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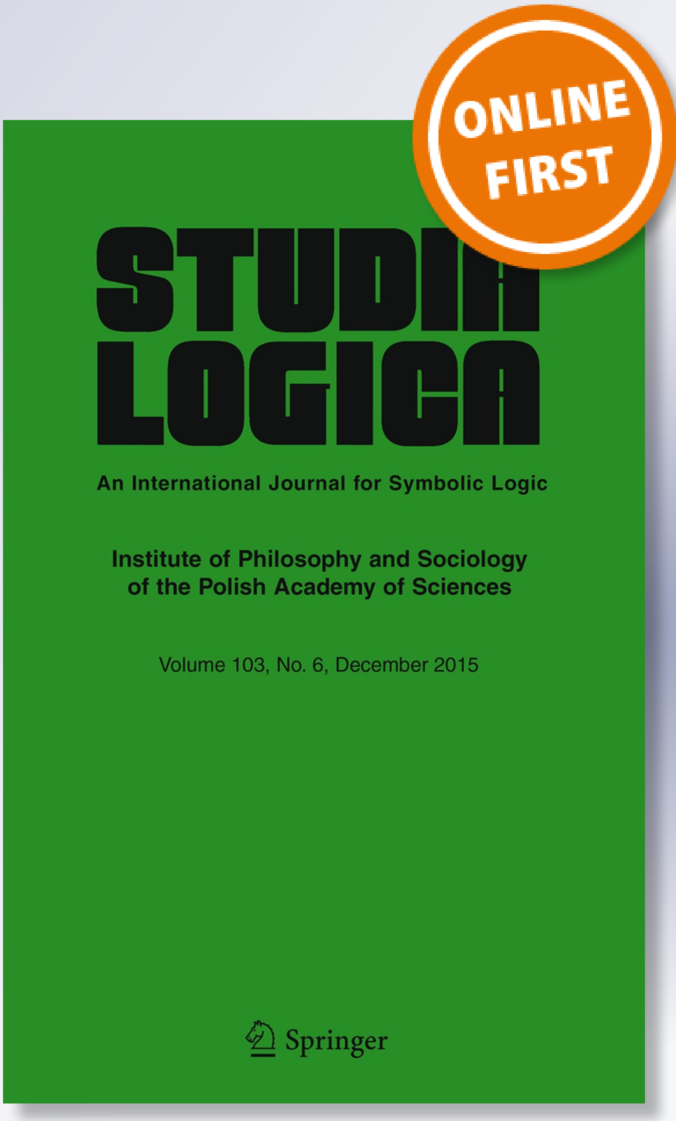
Studia Logica
An International Journal for Symbolic Logic

ISSN 0039-3215

Stud Logica
DOI 10.1007/s11225-016-9654-3



Springer

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Abstract. Franco Montagna, a prominent logician and one of the leaders of the Italian school on Mathematical Logic, passed away on February 18, 2015. We survey some of his results and ideas in the two disciplines he greatly contributed along his career: provability logic and many-valued logic.

Keywords: Provability Logic, Many-valued Logic, Franco Montagna.

1. Introduction

Figure 1. Franco Montagna at a conference in Prague

This paper is a tribute to the memory of our long-time friend and colleague Franco Montagna (1948–2015). Franco Montagna graduated in Mathematics in 1972 from the University of Pavia (Italy) under the supervision of Roberto Magari and then moved to Siena where, with Magari and other colleagues, he founded and led, from 1987 to 1990, a specialization school in Mathematical Logic. In 1987, Franco was appointed full professor of Mathematical Logic; from 1994 to 2002 he served as the coordinator of the PhD program in Mathematical Logic and Theoretical Computer Science at the University of Siena. He supervised nine PhD students: Antonella Mancini, Sandra Fontani, Alessandro Agostini, Claudio Marini, Lorenzo Carlucci, Luca Spada, Tommaso Flaminio, Simone Bova

Presented by **Daniele Mundici**; *Received* October 28, 2015

DOI: 10.1007/s11225-016-9654-3

and Martina Fedel. He also served as a host for several postdoc fellows and visitors.

Over the years of his work at the University of Siena until his untimely death, he contributed in many ways to make that small Italian town a special place for Mathematical Logic. Several international workshops that he hosted at *Certosa di Pontignano* are being remembered by their participants for the unique atmosphere of Italian hospitality and the beautiful countryside of Tuscany.

Franco himself, with his quiet and introvert demeanour, in personal contacts was an example of modesty and sincerity. He was particularly modest about his own achievements. However, along his career, Franco published about 120 papers that appeared in international journals of Logic, Algebra and Computer Science and ranged over many different topics. In this paper we would like to review some of his main scientific contributions, mostly focusing on two topics that attracted Franco's attention during different periods of his life.

Franco's early work (from the second half of the 1970s until the end of the 1990s) follows the line of study initiated by his teacher Roberto Magari and focused on Provability Logic. From the end of the 1990s, Franco switched his main interests to many-valued logics and uncertain reasoning. Franco was attracted by the link between many-valued logics and probability theory and, in particular, by the generalization of de Finetti's coherence criterion to a nonclassical logical setting.

In this review we could not possibly cover all of Franco's contributions to provability logic and many-valued logics. The current selection of topics naturally reflects biases of the authors. We tried to combine a survey with some glimpses into Franco's own arguments and ideas. This task was not easy, and we acknowledge that sometimes one side goes at the expense of the other. We hope that Franco's legacy will be kept alive through other expositions and research papers based on his ideas.

1.1. Half of the 1970s to the end of the 1990s: Provability logic

In the 1970s, Magari introduced a class of algebraic structures, which he called *diagonalizable algebras* and which are nowadays mostly called *Magari algebras*. The main motivation to study these objects comes from the interest and applications in formal arithmetic and proof theory, and in the representation of the phenomenon of self-reference exhibited by the famous incompleteness theorems of Kurt Gödel. Magari and his students embarked on an ambitious project to approach the phenomenon of self-reference and provability in formal arithmetic from an algebraic point of view. Interesting results soon followed. Thus, the Italian school of provability logic emerged, and in the years to follow it would become one of the main centers of this discipline in the world.

From the second half of the 1970s until the end of the 1990s Franco Montagna's main contributions were in the field of provability logic. He provided a number of fruitful ideas and initiated several lines of study that determined the direction of efforts of other researchers in provability logic and were successfully completed later. To mention a few of his results (grouped thematically rather than in the order of importance):

- A proof of the so-called *uniform* version of Solovay's arithmetical completeness theorem for provability logic. This theorem was independently discovered by S. Artemov, G. Boolos, A. Visser.
- A proof that the first-order theory of the class of diagonalizable algebras is undecidable. This work preceded the negative results on the first-order theory of the diagonalizable algebra of PA by V. Shavrukov.
- First negative results on predicate provability logic; in particular, a proof that the predicate version of Gödel–Löb provability logic is not arithmetically complete, and that PA and ZF have different predicate provability logics. This work preceded the results by S. Artemov and V. Vardanyan on the non-invariance and the undecidability of predicate provability logics.
- The study of non-standard provability predicates from the algebraic and provability logical point of view. In one instance, this is the study of the so-called Feferman provability, taken up by A. Visser and V. Shavrukov. In another instance, this is the bimodal study of provability in PA together with what Franco describes as a 'generic finite subtheory of PA'. The second development influenced the emergence and extensive study of *interpretability logic* to which several logicians (including A. Visser, D. de Jongh, F. Veltman, A. Berarducci, V. Shavrukov, K. Ignatiev, P. Hájek and Franco Montagna himself) later gave significant contributions.
- The study of new fixed-point constructions and speed-up results in arithmetic by means of provability logic (jointly with D. de Jongh, P. Hájek, P. Pudlák, A. Carbone and others).

This list is not exhaustive, but gives the gist of what seems to be his most memorable and influential contributions to provability logic. However, some of his no less remarkable results of the same period extending outside the field of provability logic should also be mentioned. Among them the study of positive equivalence relations and a recursion-theoretic characterization of the provable equivalence relation in PA, the study of fixed-point algebras, contributions to the theory of inductive learning, recursive progressions of theories, a universal-algebraic treatment of the theories of quantifier-free arithmetic, the study of weak set theories, and several others.

1.2. End of the 1990s onwards: Many-valued logic and generalized probability theory

Many-valued logics are systems where propositions can be evaluated in intermediate values between the classical 0 (for false) and 1 (for true). Even though the birth of these logics can be traced back to the 1930's [72], it is since 1998, when P. Hájek published his monograph [53], that *Mathematical Fuzzy Logic* (MFL) has become a well-established sub-discipline of Mathematical Logic (see also [25, 26]). MFL aims at studying those many-valued logics in which truth-values are comparable (the real unit interval $[0, 1]$ is usually taken as truth-values set), connectives are interpreted by $[0, 1]$-valued operations (of a given arity) and, in particular, the connective of *strong conjunction* (or *fusion* operator) is interpreted by a (left-)continuos t-norm (cf. [64] and Definition 3.1 below).

Since 1998, MFL encompassed formalisms that, while maintaining the fundamental features framing its common ground, have increased in generality. Amongst them we recall Hájek *basic logic* BL [53] (the logic of continuous t-norms) and its corresponding variety of algebras $\mathbb{BL}$, and Esteva and Godo's *monoidal t-norm based logic* MTL [38] (the logic of left-continuous t-norms) and its corresponding variety $\mathbb{MTL}$.

