[dr, "g"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "t"'] \arrow[dl, "s_2"] \\
 & Y_2 \arrow[u] &
\end{tikzcd}$$

Need to show that it's an equivalence relation, define composition (show that it's well-def).

In order to show transitivity of $\sim$, need to show that

$$\begin{tikzcd}
 & Y_1 & \\
X \arrow[ur] \arrow[r] \arrow[dr] & Y_3 \arrow[u] \arrow[d] & Y \arrow[ul] \arrow[l] \arrow[dl] \\
 & Y_2 &
\end{tikzcd}$$

gives equivalent morphisms!

Two steps: 1) use

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_1 \arrow[d, dashed] \\ Y_2 \arrow[r, dashed] & Y_4 \end{tikzcd}$$

2) need to coequalize $Y \to Y_3 \rightrightarrows Y_4 \dashrightarrow Y_5$.

Application: construct $\mathcal{A}/\mathcal{B}$, $\mathcal{B}$ - Serre subcategory (full, closed under extensions).

Defined $\mathcal{B}$-iso's as $f: A \to B$ s.t. $\operatorname{Ker} f$, $\operatorname{Coker} f \in \mathcal{B}$.
One checks that $\mathcal{B}$-iso - both left & right localization system. $\mathcal{A}/\mathcal{B}$ & $\mathcal{A}[\mathcal{B}\text{-iso}^{-1}]$ satisfy the same UP.

<u>Q</u>: Why is $\mathcal{A}[\mathcal{B}\text{-iso}^{-1}]$ abelian? Easiest way - check that $\mathcal{A}/\mathcal{B}$ is abelian (you have explicit descriptions of $\mathcal{B}$-iso / $\mathcal{B}$-mono / $\mathcal{B}$-epi).

Finally, we defined $\mathcal{D}(\mathcal{A}) = C(\mathcal{A})[Q_{is}^{-1}]$

category of complexes

class of quasi-isom's

1. Complexes in abelian categories

<u>Def</u> $C(\mathcal{A})$ consists of objects

$$\cdots \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \qquad d^{i+1} \circ d^i = 0 \quad \forall i$$

morphisms

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r, "d"] \arrow[d, "f^i"] \arrow[dr, phantom, "\circlearrowleft"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

composition is obvious.

<u>Exc</u> $C(\mathcal{A})$ is abelian if $\mathcal{A}$ is abelian.

<u>Sketch</u> $\oplus$ are defined term-wise: $A^\bullet, B^\bullet \in C(\mathcal{A})$, then

$$A^\bullet \oplus B^\bullet : \quad \to A^i \oplus B^i \longrightarrow A^{i+1} \oplus B^{i+1} \longrightarrow$$

Ker & Coker are also term-wise!

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & K^i \arrow[r] \arrow[d] & K^{i+1} \arrow[r] \arrow[d] & {} \\
\arrow[r] & A^i \arrow[r] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & {} \\
\arrow[r] & B^i \arrow[r] \arrow[d] & B^{i+1} \arrow[r] \arrow[d] & {} \\
\arrow[r] & C^i \arrow[r] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 &
\end{tikzcd}$$

Ker f

Coker f

Please, check all the details.

$A^\bullet \in C(\mathcal{A}) \rightsquigarrow H^i(A^\bullet) = \ker d^i / \operatorname{Im} d^{i-1}$

$\ker d^i = Z^i(A^\bullet)$ called cycles

$\operatorname{Im} d^{i-1} = B^i(A^\bullet)$ called boundaries

Get a collection of functors $H^i : C(\mathcal{A}) \to \mathcal{A}$.