Franco Montagna started his research activity in this discipline in 1999 and one of his first papers on MFL [85], titled *"An algebraic approach to propositional fuzzy logics"*, revealed his approach to MFL to be algebraic (and hence semantical) in nature. In this frame, Franco, in a series of coauthored papers, provided important results concerning the structure of totally ordered BL-algebras [1, 2], he introduced a single totally ordered BL-algebra that generates $\mathbb{BL}$, he proved strong standard completeness[1] for Esteva and

[1] A logic L (for instance any schematic extension of MTL) is said to be *standard complete* if L is complete with respect to those totally ordered L-algebras the carrier of which is the real unit interval. *Strong completeness* means that completeness is preserved by taking deductions from infinite theories, i.e. if Γ is a denumerable set of formulas and φ is a formula, then $\Gamma \models \varphi$ in all L-algebras implies $\Gamma \vdash \varphi$ in L.

Godo's MTL [60], he studied expansion of MV-algebras (the equivalent algebraic semantics of Łukasiewicz logic) by means of a product operator [85, 89], he introduced GBL-algebras [61], a further generalization of BL-algebras, etc. In Section 3 we will see some details concerning the decomposition of totally ordered BL-algebras and the strong standard completeness theorem for MTL.

States of MV-algebras have been introduced by D. Mundici in [101]. Given an MV-algebra $\mathbf{A}$, its states are normalized and additive functions mapping A in the real unit interval $[0, 1]$ (see [101, 104] and Section 4 for details). States play for MV-algebras the same rôle as finitely additive probability measures do for boolean algebras. Moreover, as shown by Mundici in [103], states are also related to probability. Indeed, he shows that a *book* over many-valued events can be extended to a state iff there is no *Dutch-book* for it.

Franco Montagna investigated state theory intensively (see [40, 58, 92, 41, 93]) and his papers on this topic covered several areas ranging from the foundational aspects to a purely algebraic approach of MV-algebraic states. In Section 4 we will present his main results, while Subsection 4.3 is dedicated to recap on further contributions and generalizations he provided until his last research topic in this subject: the generalization of *strict coherence* to many-valued events.

Beyond the two main research topics mentioned above, namely the algebraic study of many-valued logics and state theory, Franco's contributions to MFL covered several areas ranging from proof theory [21, 22], to higher order logics [56, 87, 88], computational complexity [9, 17, 95], game semantics [27, 94], categorical approach to many-valued logics [99, 100], and others.

2. Provability logic

2.1. Basic provability logic

Let T be a recursively enumerable (r.e.) extension of a sufficiently strong fragment of Peano arithmetic. For such a fragment one usually takes elementary arithmetic EA (or $\mathrm{I}\Delta_0 + \exp$). For simplicity, we can assume that T is formulated in the language of PA, however most of the results mentioned below hold under the weaker assumption that EA is relatively interpretable in T. We assume throughout this paper a fixed natural and elementary Gödel numbering of the language of T (variables, terms, formulas). Following [42], given an r.e. representation of the set of axioms of T one can naturally write out an arithmetical Σ_1-formula $\mathrm{Pr}_T(x)$, representing the set of Gödel numbers of theorems of T, or the *provability predicate for T*. The consistency of T is then expressed by the formula $\neg\mathrm{Pr}_T(\ulcorner\bot\urcorner)$, abbreviated Con_T, where $\bot$ denotes logical absurdity (or any refutable sentence such as $0 = 1$). A theory T is called Σ_1-*sound* iff all its Σ_1-theorems are true in the standard model of arithmetic.

The provability predicate enjoys three natural properties called the *Hilbert–Bernays–Löb derivability conditions* [71]:

D1. $T \vdash \phi$ implies $\mathrm{EA} \vdash \mathrm{Pr}_T(\ulcorner\phi\urcorner)$;

D2. $\mathrm{EA} \vdash \mathrm{Pr}_T(\ulcorner\phi \to \psi\urcorner) \to (\mathrm{Pr}_T(\ulcorner\phi\urcorner) \to \mathrm{Pr}_T(\ulcorner\psi\urcorner)$;

D3. $\mathrm{EA} \vdash \mathrm{Pr}_T(\ulcorner\phi\urcorner) \to \mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\phi\urcorner)\urcorner)$.

A well-known consequence of D1–D3 and the diagonalization lemma is a theorem of Löb stating that, for each formula ϕ, $T \vdash \mathrm{Pr}_T(\ulcorner\phi\urcorner) \to \phi$ iff $T \vdash \phi$. This theorem can be seen as a general version of Gödel's second incompleteness theorem. In particular, substituting $\bot$ for ϕ we observe that the consistency of T, that is, Con_T, is provable in T only if T is inconsistent. A formalization of Löb's theorem is provable in EA, which yields the following well-known statement.

Proposition 2.1. $\mathrm{EA} \vdash \mathrm{Pr}_T(\ulcorner\mathrm{Pr}_T(\ulcorner\phi\urcorner) \to \phi\urcorner) \leftrightarrow \mathrm{Pr}_T(\ulcorner\phi\urcorner)$.

Macintyre and Simmons [73] and Magari [74] took a very natural algebraic perspective on the phenomenon of formal provability which eventually led the latter to the concept of *diagonalizable algebra*. Recall that the Lindenbaum–Tarski algebra of a theory T is the set of all T-sentences Sent_T modulo provable equivalence in T, that is, the structure $\mathcal{L}_T = \mathrm{Sent}_T/{\sim_T}$ where, for all $\phi, \psi \in \mathrm{Sent}_T$,

$$\phi \sim_T \psi \iff T \vdash \phi \leftrightarrow \psi.$$

Since we assume T to be based on classical propositional logic, $\mathcal{L}_T$ is a boolean algebra with operations $\wedge$, $\vee$, $\neg$. Constants $\bot$ and $\top$ are identified with the sets of refutable and of provable sentences of T, respectively. The standard ordering on $\mathcal{L}_T$ is defined by

$$[\phi] \leq [\psi] \iff T \vdash \phi \to \psi \iff [\phi \wedge \psi] = [\phi],$$

where $[\phi]$ denotes the equivalence class of ϕ.

As a consequence of the first two derivability conditions the formula Pr_T is *extensional*, in other words $\phi \sim_T \psi$ implies $\mathrm{Pr}_T(\ulcorner\phi\urcorner) \sim_T \mathrm{Pr}_T(\ulcorner\psi\urcorner)$.

Hence, the provability predicate for T correctly defines an operator $\Box_T$ acting on the equivalence classes of $\mathcal{L}_T$:

$$\Box_T : [\phi] \mapsto [\mathrm{Pr}_T(\ulcorner \phi \urcorner)].$$

The boolean algebra $\mathcal{L}_T$ enriched by the operator $\Box_T$ is called the *Magari algebra of* T. This structure satisfies the following identities (where we write $\Box$ for $\Box_T$):

M1. $\Box\top = \top$

M2. $\Box(x \wedge y) = \Box x \wedge \Box y$

M3. $\Box(\Box x \to x) = \Box x$

The first two identities correspond to the first two derivability conditions, whereas the third one is a formalization of Löb's theorem (Proposition 2.1). The third derivability condition corresponds to the identity $(\Box x \to \Box\Box x) = \top$ that holds in $(\mathcal{L}_T, \Box_T)$, however it follows from the three identities above.

Definition 2.2. A *modal algebra* is a boolean algebra B enriched by an operator $\Box : B \to B$ satisfying identities *M1* and *M2*. A *Magari algebra* is a modal algebra satisfying *M3*.

Algebras of the form $\mathcal{M}_T = (\mathcal{L}_T, \Box_T)$, for a consistent r.e. arithmetical theory T, are primary examples of Magari algebras.

Terms in the language of modal algebras are naturally identified with formulas of propositional logic enriched by a unary connective $\Box$. If $\phi(\vec{x})$ is such a formula and $\mathcal{M}$ a Magari algebra, we write $\mathcal{M} \models \phi$ iff $\forall \vec{x}\, (t_\phi(\vec{x}) = \top)$ is true in $\mathcal{M}$, where t_ϕ is the term corresponding to ϕ. Since any identity in Magari algebras can be equivalently written in the form $t = \top$, for some term t, the axiomatization of identities of $\mathcal{M}$ amounts to axiomatizing modal formulas valid in $\mathcal{M}$. The *logic of* $\mathcal{M}$, $\mathrm{Log}(\mathcal{M})$, is the set of all modal formulas valid in $\mathcal{M}$, that is, $\mathrm{Log}(\mathcal{M}) = \{\phi : \mathcal{M} \models \phi\}$. The logic of a class of modal algebras consists of all modal formulas valid in every algebra of the class.

Gödel–Löb provability logic **GL** can be defined as the logic of the class of all Magari algebras. Its standard Hilbert-style axiomatization consists of the axioms of classical propositional logic together with the following principles (for all modal formulas ϕ, ψ):

L1. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;

L2. $\Box(\Box\phi \to \phi) \to \Box\phi$.

The inference rules of **GL** are modus ponens and *necessitation*: from ϕ infer $\Box\phi$. The modal logic **K** is obtained by dropping Axiom L2 from the above axiomatization of **GL**.

PROPOSITION 2.3. (i) ***K*** $\vdash \phi$ *iff* $\mathcal{M} \models \phi$ *for every modal algebra* $\mathcal{M}$*;*
(ii) **GL** $\vdash \phi$ *iff* $\mathcal{M} \models \phi$ *for every Magari algebra* $\mathcal{M}$*.*

Apart from its algebraic semantics, **GL** also has a very convenient Kripke semantics. Kripke models are both a source of examples and an important tool in the study of Magari algebras and provability logics.