<u>Prop</u> Let $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ be a short exact sequence in $C(\mathcal{A})$. There exist morphisms $\delta : H^i(Z^\bullet) \to H^{i+1}(Z^\bullet)$ s.t. the following is exact

$$\cdots \xrightarrow{\delta} H^i(X^\bullet) \to H^i(Y^\bullet) \to H^i(Z^\bullet) \xrightarrow{\delta} H^{i+1}(X^\bullet) \to \cdots$$

and given

$$\begin{array}{ccccccccc} 0 & \to & X_1^\bullet & \to & Y_1^\bullet & \to & Z_1^\bullet & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X_2^\bullet & \to & Y_2^\bullet & \to & Z_2^\bullet & \to & 0 \end{array}$$

there is a morphism of LES's $\Leftarrow\!\Rightarrow$

all the squares

$$\begin{tikzcd} H^i(Z_1^\bullet) \arrow[r, "\delta"] \arrow[d] & H^{i+1}(X_1^\bullet) \arrow[d] \\ H^i(Z_2^\bullet) \arrow[r, "\delta"] & H^{i+1}(X_2^\bullet) \end{tikzcd}$$

commute.

<u>Pf</u> $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ is a SES, thus

$$\begin{tikzcd} 0 \arrow[r] & Z^i(X^\bullet) \arrow[r] \arrow[d] & Z^i(Y^\bullet) \arrow[r] \arrow[d] & Z^i(Z^\bullet) \arrow[d] & \\ 0 \arrow[r] & X^i \arrow[r] \arrow[d] & Y^i \arrow[r] \arrow[d] & Z^i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^{i+1} \arrow[r] \arrow[d] & Y^{i+1} \arrow[r] \arrow[d] & Z^{i+1} \arrow[r] \arrow[d] & 0 \\ & X^{i+1}/B^{i+1}(X^\bullet) \arrow[r] & Y^{i+1}/B^{i+1}(Y^\bullet) \arrow[r] & Z^{i+1}/B^{i+1}(Z^\bullet) \arrow[r] & 0 \end{tikzcd}$$

Stich these sequences into another 5-lemma type diagram:

$$
\begin{tikzcd}
 & H^i(X) \arrow[r] \arrow[d] & H^i(Y) \arrow[r] \arrow[d] & H^i(Z) \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
 & X^i/B^i(X) \arrow[r] \arrow[d, "d"'] & Y^i/B^i(Y) \arrow[r] \arrow[d, "d"'] & Z^i/B^i(Z) \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z^{i+1}(X) \arrow[r] \arrow[d] & Z^{i+1}(Y) \arrow[r] \arrow[d] & Z^{i+1}(Z) \arrow[d] & \\
 & H^{i+1}(X) \arrow[r] & H^{i+1}(Y) \arrow[r] & H^{i+1}(Z) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & y \arrow[r, mapsto, "\text{pick}"] \arrow[d, mapsto] & x \\
z \arrow[r, mapsto, "\text{exists}"] \arrow[d, mapsto] & d(y) & \\
\delta(x) & &
\end{tikzcd}
$$

How is δ constructed?

Conclusion:

$$
\begin{tikzcd}
0 \arrow[r] & X^i \arrow[r] \arrow[d] & Y^i \arrow[r] \arrow[d] & Z^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{i+1} \arrow[r] & Y^{i+1} \arrow[r] & Z^{i+1} \arrow[r] & 0
\end{tikzcd}
$$

$z \in H^i(Z)$ $\rightsquigarrow$ 1) pick a representative $\tilde{z} \in Z^i$
2) pull it back to Y^i
3) apply d_Y^i
4) pull it back to X^{i+1}. $\square$

Recall that long time ago we defined cones of morphisms.

<u>Shift functors</u> $[n] : C(\mathcal{A}) \to C(\mathcal{A})$

$$(X^\bullet[n])^i = X^{i+n}, \quad d_{X[n]} = (-1)^n d_X .$$

Gives an action of $\mathbb{Z}$ on $C(\mathcal{A})$: $\mathbb{Z} \to \mathrm{Aut}(C(\mathcal{A}))$.