Recall that a *Kripke frame* is a structure (W, R) where W is a nonempty set and R is a binary relation on W. A *Kripke model* $\mathcal{W} = (W, R, v)$ consists of a Kripke frame together with an evaluation $v : \mathrm{Var} \to \mathcal{P}(W)$ assigning to each variable p a set $v(p) \subseteq W$ of nodes where p is stipulated to be true. The evaluation of variables is extended to the evaluation of arbitrary modal formulas inductively according to the following clauses, where $\mathcal{W}, x \models \phi$ reads ϕ *is true at* x *in* $\mathcal{W}$:

K1. $\mathcal{W}, x \models p \iff x \in v(p)$;

K2. $\mathcal{W}, x \models \phi \wedge \psi \iff (\mathcal{W}, x \models \phi$ and $\mathcal{W}, x \models \psi)$;

K3. $\mathcal{W}, x \models \neg\phi \iff \mathcal{W}, x \not\models \phi$;

K4. $\mathcal{W}, x \models \Box\phi \iff \forall y \in W\ (xRy \Rightarrow \mathcal{W}, y \models \phi)$.

We write $\mathcal{W} \models \phi$ if $\mathcal{W}, x \models \phi$, for each $x \in W$. The set of modal formulas true in all Kripke models coincides with the set of theorems of basic modal logic **K**. By a well-known result of Segerberg [112], **GL** is sound and complete w.r.t. the class of all transitive and upwards well-founded Kripke frames. Recall that a Kripke frame $\mathcal{W} = (W, R)$ is *upwards well-founded*, if there is no infinite sequence of nodes $a_i \in W$ such that $a_0 R a_1 R a_2 \ldots$. In fact, **GL** can also be characterized by the class of frames that are finite irreflexive trees.

PROPOSITION 2.4 (Segerberg). *For all formulas* ϕ*, the following conditions are equivalent:*

(i) **GL** $\vdash \phi$*;*

(ii) $\mathcal{W} \models \phi$*, for each model* $\mathcal{W} = (W, R, v)$ *such that* (W, R) *is transitive and upwards well-founded;*

(iii) $\mathcal{W} \models \phi$*, for each model* $\mathcal{W} = (W, R, v)$ *such that* (W, R) *is a finite irreflexive tree.*

With every Kripke frame (W, R) we can associate a modal algebra consisting of the boolean algebra $\mathcal{P}(W)$ of all subsets of W together with the operator $\Box_R : \mathcal{P}(W) \to \mathcal{P}(W)$ defined by

$$\Box_R(X) = \{x \in W : \forall y \in W\ (xRy \Rightarrow y \in X)\}, \tag{1}$$

for all $X \in \mathcal{P}(W)$. If (W, R) is transitive and upwards well-founded, then the corresponding modal algebra $\mathcal{M} = (\mathcal{P}(W), \Box_R)$ is, in fact, a Magari algebra. Moreover, the truth of formulas in the algebra and in the Kripke model agree in the following sense: For any modal formula $\phi(\vec{x})$ and any evaluation v on W,

$$\mathcal{M} \models t_\phi(v(x_1), \ldots, v(x_n)) = \top \iff (W, R, v) \models \phi(x_1, \ldots, x_n).$$

Hence, the logics of the Kripke frame and of the associated algebra are the same.

2.2. Uniform arithmetical completeness theorem

Having isolated the class of Magari algebras, an immediate and challenging problem was to check whether the identities of Magari algebras exhaust all the identities valid in the Magari algebra of PA. A negative answer to this question would have meant that there were some 'unaccounted for' identities of $\mathcal{M}_{\mathrm{PA}}$ and that the notion of Magari algebra was not yet the right one. In the universal algebraic terms, this is the question whether $\mathcal{M}_{\mathrm{PA}}$ is functionally free in the variety of all Magari algebras. A logical form of the same problem is the question whether the modal logic **GL** is *arithmetically complete*. One of the very first papers by Franco Montagna, in fact, showed that the free Magari algebra on n generators is not functionally free in the class of Magari algebras; the same paper contained some ideas of approaching the main problem above.

However, the great impetus came very soon in 1976 from Robert Solovay who gave a positive answer to this problem [121]. The following formulation incorporates two later improvements. The first one is the extension of the result to the class of r.e. theories containing EA, due to Dick de Jongh, Mark Jumelet and Franco Montagna [29]. The second one is due to Albert Visser, who observed that Solovay's theorem not only holds for PA and Σ_1-sound theories T, but also under a weaker (and in fact necessary) condition of having *infinite characteristic*.

Recall that the *characteristic* of a Magari algebra $\mathcal{M}$ is the least number n such that $\Box^{n+1}\bot = \top$ holds in $\mathcal{M}$. If such a number does not exist we say that $\mathcal{M}$ has infinite characteristic. The same terminology is applied to theories T whenever the Magari algebra $\mathcal{M}_T$ of T enjoys the corresponding property. Σ_1-sound theories such as PA have infinite characteristic.

Proposition 2.5 (Solovay). *If T is of infinite characteristic, then* $\mathrm{Log}(\mathcal{M}_T) = \mathbf{GL}$.

Solovay's method of proof gives more than was just stated. Given a modal formula $\phi(x_1, \ldots, x_n)$ with all the variables shown and arithmetical sentences $A_1, \ldots, A_n$ let $\phi^*(A_1, \ldots, A_n)$ denote the result of substituting $A_1, \ldots, A_n$ for $x_1, \ldots, x_n$ in ϕ and of translating $\Box$ as $\mathrm{Pr}_T(\ulcorner \cdot \urcorner)$. Thus, $\phi^*(A_1, \ldots, A_n)$ is an arithmetical sentence and

$$\mathcal{M}_T \models t_\phi([A_1], \ldots, [A_n]) = \top \iff T \vdash \phi^*(A_1, \ldots, A_n).$$

Solovay's method shows that there is a recursive procedure that, for any modal formula $\phi(x_1, \ldots, x_n)$ unprovable in $\mathbf{GL}$, produces a sequence of arithmetical sentences $A_1, \ldots, A_n$ such that $T \nvdash \phi^*(A_1, \ldots, A_n)$. Moreover, these sentences are boolean combinations of arithmetical Σ_1-sentences.[2] We call these sentences *Solovay's evaluation of variables of ϕ.*

Franco Montagna used this additional information to prove the following theorem [80], which is a very natural strengthening of Solovay's theorem.

Theorem 2.6 (Uniform arithmetical completeness). *If T is an r.e. extension of* EA *of infinite characteristic, then the free Magari algebra on countably many generators is embeddable into $\mathcal{M}_T$.*

We sketch the ingenious idea of his proof, which is rather different from the other (independently found) proofs of this theorem. Unlike the proofs involving the Solovay construction on an infinite Kripke model, this proof uses diagonalization on top of the existing construction. An advantage is that it applies to a somewhat wider class of arithmetical theories T.[3]

Proof. Let $f(v, i) = \ulcorner A_i \urcorner$, in case v is the Gödel number of a modal formula $\phi(x_0, \ldots, x_{n-1})$ such that $\mathbf{GL} \nvdash \phi$, $i < n$, and A_i is Solovay's evaluation of variable x_i in ϕ. Otherwise, let $f(v, i) = 0$. Clearly, f is a recursive function representable in EA.

[2] The fact that Solovay's construction on a finite model produces boolean combinations of Σ_1-sentences was not so apparent at the beginning: Franco in his paper talks about Σ_2-sentences. This property must have already been known to Japaridze [59], however it is particularly obvious from the simple variant of Solovay's construction presented in [29].

[3] A proof by Albert Visser also exploited a similar idea.

The required embedding will be given by a sequence of arithmetical sentences of the form $B(0), B(1), \ldots, B(\overline{n}), \ldots$ We are going to define the formula B by means of the arithmetical fixed point theorem.

Let $H(x, v, z)$ hold if $x = \ulcorner B \urcorner$, where B is an arithmetical formula in one free variable, v is the Gödel number of a modal formula ϕ such that $\mathbf{GL} \nvdash \phi$, z is the Gödel number of a T-proof of $\phi^*(B(0), \ldots, B(\overline{i-1}))$, for an appropriate i, and there is no $y < z$ which is a T-proof of $\psi^*(B(0), \ldots, B(\overline{j-1}))$, for any $\mathbf{GL}$-unprovable ψ. The relation H is recursive and, for each x, there is at most one pair (v, z) satisfying $H(x, v, z)$ (so, H is the graph of a partial recursive function). As above, we fix an arithmetical formula representing this function in EA.

Let $\mathrm{Tr}(x)$ denote the partial truth definition for boolean combinations of Σ_1-sentences. Let $B(x)$ be defined as a solution of the following fixed point equation in EA:

$$\mathrm{EA} \vdash B(x) \leftrightarrow \forall v, z\, (H(\ulcorner B \urcorner, v, z) \to \mathrm{Tr}(f(v, x))).$$

We show that B is as required.

Assume there is a modal formula ϕ such that $T \vdash \phi^*(B(0), \ldots, B(\overline{n-1}))$. We can select such a formula ϕ with the smallest T-proof. The value of the partial function represented by H is provably unique, and hence

$$\mathrm{EA} \vdash \forall v, z\, (H(\ulcorner B \urcorner, v, z) \to v = \ulcorner \phi \urcorner).$$

It follows that

$$\mathrm{EA} \vdash B(x) \leftrightarrow \mathrm{Tr}(f(\ulcorner \phi \urcorner, x)).$$

Let $A_0, \ldots, A_{n-1}$ denote Solovay's evaluation of variables of ϕ. Then, provably in EA, $f(\ulcorner \phi \urcorner, \overline{m}) = \ulcorner A_m \urcorner$, for each $m < n$. Hence,

$$\mathrm{EA} \vdash B(\overline{m}) \leftrightarrow \mathrm{Tr}(\ulcorner A_m \urcorner) \leftrightarrow A_m.$$

However, we have $T \nvdash \phi^*(A_0, \ldots, A_{n-1})$ by Solovay, a contradiction. ■

The uniform arithmetical completeness theorem has found a few useful applications in the study of other questions in provability logic. In particular, Sergei Artemov used this (independently proven) result to show that all semi-normal extensions of basic provability logic **GL** by closed modal formulas are arithmetically complete [4, 5]. These results, in turn, eventually led to a complete classification of arithmetically complete propositional provability logics relative to an arbitrary metatheory [10] (see [11] for a detailed exposition).

On the other hand, the uniform arithmetical completeness theorem naturally led to the more general problem of characterizing the subalgebras of the Magari algebras of theories. This problem has been taken up in an extensive study of Volodya Shavrukov who answered this question for r.e. subalgebras [114]; further improvements were obtained by Domenico Zambella [132] including a generalization to arbitrary subalgebras [133].

2.3. First order theory of Magari algebras

By Solovay's theorem the logic of the Magari algebra $\mathcal{M}_{\mathrm{PA}}$ coincides with **GL**. Hence, the equational theory of the class of Magari algebras is decidable. Franco Montagna initiated the study of the first order theories of Magari algebras. He published two main results. The first one is an observation that the first order theory of the class of all Magari algebras is, in a sense, interpretable in PA [81]. The second one is the theorem that the first order theory of the class of all Magari algebras is undecidable [82]. Craig Smoryński [119] showed that this theory is recursively inseparable from the set of all sentences refutable on finite Magari algebras. Artemov an Beklemishev [8] showed that the first order theory of the free Magari algebra on n generators is decidable iff $n = 0$. This means that the Lindenbaum Magari algebra of the closed fragment of provability logic **GL** is decidable (and is, in fact, equivalent to the weak monadic theory of $(\mathbb{N}, <)$), whereas the decidability fails if at least one propositional variable is present.

This development brings us to the more difficult question whether the first order theory of the specific Magari algebra, the Magari algebra of PA, is decidable. The problem, formulated for the first time apparently in Montagna's paper [81], had become one of the central open problems in provability logic until a negative solution was obtained in 1994 by Volodya Shavrukov [116]. It is still an open problem whether the $\forall^*\exists^*$ fragment of that theory is decidable. The decidability of the purely universal theory of $\mathcal{M}_{\mathrm{PA}}$ (as well as of its purely existential theory) follows from the so-called second arithmetical completeness theorem of Solovay [121].

We give an account of Montagna's basic result that started off this long line of developments. Let MA denote the class of all Magari algebras. Let $\mathrm{Th}(\mathsf{MA})$ denote the set of all first order sentences in the language of Magari algebras true in each Magari algebra.

THEOREM 2.7. $\mathrm{Th}(\mathsf{MA})$ *is undecidable.*

PROOF. Franco's proof is based on Tarski's method of interpretations. It is sufficient to interpret the standard model of arithmetic $(\mathbb{N}, +, \cdot)$ in *some*

Magari algebra $\mathcal{M}$. In other words, it is sufficient to specify an algebra $\mathcal{M}$, and formulas $D(x)$, $E(x,y)$, $P(x,y,z)$, $T(x,y,z)$ in the first order language of Magari algebras such that in $\mathcal{M}$

- The formula $D(x)$ defines a countable subset $D \subseteq M$,
- E defines an equivalence relation on D;
- The predicates P and T respect the equivalence E and define the graphs of addition and multiplication on the natural numbers on the quotient D/E.

(The set of arithmetical sentences true in all Magari algebras under such an interpretation is a certain subtheory $S \subseteq \mathrm{Th}(\mathbb{N})$. If S were decidable, so would be its (consistent) extension $Q \cup S$ by finitely many axioms of Robinson's arithmetic Q, however Q is essentially undecidable.)

The algebra $\mathcal{M}$ will be the Magari algebra associated with a particular infinite upwards well-founded Kripke frame (W,R). Let $W = W_0 \cup W_1$ where $W_0 = \mathbb{N}$ and $W_1 = \{\langle\{m,n\},k\rangle : m,n,k \in \mathbb{N},\ m \neq n,\ k > m,n\}$. We let uRv iff for some $m,n,k \in \mathbb{N}$ $u = \langle\{m,n\},k\rangle$, $v \in W_0$ and either $v \in \{m,n\}$, or $v > k$. Thus, each node of W has either depth 0 or 1; the nodes of W_0 are the nodes of depth 0, and W_1 consists of the nodes of depth 1. Also notice that each node $u \in W_1$ can see all but finitely many nodes of W_0.

The Magari algebra associated with (W,R) is the structure $\mathcal{M} = (\mathcal{P}(W), \cap, \cup, -, \Box_R)$, where $\Box_R$ is defined as in (1). The domain of the interpretation D consists of all finite subsets of W_0. The formula $E(x,y)$ will mean that the sets x and y have the same number of elements. The formula $P(x,y,z)$ will mean that the set z has the same number of elements as the disjoint union of x and y. The formula $T(x,y,z)$ will mean that the set z can be split into $|y|$ many equivalence classes of cardinality $|x|$. It remains for us to show that these relations are first order definable in $\mathcal{M}$.

It is easy to check that in $\mathcal{M}$ we can talk about singletons (these are the atoms of $\mathcal{M}$). Also, we are able to talk about upwards closed subsets of W and *connected* subsets of W (these are the sets that cannot be nontrivially split into two disjoint upwards closed subsets of W).

Let $\mathrm{At}(x)$ denote the formula expressing that x is an atom. And let $\mathrm{Cmp}(u,v)$ denote the formula expressing that u is a maximal connected subset *(connected component)* of v.

To distinguish between the operations of the signature of Magari algebra and the first order logical connectives we now use set-theoretic notations for the former. The terms $\top$, $\bot$, $\Box\bot$, $\Diamond\top$ denote the sets W, $\emptyset$, W_0, W_1

respectively. We define:

$$D(x) := (x \subseteq \Box\bot \wedge \Diamond x \neq \Diamond\top).$$

The formula $x \subseteq \Box\bot$ means that x is a subset of W_0. Since every node of W_1 can see all but finitely many nodes of W_0, the formula $D(x)$ defines the set of finite subsets of W_0.

Let $u \oplus v$ abbreviate $(u - v) \cup (v - u)$. Let $E(u, v)$ denote the formula

$$D(u) \wedge D(v) \wedge \exists z\,(u \oplus v \subseteq z \wedge \forall w\,(\mathrm{Cmp}(w, z) \to \mathrm{At}(u \cap w) \wedge \mathrm{At}(v \cap w))).$$

The formula $E(u, v)$ holds iff $u, v \in W_0$ and $|u| = |v|$. Suppose $E(u, v)$. If $u = v$ the claim is trivial. Otherwise, $u \oplus v \neq \emptyset$ is split by the connected components of z into disjoint pairs of points $\{a, b\}$ such that $a \in u - v$ and $b \in v - u$ and a, b belong to the same component. This yields a bijection between $u - v$ and $v - u$, and hence between u and v.

Conversely, if $|u| = |v|$ we can consider the sets $u - v = \{a_1, \ldots, a_n\}$ and $v - u = \{b_1, \ldots, b_n\}$ and define

$$z := (u \oplus v) \cup \{\langle\{a_1, b_1\}, h\rangle, \ldots, \langle\{a_n, b_n\}, h\rangle\},$$

where $h = \max(u \oplus v) + 1$.

Having defined $E(u, v)$ it is now easy to define the formulas P and T. We let $P(x, y, z)$ state that $x, y, z \in D$ and

$$\exists u \in D\,(E(u, y) \wedge x \cap u = \bot \wedge E(z, x \cup u)).$$

We let $T(x, y, z)$ state that $x, y, z \in D$ and there are $u, v \in D$ such that $E(x, u)$, $E(z, v \cap \Box\bot)$, $u \subseteq v$ and

$$\forall w\,(\mathrm{Cmp}(w, v) \to \mathrm{At}(w \cap u) \cap E(w \cap \Box\bot, y)).$$

In other words, the set u intersects every connected component of v at a singleton, and the intersection of every such component with W_0 has cardinality $|y|$. It follows that $|z| = |v \cap \Box\bot| = |y| \cdot |u| = |y| \cdot |x|$. ■

More recently, the study of the first order theories of Magari algebras and related structures has been revived in several directions. Shavrukov [117] shows that the lattice of Σ_1-sentences in PA has undecidable first order theory. On the other hand, by Lindström and Shavrukov [70], the $\forall^*\exists^*$-theory of that lattice is decidable.

Fedor Pakhomov [107, 109] studied the first order theories of the algebras of the closed fragment of the polymodal provability logic **GLP** and of their natural substructures related to ordinal notation systems. In particular, he showed that the Lindenbaum **GLP**-algebra of the closed fragment of **GLP** (restricted to the language with finitely many modalities) enjoys a decidable first order theory [108]. This result significantly generalizes a theorem on the decidability of the first order theory of the free 0-generated Magari algebra [8].

2.4. Predicate provability logic

Franco Montagna's 1984 paper [83] on the predicate provability logic is one of our favorites. After the celebrated arithmetical completeness results on propositional provability logic by Robert Solovay the question of characterizing the predicate provability logic became a central question in this area. In this paper Franco undertakes the first systematic study of this problem and shows that many significant positive results known for the propositional provability logic fail for the predicate provability logic. In particular, the predicate version of modal logic **GL**, denoted **QGL**, is not complete with respect to any class of Kripke frames, it does not enjoy the fixed point property, and it is not arithmetically complete.