<u>Cone</u> $f : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ $\rightsquigarrow$ $C^\bullet(f) \in C(\mathcal{A})$

$$C^i(f) = X^{i+1} \oplus Y^i, \quad d = \begin{pmatrix} -d_X^{i+1} & 0 \\ f^i & d_Y^i \end{pmatrix}$$

<u>Exc</u> Check that $d^2 = 0$

More importantly, there is a SES of complexes:

$$0 \to Y^\bullet \to C^\bullet(f) \to X[1] \to 0$$

The associated LES of cohomology is

$$\dots \to H^i(Y^\bullet) \to H^i(C^\bullet(f)) \to \underset{\substack{\| \\ H^{i+1}(X)}}{H^i(X[1])} \xrightarrow{\delta} H^{i+1}(Y) \to \dots$$

Exc Using the explicit description of δ, check that for $0 \to Y \to C(f) \to X[1] \to 0$ the connecting homomorphism is $\delta = f_*$, the morphism $H^i(X) \to H^i(Y)$!

Recall that $f: X \to Y$ is a quasi-iso if $H^i(f): H^i(X) \to H^i(Y)$ - iso $\forall\, i \in \mathbb{Z}$.

We conclude that $f : X \to Y$ – quasi-iso $\Leftrightarrow$

$\Leftrightarrow$ $H^i(C(f)) = 0 \quad \forall i.$

$$\overset{\sim}{\Rightarrow} \quad \dots \to \underbrace{H^i(C(f))}_{\text{must be } 0!} \to H^{i+1}(X) \xrightarrow[\sim]{H^i(f)} H^{i+1}(Y) \to H^{i+1}(C(f)) \to \dots$$

Rmk the "analogy" with quotients by Serre subcategories. We want to invert quasi-isom's, looks like it is enough to kill acyclic complexes.

$$f : X \to Y \qquad \rightsquigarrow 0 \to Y \to C(f) \to X[1] \to 0$$

if qis $\Rightarrow$ must go to 0?

Also remark that $X^\bullet \in C(\mathcal{A})$ is acyclic (all $H^i(X^\bullet) = 0$) iff $0 \to X^\bullet$ is a quasi-isomorphism.

Inverting qis's should kill all acyclics.

Problem In $C(\mathcal{A})$ qis's do not form a localization system (at least, not obvious).

2. Homotopy category

Given $X^\cdot, Y^\cdot \in C(\mathcal{A})$, one can produce a <u>complex</u>

$$\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot) \in C(\mathcal{A}b):$$

$$\mathrm{Hom}^n(X^\cdot, Y^\cdot) = \prod \mathrm{Hom}(X^i, Y^{i+n}) \leftarrow \text{collection of } X^i \to Y^{i+n},$$

no relations / commutation with d's.

Given $\varphi \in \mathrm{Hom}^n(X^\cdot, Y^\cdot)$, put $d\varphi$ by the rule

$$\begin{tikzcd} \ldots \arrow[r] & X^i \arrow[r] \arrow[dl, "\varphi^i"'] & X^{i+1} \arrow[r] \arrow[dl, "\varphi^{i+1}"'] & \ldots \\ Y^{i+n} \arrow[r] & Y^{i+n+1} \arrow[r] & \ldots \end{tikzcd}$$

$$\boxed{(d\varphi)^i = d\varphi^i - (-1)^i \varphi^{i+1} d}$$

Exc Check that $\mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$ is indeed a complex.

Look at its lower terms.

$$Z^0(\mathrm{Hom}^\bullet(X, Y)) = \{ \varphi^i : X^i \to Y^i \mid d\varphi^i - \varphi^{i+1} d = 0 \}$$

$$\shortparallel$$

$$\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$$

$$B^0(\mathrm{Hom}^\bullet(X, Y)) = \{ dh^i + h^{i+1} d : X^i \to Y^i \mid h^i : X^i \to Y^{i-1} \}$$

morphisms homotopic to zero!

We discussed that there is an equivalence relation on $\mathrm{Hom}(X, Y)$ $f \sim g \iff f - g$ is homotopic to zero.

Morphisms homotopic to 0 form an ideal:

$f \sim g \Rightarrow hf \sim hg$, $fw \sim gw$

Observation $f \sim 0 \Rightarrow H^i(f) = 0$ for all $i \in \mathbb{Z}$.