It is the latter proposition that we are going to sketch here. **QGL** is formulated in the language of predicate logic augmented by the modality $\Box$. As in Section 2.1 we consider an arithmetical r.e. theory containing EA together with its provability predicate Pr_T. *Arithmetical realization* ϕ^* of a predicate modal formula ϕ is a function that maps each atomic formula $P(x_1,\ldots,x_n)$ to an arithmetical formula $P^*(x_1,\ldots,x_n)$ of the same arity, commutes with the boolean connectives and translates $\Box$ as the formal provability predicate in T:

$$(\Box\phi(x_1,\ldots,x_n))^* = \mathrm{Pr}_T(\ulcorner\phi^*(\dot{x}_1,\ldots,\dot{x}_n)\urcorner).$$

Here $\ulcorner A(\dot{x})\urcorner$ denotes the standard elementary term for the function $n \mapsto \ulcorner A(\overline{n})\urcorner$.

We call a predicate modal formula ϕ *T-valid* if $T \vdash \phi^*$, for every arithmetical realization $*$ in T. The *predicate provability logic of T* is the set of all T-valid formulas ϕ. Clearly, the predicate provability logic of T contains **QGL** and is closed under the rules of predicate logic, necessitation and substitution. Franco Montagna proved the following theorem showing that **QGL** is properly weaker than the predicate provability logic of PA.

THEOREM 2.8. **QGL** *does not contain the predicate provability logic of* PA.

Proof. Let T be a finitely axiomatizable theory such that

$$\mathrm{PA} + \mathrm{Con}_T \vdash \mathrm{Con}_{\mathrm{PA}+\mathrm{Con}_{\mathrm{PA}}}.$$

For example, as such a theory T one can take Gödel–Bernays set theory GB. We assume without loss of generality that **QGL** contains the language of T. Let $[T]$ denote the universal closure of the conjunction of the axioms of T and let A denote the formula $\Diamond[T] \to \Diamond\Diamond\top$. We claim that A is PA-valid but unprovable in **QGL**.

Since the arithmetical interpretation $(\cdot)^*$ is faithful to the rules of predicate logic, $T \vdash B$ implies $[T]^* \vdash B^*$, for each $\Box$-free formula B. This argument is formalizable in PA, in particular, taking $B = \bot$ we obtain

$$\mathrm{PA} \vdash \mathrm{Con}_{[T]^*} \to \mathrm{Con}_T.$$

Consider any arithmetical realization of A. Then

$$A^* = (\mathrm{Con}_{\mathrm{PA}+[T]^*} \to \mathrm{Con}_{\mathrm{PA}+\mathrm{Con}_{\mathrm{PA}}}).$$

We deduce:

$$\begin{aligned} \mathrm{PA} \vdash \mathrm{Con}_{\mathrm{PA}+[T]^*} &\to \mathrm{Con}_{[T]^*} \\ &\to \mathrm{Con}_T \\ &\to \mathrm{Con}_{\mathrm{PA}+\mathrm{Con}_{\mathrm{PA}}}. \end{aligned}$$

Hence, A is PA-valid.

To show that A is unprovable in **QGL** we use a Kripke model argument. Consider a Kripke model with two nodes $W = \{0, 1\}$ such that $0R1$. Fix a classical model M of T and assume that M is assigned both to 0 and to 1. We extend the evaluation of closed atomic formulas in M (in the language with constants $\underline{a}$ for all elements $a \in M$) at each node to all predicate formulas by using the clauses K1–K4 together with the following clause for the universal quantifier:

K5. $\mathcal{W}, x \models \forall u\, \phi(u) \iff \forall a \in M\, \forall y \in W\ (xRy \Rightarrow \mathcal{W}, y \models \phi(\underline{a}))$.

Since (W, R) is well-founded, it is easy to see that this model validates **QGL**. On the other hand, since $M \models [T]$ we also have $\mathcal{W}, 1 \models [T]$. Hence, $\mathcal{W}, 0 \models \Diamond[T]$ but $\mathcal{W}, 0 \not\models \Diamond\Diamond\top$. ■

In a similar vein, Franco shows that the predicate provability logic of PA is different from the one of ZF. Let T be a finite subtheory of ZF strong enough to prove $\mathrm{PA} + \mathrm{Con}_{\mathrm{PA}}$. By the previous argument, the formula A as above is PA-valid, however it is not ZF-valid. Consider the identical realization of the language of T in ZF. Then, the realization of A in ZF is equivalent to

$$\mathrm{Con}_{\mathrm{ZF}+T} \to \mathrm{Con}_{\mathrm{ZF}+\mathrm{Con}_{\mathrm{ZF}}},$$

which is unprovable in ZF by Gödel's second incompleteness theorem, since $\mathrm{ZF} \vdash T$.

Franco Montagna ends the discussion of the arithmetical incompleteness of **QGL** by asking a notable question whether the predicate provability logic of PA is recursively enumerable. He also makes a conjecture that the set of all predicate modal formulas that are T-valid in *every* Σ_1-sound r.e. extension of PA coincides with **QGL**.

Franco's paper paved the way for a breakthrough in the study of predicate provability logics that came in 1985 in the papers by Sergei Artemov [6, 7] and Valery Vardanyan [125]. Artemov showed that the truth provability logic of PA is non-arithmetical. Later Boolos and McGee [15] improved this result by showing that this logic is Π^1_1-complete. Vardanyan gave a negative answer to Franco's original question: the predicate provability logic of PA is Π^0_2-complete. The wealth of negative results obtained on predicate provability logics by the end of 1980s more or less closed this fruitful area of study. However, see [130] for more recent improvements.

2.5. Feferman's provability predicate

Two early papers by Franco Montagna sparkled two important lines of research in provability logic. Both of them concerned the concept of provability in finite subtheories of PA, although in somewhat different ways.

The first paper of the two initiated the bimodal study of the so-called Feferman provability predicate [79]. It was one of the earliest papers that aimed at characterizing two different provability predicates simultaneously. Moreover, in this paper Franco investigated the predicate that was both curious and useful in the study of provability and interpretability in arithmetic.

The second paper [84] dealt with bimodal systems describing provability in PA together with provability in its 'arbitrarily large' finite subtheory. This paper, among other things, introduced one of the first natural examples of an arithmetically complete system of bimodal provability logic describing the provability predicates in two different r.e. theories. (Another paper where the same system was introduced as the joint provability logic of PA and ZF was published a year earlier by Timothy Carlson [19].) This paper also was a precursor of a line of papers dedicated to the investigation of interpretability by means of modal logic. In this section we describe Franco's contributions in the study of Feferman provability and in the next one say a few words on the other paper.

Feferman's predicate $\mathrm{Pr}^F(x)$ was invented by Solomon Feferman in order to illustrate the condition for the numeration of a formal theory to be Σ_1 in Gödel's second incompleteness theorem [42]. Fix an increasing sequence of finite subsystems of PA, denoted $\mathrm{PA}\restriction n$, such that $\bigcup_{n\geq 0} \mathrm{PA}\restriction n = \mathrm{PA}$. Usually, the exact choice of this sequence does not matter and one takes $\mathrm{PA}\restriction n$ to denote the theory axiomatized by the axioms of Peano arithmetic whose Gödel numbers are smaller than, or equal to, n. Formula $\mathrm{Pr}^F(x)$ expresses the statement that *for some n the formula with the Gödel number x is provable in* $\mathrm{PA}\restriction n$ *and* $\mathrm{PA}\restriction n$ *is consistent.* Reflexive theories such as PA are able to prove the consistency of each of their finite subtheories, hence

$$\forall n\, \mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA}\restriction n).$$

It follows that $\mathrm{Pr}^F(x)$ defines in the standard model of arithmetic exactly the set of PA-provable formulas, that is, the same set as $\mathrm{Pr}_{\mathrm{PA}}(x)$. However, Feferman showed that, in contrast with the standard provability predicate, Gödel's second incompleteness theorem does not hold for $\mathrm{Pr}^F(x)$, that is,

$$\mathrm{PA} \vdash \mathrm{Con}^F_{\mathrm{PA}},$$

where $\mathrm{Con}^F_{\mathrm{PA}}$ denotes $\neg\mathrm{Pr}^F(\ulcorner\bot\urcorner)$.

Feferman's provability predicate is a representative of a host of other provability predicates externally numerating PA, but internally exhibiting an unexpected and sometimes pathological behavior (Rosser's, Kreisel's, etc.). Apart from being important as arguments in the philosophical discussions around Gödel's theorems, some of these 'Peano's smart children'[4] have demonstrated their technical usefulness in the study of formal arithmetic. In particular, Feferman's trick allowed to prove for essentially reflexive theories the following characterization of relative interpretability [42]: T is interpretable in S iff there is a binumeration α of T in S for which $S \vdash \mathrm{Con}_\alpha$. For $T = S = \mathrm{PA}$ this theorem is witnessed by Feferman's binumeration of PA.

Of various of Peano's offspring, Feferman's provability seems to be best suited for a provability logic analysis. It happens to be more invariant and stable than the other notions such as Rosser's provability predicate. In particular, $\mathrm{Pr}^F(x)$ is extensional, that is, $\phi \sim_{\mathrm{PA}} \psi$ implies $\mathrm{Pr}^F(\ulcorner\phi\urcorner) \sim_{\mathrm{PA}}$

[4]The term was coined by Albert Visser [126].

$\mathrm{Pr}^F(\ulcorner\psi\urcorner)$. Hence, it defines a second operator (usually denoted $\triangle$) on the Magari algebra of Peano arithmetic. Franco Montagna [79] was the first to tackle the problem of characterizing the identities of that structure, in other words, the bimodal logic of Gödel's and Feferman's provability predicates. A full answer to that question had to wait a number of years, though.

Franco Montagna isolated a few key principles relating Feferman's and Gödel's provability predicates (which we now formulate in logical rather than algebraic terms). Apart from the axioms L1 and L2 of **GL** the following principles are satisfied:

F1. $\triangle(A \to B) \to (\triangle A \to \triangle B)$;

F2. $\neg\triangle\bot$;

F3. $\triangle A \to \Box A$;

F4. $\Box A \wedge \neg\Box\bot \to \triangle A$;

F5. $\Box A \to \triangle\Box A$.

Both $\Box$ and $\triangle$ enjoy the corresponding necessitation rules $A/\Box A$ and $A/\triangle A$.

Notice that we do have F5, though the third derivability condition $\triangle A \to \triangle\triangle A$ fails for Feferman's provability: otherwise we would have obtained Löb's principle for $\triangle$ by the usual diagonalization argument, contradicting F2.

Montagna studied a few basic properties of bimodal algebras satisfying the identities corresponding to the above principles of Feferman provability. Thus, he showed that on any Magari algebra one can define an operation $\triangle$ satisfying the above principles. He also studied the question of definability and uniqueness of fixed points in these algebras. He observed that the situation here is very much different from the case of Magari algebras and obtained some partial characterizations of classes of formulas having fixed points in the logic in question.

As in many of Franco's contributions, the main impact of his paper was in posing new questions that were both deep and solvable and that eventually led to notable advances.[5] In the paper under discussion, Franco formulated three problems that later attracted particular attention by the other researchers. The first problem was the question whether the sentence asserting its own Feferman's unprovability, that is, the analogue of Gödel's sentence, was provably unique. The second problem was the question whether the sentence asserting its own Feferman's provability, that is, the analogue

[5] Many of his early papers end up with lists of open problems. It is noteworthy how many of these questions turned out to be very good ones.

of Henkin's sentence, was either provable or disprovable (clearly, both $\bot$ and $\top$ are equivalent to the assertions of their own Feferman provability). The third problem was the natural question whether there were any other valid principles of the bimodal logic of Feferman's and Gödel's provability predicates.

Albert Visser wrote an influential paper [126] in which he gave a powerful case for the modal-logical study of various non-standard provability predicates and presented them in a wider context. Feferman's provability predicate was an important, though not unique, particular case. Among other things, Visser gave a negative answer to Franco's second question showing that there are infinitely many pairwise inequivalent sentences asserting their own Feferman's provability.

As it happens, the answers to the other two of Franco's questions turned out to be dependent on the details so far left behind the scene, namely on the exact content of the theories in the sequence $\mathrm{PA}\restriction n$. This dependence first came to light in the work of Craig Smoryński [120] who showed that the answer to the first question was positive, under a particularly natural choice of a sequence of finite subtheories of PA defining the formula $\mathrm{Pr}^F(x)$. The natural choice of $\mathrm{PA}\restriction n$, in this case, is the system axiomatized by the schema of induction restricted to Σ_n-formulas, more commonly known as $\mathrm{I}\Sigma_n$. Smoryński also gave some examples showing that the condition that the sequence of theories be sufficiently fast growing (in terms of provability of reflection principles) was substantial.

Further progress in the remaining problems was obtained in the insightful work of Shavrukov [115]. He found two additional principles of Feferman's provability:

F7. $\Box A \to \Box \triangle A$;

F8. $\triangle A \to \triangle((\triangle B \to B) \vee \triangle A)$.

The first of the two principles holds generally and was already present in the work of Visser. The second one holds for the natural sequence of subsystems of PA described above but not generally. By adapting Solovay's proof of arithmetical completeness theorem for **GL**, Shavrukov showed that the system axiomatized by F1–F8 was decidable and arithmetically complete. This gave a full answer to Montagna's third question mentioned above (which was clearly the main question of the three).[6] Furthermore, Shavrukov gave

[6] One can ask what principles of Feferman provability hold independently of the chosen sequence of subsystems of PA. To the best of our knowledge this question is open though possibly not difficult given the present day understanding of this area.

a tricky counterexample showing that under a suitable choice of a sequence of finite subsystems of PA there may exist inequivalent sentences asserting their own Feferman's unprovability. This completed the answer to Franco's first question.

In the more recent years a new generation of curious children of Peano was born. These were needed to obtain rather natural consistency assertions of strength below the standard Gödel's $\mathrm{Con}_{\mathrm{PA}}$. They appeared under the name *slow consistency* in the paper [49]. Similar constructions occurred in Visser [129] in the context of weak fragments of PA. However, a full-fledged modal logical study of slow provability is still awaiting its inquisitive researcher.

2.6. Provability in finite subtheories of PA and interpretability logic

Suppose a sufficiently strong arithmetical r.e. theory T be given. Interpretability logic deals with the study of binary modality $\phi \rhd_T \psi$ to mean that there is a relative interpretation of $T + \psi$ in $T + \phi$. The study of interpretability in arithmetic became a well-established field of study in the 1960s, after the work of Feferman, Kreisel and Orey. In the 1970s and 80s Hájek, Guaspari, Solovay, Friedman, Lindström, Pudlák and others contributed in various ways. Among other things, various characterizations of interpretability for important classes of theories were obtained, lattices of interpretability types were studied, relationships with bounded arithmetic theories and speed-up of proofs were established. The study of interpretability as a binary modality, that is, of the logic of interpretability started with a paper by Švejdar [124]. Logicians in the Netherlands (Visser, de Jongh, Veltman and their students) powerfully joined in, which soon led to considerable progress in the area.

Franco Montagna's contributions to the the modal study of interpretability are not fully represented by his own publications. A well-known example is the so-called *Montagna's principle*. This is the axiom

$$\phi \rhd \psi \to (\phi \wedge \Box\theta) \rhd (\psi \wedge \Box\theta).$$

This axiom appears for the first time in the paper by Albert Visser [127] which contained the first systematic exposition of interpretability logic and a characterization of the interpretability logic for sufficiently strong finite sequential theories. At that time, the problem of characterizing the interpretability logic for essentially reflexive theories, such as Peano arithmetic, remained open. Montagna's principle was suggested as the main additional axiom of interpretability logic for PA. When introducing this axiom, Albert Visser refers to his correspondence with Franco. However, in his own review of Visser's paper, Franco mentions that this principle has actually been known to Per Lindström. Be it as it may, the name stuck, and Montagna's principle is still called M by Franco's name. The logic **ILM** of interpretability over PA was provided a complete Kripke semantics in the paper by Dick de Jongh and Frank Veltman [30] and subsequently shown to be arithmetically complete by Alessandro Berarducci [12] and Volodya Shavrukov [113] (independently).

Montagna's paper [84] was one of the earliest publications related to interpretability logic and predated the subsequent work of Visser, de Jongh and Veltman, and later Berarducci and Shavrukov. Franco's approach to the study of interpretability was via a bimodal logic representing both provability in PA (by $\Box$) as well as provability in its 'arbitrarily large' finite subtheory (by the additional modality $\triangle$). These considerations were motivated by the fact that $\phi \rhd_{\mathrm{PA}} \psi$ is provably in PA equivalent to the statement that

$$\forall n\, \mathrm{Pr}_{\mathrm{PA}}(\ulcorner \phi \to \mathrm{Con}_{\mathrm{PA}\restriction n+\psi} \urcorner).$$

The bimodal logic of Montagna only allowed for an outer universal quantifier over (sufficiently large) n, so it was able to express certain facts about interpretability on the metalevel (by $\vdash \phi \to \neg\triangle\neg\psi$), however it did not represent the iterations of interpretability modality. Nevertheless, Franco used it to obtain certain memorable characterizations. For example, he characterized the set of all pairs of formulas (ϕ, ψ) in the language of Gödel–Löb logic such that $\mathrm{PA} + \phi$ interprets $\mathrm{PA} + \psi$ under every substitution of arithmetical sentences for propositional variables, and showed that this set is decidable. Another result from that paper is the theorem on the uniqueness of fixed points and their explicit computation for the interpretability logic of PA in the case that the fixed point equation does not contain parameters (variables other than the one for which the fixed point is substituted). This result has later been generalized to arbitrary modalized formulas of **ILM** by de Jongh and Visser [31]. Despite the results of Franco's paper being superseded by subsequent work, they were truly innovative and did serve as an important source of inspiration for subsequent researchers.

Finally, we should mention a later contribution of Montagna (jointly with Petr Hájek) to the study of interpretability logic **ILM** [54, 55]. They showed that this logic is complete under another natural arithmetical semantics, namely the interpretation of $\phi \rhd \psi$ as the formalization of the Π^0_1-conservativity statement: *every* Π^0_1*-sentence provable in* $T+\psi$ *is provable in*

$T+\phi$. It is well-known that for the extensions of Peano arithmetic in its own language Π_1^0-conservativity is equivalent to interpretability. However, this is not so, in general, for subsystems of PA where the two notions diverge. Hájek and Montagna showed that **ILM** is complete under the arithmetical interpretation of $\rhd$ as Π_1^0-conservativity for sound arithmetical theories extending $I\Sigma_1$.