Thus, $f \sim g$, then $H^i(f) = H^i(g)$ for all $i \in \mathbb{Z}$.

Define $K(\mathcal{A})$ as the category whose objects are complexes, morphisms – morphisms $/\sim$.

$$\mathrm{Hom}_{K(\mathcal{A})}(X, Y) = H^0(\mathrm{Hom}^\bullet(X, Y)).$$

Problem 5 Show that in $K(\mathcal{A})$ quasi-isomorphisms form a localization system.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"', "\text{qis}"] & Y \arrow[d, dashed, "t"] \\ Z \arrow[r, dashed, "g"'] & W \end{tikzcd}$$

$tf \sim gs$ homotopic, not equal

Exc Check that $K(\mathcal{A})[\mathrm{Qis}^{-1}] = D(\mathcal{A}) = C(\mathcal{A})[\mathrm{Qis}^{-1}]$!

<u>Hint</u> $K(\mathcal{A})$ also satisfies some UP.

3. Resolutions

Recall that we have a functor $\mathcal{A} \to C(\mathcal{A})$, exact, which sends $X \mapsto \ldots \to 0 \to X \to 0 \to \ldots$

0th term

<u>Def</u> A resolution of X is any complex

$R^\bullet \in C(\mathcal{A})$ qis to X.

<u>Def</u> A projective resolution of X - resolution of the form

$$\to P^{-3} \to P^{-2} \to P^{-1} \to P^0 \to 0 \to \ldots, \qquad P^i\text{-proj.}$$

an injective resolution of X - resolution of the form

$$\ldots \to 0 \to I^0 \to I^1 \to \ldots \qquad I^i\text{-injective.}$$

<u>Recall</u> $\mathcal{A}$ has enough projectives / injectives if $\forall A \in \mathcal{A}$ $\exists$ $P \twoheadrightarrow A$ / $A \twoheadrightarrow I$ with P - projective / I - injective.

If $\mathcal{A}$ has enough projectives / injectives $\Rightarrow$ $\Rightarrow$ every object has a projective / injective resolution.

<u>Prop</u> Mod-A & A-Mod have enough projectives.

<u>Pf</u> Free modules are projective, every module is a quotient of a free module.

<u>Thm</u> A-Mod & Mod-A have enough injectives.

Side remark: projectives are easier to deal with (morally), but in many important situations sheaves of Ab e.g., there is not enough of them.

Prop (Baer's criterion)

$I \in A\text{-Mod}$ is injective $\Leftrightarrow$ satisfies the UP for all $I \hookrightarrow A$, I - ideal.

Pf $\Rightarrow$ immediate

$\Leftarrow$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\iota"] \arrow[d, "f"'] & Y \\ & I & \end{tikzcd}$$

consider the maximal submodule in Y containing X for which the ext exists (Zorn).

WLOG we may assume that f does not extend to any larger submodule.

If $Y \neq X \Rightarrow \exists\, y \in Y \setminus X$.

Consider $0 \to I \to X \oplus A \to \underset{\underset{X+(g)}{\cup}}{X'} \to 0$

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr, bend left, "f"] & \\
 & & I \\
A \arrow[r] \arrow[ur, dashed, phantom, "\exists"] & X' \arrow[ur, dashed] &
\end{tikzcd}$$

This square is both cartesian and cocartesian. $\square$

<u>Rmk</u> A very similar argument (tough on ordinals) shows that in any Grothendieck category injective envelopes exist!

Application to A-Mod:

1) Check that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathcal{A}b$!

2) $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ are adjoint & F is exact, then G sends injectives to injectives.

3) Consider functors

$$F: A\text{-Mod} \to (\text{Mod-}A)^\circ \qquad G: (\text{Mod-}A)^\circ \to A\text{-Mod}$$

$$M \mapsto \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$$

Check that they are adjoint & exact.

Use that $(\text{Mod-}A)^\circ$ has enough injectives:

same as projectives in Mod-A.