The modal study of formalized interpretability and conservativity remains an alive field to this day. One of the main problems in this area, the axiomatization of the interpretability logic of the class of all sufficiently strong sequential theories, was formulated very early on in Visser's work and it still remains open. For an overview and more recent progress in this direction see [128, 52].

By the end of 1990s the field of provability logic reached a stage in its development when many of the deep problems posed by its originators in the 1970s had been solved and the remaining ones looked intractable. In the middle of the same decade, some of the most powerful proponents of the discipline died[7]. Other important figures, like Craig Smoryński, left the scene for other reasons. There was a need for rethinking, setting new goals and searching for new applications.

Over the same period, methods of Logic in Computer Science are becoming increasingly important. Following the trend, since the end of the 1980s Petr Hájek, Jeff Paris and other logicians are getting increasingly involved in the study of uncertain reasoning. At the beginning of the 1990s Petr Hájek, an old friend and colleague of Franco, started a collaboration with Francesc Esteva and Lluis Godo aimed at providing a formal approach to fuzzy set theory. In 1998 Hájek publishes his monograph *Metamathematics of Fuzzy Logic* [53], which brings out the mathematical content of this subject to a wider community of mathematical logicians.

By the end of 1990s Franco joins Hájek in the investigation of uncertain reasoning and many-valued logic. His own education in the school of universal algebra under Roberto Magari makes the switch natural and easy. At this point we would like to follow Franco and leave the subject of provability logic for what will become the main topic of the next one and a half decades of his scientific life.

[7] Roberto Magari (†1994), George Boolos (†1996).

3. Algebraic analysis of many-valued logics

Definition 3.1. A *t-norm* is a commutative, associative, monotone operation $\odot : [0,1]^2 \to [0,1]$ satisfying the equations $1 \odot x = x$ and $0 \odot x = 0$ for all $x \in [0,1]$. A t-norm $\odot$ is said to be (*left-*)*continuous* if so is with respect to the usual topology of $[0,1]$.

Given a left-continuous t-norm $\odot$, its *residuum* $\to: [0,1]^2 \to [0,1]$ is defined by the following stipulation: for all $x, y \in [0,1]$,

$$x \to y = \sup\{z \in [0,1] \mid x \odot z \leq y\}$$

(cf. [53, 51]). Hájek proved in [53, Lemma 2.1.4] that, indeed, $x \to y = \max\{z \in [0,1] \mid x \odot z \leq y\}$. Any such pair $(\odot, \to)$ is called a *residuated pair*. The following are the main examples of (continuous) t-norms and their residua.

- Łukasiewicz t-norm and its residuum: $x \odot y = \max\{0, x + y - 1\}$, $x \to y = \min\{1, 1 - x + y\}$,
- Gödel-Dummett t-norm and its residuum: $x \odot y = \min\{x, y\}$, $x \to y = 1$ if $x \leq y$ and $x \to y = y$ otherwise,
- Product t-norm and its residuum: $x \odot y = x \cdot y$, $x \to y = 1$ if $x \leq y$ and $x \to y = y/x$ otherwise.

In his monograph [53] Hájek introduced, for every fixed continuous t-norm $\odot$, a propositional calculus $L(\odot)$ where $\odot$ is taken to provide a semantics for a (strong) conjunction connective, its residuum $\to$ becomes the truth function of the implication, and truth values for formulas range in the real unit interval $[0,1]$, where 0 stands for *false* and 1 stands for *true*. Further operations are definable in $L(\odot)$ as follows: for every $x, y \in [0,1]$, $\neg x = x \to 0$, $x \wedge y = x \odot (x \to y)$ and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$.

The *Basic Logic* (*BL* for short) was introduced by Hájek in order to capture the tautologies which are common to all continuous t-norms. (We refer to [18, 53] for an exhaustive treatment of BL). BL is algebraizable in the sense of Block and Pigozzi [14] and its equivalent algebraic semantics is the variety of *BL-algebras*. These structures can be defined in several equivalent ways. We decide to define BL-algebras in the following manner.

Definition 3.2. An algebra $\mathbf{A} = (A, \odot, \to, \vee, \wedge, \bot, \top)$ of type $(2,2,2,2,0,0)$ is a *BL-algebra* provided that the following hold:

(i) $(A, \odot, \top)$ is a commutative monoid,

(ii) $(A, \vee, \wedge, \bot, \top)$ is a bounded distributive lattice with top element $\top$ and bottom element $\bot$,

(iii) $\rightarrow$ is a binary operation such that the following residuation property holds: $x \odot y \leq z$ iff $x \leq y \rightarrow z$.

(iv) $\mathbf{A}$ is *divisible*: $x \wedge y = x \odot (x \rightarrow y)$,

(v) $\mathbf{A}$ is *prelinear*: $(x \rightarrow y) \vee (y \rightarrow x) = \top$

Notice that the above definition also presents BL-algebras as commutative, integral, bounded, divisible, prelinear residuated lattices. Whenever the lattice order of a BL-algebra $\mathbf{A}$ is a total order, we will call it a *BL-chain*.

As shown by Hájek [53, Lemma 2.3.10], the class of BL-algebras forms a variety which we will denote $\mathbb{BL}$. Furthermore, every subdirectly irreducible algebra in $\mathbb{BL}$ is a BL-chain. Hence $\mathbb{BL}$ is generated by its totally ordered components.

Remark 3.3. Among BL-algebras, the class of MV-algebras surely is the best studied (the monographs [23, 104] are sufficient to justify the wide interest that many mathematicians and algebraic-logicians have in these structures). The class of these structures, which were introduced by Chang in [20], can now be regarded as that subvariety of $\mathbb{BL}$ given by the equation $\neg\neg x = x$. We will meet these algebras in Section 4. For MV-algebras, we will use a signature that differs from the one introduced in Definition 3.2. Indeed, following the standard notation, MV-algebras will be presented as algebras of the form $(A, \oplus, \neg, \bot)$ of type $(2, 1, 0)$. Other operations and constants (including the usual ones of BL-algebras) are hence defined as follows: $\top = \neg\bot$, $x \rightarrow y = \neg x \oplus y$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \vee y = (x \rightarrow y) \rightarrow y$, $x \wedge y = \neg(\neg x \vee \neg y)$, $x \ominus y = \neg(x \rightarrow y)$. The real unit interval $[0, 1]$ with operations $x \oplus y = \min\{1, x + y\}$, $\neg x = 1 - x$ and the constant 0, forms an MV-algebra called the *standard MV-algebra* and denoted by $[0, 1]_{MV}$.

Typical examples of BL-algebras are obtained considering the real unit interval $[0, 1]$ endowed with operations $\odot, \rightarrow, \vee, \wedge$ where $\odot$ is a continuous t-norm, $\rightarrow$ is its residuum, and $\vee$ and $\wedge$ are the usual lattice operations of max and min. These algebras are called *standard* and a fundamental issue in many-valued logics is to prove a given formalism (for instance BL) to be *standard complete*, i.e., to be complete with respect to its standard algebras. The standard completeness theorem for BL was proved by Cignoli, Esteva, Godo and Torrens [24]. In algebraic terms this result shows that $\mathbb{BL}$ is generated by its linearly ordered components whose lattice reduct is the real unit interval $[0, 1]$. In Subsection 3.1 we will present one of the main contributions of Franco Montagna to the study of BL-algebras: a *single chain completeness theorem* for the logic BL, namely a single standard algebra which is generic for $\mathbb{BL}$. In Subsection 3.2, we will present an algebraic method that nowadays is called the Jenei-Montagna method. By means of that construction, Jenei and Montagna proved in [60] the standard completeness for a weaker logic than BL (called MTL [38]).

3.1. Decomposition of BL-chains and a generic BL-chain

In a series of papers that started with [2], Franco Montagna investigated the algebraic properties of BL-algebras focusing, in particular, on BL-chains. Starting with the intended aim of shedding a new light on the structure of the lattice of subvarieties of $\mathbb{BL}$, in two papers coauthored with Paolo Aglianò [2] and Isabel M. A. Ferreirim [1], Franco and his coauthors discovered an intimate relation between BL-agebras and *basic hoops*. Indeed, as the authors already noticed in [1], BL-algebras are precisely bounded basic hoops, that is, those bounded hoops which are isomorphic to subdirect products of totally ordered ones. Furthermore, every BL-chain can be decomposed as an ordinal sum of a family of Wajsberg hoops the first component of which is a Wajsberg algebra (see below). Before treating this fundamental result, we will need some preliminaries.

Definition 3.4. A *hoop* is a structure $\mathbf{H} = (H, \odot, \to, \top)$ of type $(2, 2, 0)$ such that:

(i) $(H, \odot, \top)$ is a commutative monoid,

(ii) $\to$ is a binary operation satisfying the following properties:

- $x \to x = \top$,
- $x \odot (x \to y) = y \odot (y \to x)$,
- $x \to (y \to z) = (x \odot y) \to y$.

A *bounded hoop* is an algebra $\mathbf{H} = (H, \odot, \to, \bot, \top)$, such that $(H, \odot, \to, \top)$ is a hoop and $\bot \leq x$ for all $x \in H$.
A *Wajsberg hoop* is a hoop $\mathbf{W}$ satisfying

$$(x \to y) \to y = (y \to x) \to x.$$

A *cancellative hoop* is a hoop satisfying

$$x \to (x \odot y) = y.$$

Every totally ordered Wajsberg hoop is either bounded (in which case it will be called a *Wajsberg algebra*[8]), or is cancellative.

Hoops can be obtained by means of the *ordinal sum construction*. Let $(I, \leq)$ be a linearly ordered set with minimum i_0. For all $i \in I$, let $\mathbf{H}_i = (H_i, \odot_i, \rightarrow_i, \top)$ be a hoop such that for $i \neq j$, $H_i \cap H_j = \{\top\}$. Then $\bigoplus_{i \in I} \mathbf{H}_i$ is called the *ordinal sum* of the family $\{\mathbf{H}_i\}_{i \in I}$, its universe is $\bigcup_{i \in I} H_i$, and its operations $\odot, \rightarrow$ are defined by the following stipulations:

$$x \odot y = \begin{cases} x \odot_i y & \text{if } x, y \in H_i, \\ y & \text{if } j < i,\ x \in H_i,\ \top \neq y \in H_j, \\ x & \text{if } i < j,\ \top \neq x \in H_i,\ y \in H_j, \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow_i y & \text{if } x, y \in H_i, \\ y & \text{if } j < i,\ x \in H_i,\ y \in H_j, \\ \top & \text{if } i < j,\ \top \neq x \in H_i,\ y \in H_j. \end{cases}$$

The hoops $\mathbf{H}_i$ are called *summands*.

The following analysis can be found in [2]: Let $\langle I, \leq \rangle$ be a totally ordered set; a subset $J \subseteq I$ is *connected* if for all $i, j \in J$ and $k \in I$, $i \leq k \leq j$ implies $k \in J$. A *connected partition* of $\langle I, \leq \rangle$ is a partition of I into connected subsets. A *decomposition* of a totally ordered hoop $\mathbf{H}$ is a family $D = \{\mathbf{H}_i \mid i \in I\}$ of linearly ordered hoops such that $\mathbf{H} = \bigoplus_{i \in I} \mathbf{H}_i$. Let Δ be the collection of all decompositions of a totally ordered hoop. The first thing that Montagna and Aglianò had to show is that Δ is a set. Using the axiom of choice we can assume, without loss of generality, that for every decomposition the index set I is a subset of $\mathbf{H}$ (choose $I \subseteq H$ such that $\top \notin I$ and $H_i \cap I$ has cardinality 1 for every component H_i). It follows that every element of Δ is a function from a subset I of H (the index set) into the powerset of H (because for every $i \in I$ the domain H_i of $\mathbf{H}_i$ is a subset of H). Thus, Δ is a definable subclass of the class Γ of all partial functions from H into the powerset of H. The axioms of set theory (in particular, the powerset axiom and the axiom of comprehension) guarantee that Γ is a set and therefore, again by the axiom of comprehension, Δ is in turn a set.

The set Δ can be partially ordered in the following way: if $D = \{\mathbf{H}_i \mid i \in I\}$ and $D' = \{\mathbf{K}_j \mid j \in J\}$ are in Δ, then we put $D \leq D'$ if there exists a connected partition $\{I_j \mid j \in J\}$ of I such that for every $j, j' \in J$, one has: (1) if $j < j'$, then for all $k \in I_j$ and $k' \in I_{j'}$, $k < k'$; (2) $\mathbf{K}_j = \bigoplus_{i \in I_j} \mathbf{H}_i$.

[8]It is worth noticing that Wajsberg algebras are termwise equivalent to MV-algebras [23, 53].

THEOREM 3.5. *Every totally ordered BL-algebra is the ordinal sum of a family of Wajsberg hoops whose first summand is a Wajsberg algebra.*

PROOF. Let $\mathbf{A}$ be a BL-chain, that is a totally ordered bounded basic hoop, and let $\langle \Delta, \leq \rangle$ be the poset of its decompositions and let C be a chain of decompositions in Δ. For every $a \in A \setminus \{\top\}$ and for every $D \in C$, let A^{D_a} be the unique component of D which contains a and let $A_a = \bigcap_{D \in C} A^{D_a}$. Clearly $A_a \cup \{\top\}$ is the universe of a subalgebra $\mathbf{A}_a$ of $\mathbf{A}$. Moreover, for $a, b \in A \setminus \{\top\}$, $A_a = A_b$ if and only if a and b lie in the same component of all the decompositions in C. The axiom of choice implies that there exists $I \subseteq A \setminus \{\top\}$ such that, for every $a \in A \setminus \{\top\}$, $I \cap A_a$ contains exactly one element. Then

$$\mathbf{A} = \bigoplus_{a \in I} \mathbf{A}_a,$$

and the decomposition obtained in this way is greater than or equal to every element in C. Thus, by Zorn Lemma applied to $\langle \Delta, \leq \rangle$, there is a maximal decomposition of $\mathbf{A}$ each component of which must be sum irreducible and hence, by [2, Theorem 3.6], is a totally ordered Wajsberg hoop (algebra). ■

By exploring ordinal sums of Wajsberg hoops, still in [2], the authors provided a characterization of those totally ordered BL-algebras which generate the whole $\mathbb{BL}$. In what follows we will denote by $\mathbb{BL}(n)$ the variety generated by n-generated BL-algebras. For every $n \in \mathbb{N}$ we will denote $(n)[0,1]_{MV}$ the BL-algebra given by the ordinal sum of n copies of the standard MV-algebra $[0,1]_{MV}$ (remember that MV-algebras are termwise equivalent to Wajsberg algebras), while $\omega[0,1]_{MV}$ denotes the BL-algebra given by the ordinal sum of ω copies of $[0,1]_{MV}$. The following result is contained in [2].

THEOREM 3.6. *For every $n \in \mathbb{N}$, $\mathbb{BL}(n)$ is generated by $(n+1)[0,1]_{MV}$. The variety $\mathbb{BL}$ is generated by $\omega[0,1]_{MV}$.*

Using Theorem 3.6, Franco gave an explicit representation for the free one-generated BL-algebra [86]. This representation paved the way for the representation of the free n-generated BL-algebras provided by Stefano Aguzzoli and Simone Bova in [3].

3.2. Completions of countable MTL-chains: the Jenei-Montagna method

The logic MTL and its algebraic semantics, the class of MTL-algebras, were introduced by Francesc Esteva and Lluis Godo in [38]. MTL-algebras form a variety, denoted $\mathbb{MTL}$ which coincides with that of commutative, integral, bounded, prelinear residuated lattices. Hence, every BL-algebra is an MTL-algebra, and more precisely $\mathbb{MTL}$ is obtained from $\mathbb{BL}$ by dropping the divisibility equation (see Definition 3.2 (iv)). In [38] the authors proved that, similarly to $\mathbb{BL}$, also $\mathbb{MTL}$ is generated by its linearly ordered members. In the same paper they left open the problem of showing that standard MTL-algebras (i.e. those MTL-algebras of the form $([0,1], \odot, \to, \wedge, \vee, 0, 1)$ where $[0,1]$ is the real unit interval, $\odot$ is a *left-continuous* t-norm and $\to$ its residuum) are enough to generate $\mathbb{MTL}$. In other words, they left open the problem of proving (or disproving) the standard completeness theorem for the logic MTL.

Sandor Jenei and Franco Montagna proved the standard completeness theorem for MTL in [60]. This result, apart from its relevance *per se*, is based on a technique that nowadays is known as the *Jenei-Montagna method*. This consists in a two-step completion of countable MTL-chains. In what follows we will sketch this method (we suggest the interested reader to consult [60, Theorem 3.1] for an exhaustive presentation of this construction). Later, we will point out its importance in the frame of many-valued logics.

Take an arbitrary countable MTL-chain $A = (A, \odot^A, \to^A, \leq^A, \bot^A, \top^A)$. The following steps build a standard MTL-chain $S = ([0,1], \odot, \to, \leq, 0, 1)$ and an embedding of A into S:

(i) Let $B = \{(a,q) \mid a \in A \setminus \{\bot^A\}, q \in \mathbb{Q} \cap (0,1]\} \cup \{(\bot^A, 1)\}$ equipped with the lexicographic order $\preceq$.

(ii) Define the following operation $\circ : B \times B \to B$: for all $(a,q), (b,r) \in B \times B$,

$$(a,q) \circ (b,r) = \begin{cases} \min_{\preceq}\{(a,q),(b,r)\} & \text{if } a \odot^A b = \min\{a,b\} \\ (a \odot^A b, 1) & \text{otherwise.} \end{cases}$$

Then $(B, \circ, \preceq, (\top^A, 1))$ is an ordered monoid.

(iii) The map $a \in A \mapsto (a,1) \in B$ establishes an embedding of the ordered monoid $(A, \odot^A, \leq^A, \top^A)$ into $(B, \circ, \preceq, (\top^A, 1))$.

(iv) The ordered monoid $(B, \circ, \preceq, (\top^A, 1))$ is countable and densely ordered with maximum and minimum. Hence, it is isomorphic to a monoid $(\mathbb{Q} \cap [0,1], \circ', \preceq', 1)$. Clearly $(A, \odot^A, \leq^A, \top^A)$ embeds into $(\mathbb{Q} \cap [0,1], \circ', \preceq', 1)$ as well, via an embedding h. Moreover, the residuum $\Rightarrow$ of $\circ'$ exists over the restriction $h[A]$ of h to A, and $h(a) \Rightarrow h(b) = h(a \to^A b)$.

(v) $(\mathbb{Q} \cap [0,1], \circ', \preceq', 1)$ is completed to the real unit interval $[0,1]$ by defining: for all $\alpha, \beta \in [0,1]$,

$$\alpha \odot \beta = \sup\{x \circ' y \mid x \leq \alpha, y \leq \beta, x, y \in \mathbb{Q} \cap [0,1]\}.$$

(vi) The map $\odot : [0,1] \times [0,1] \to [0,1]$ is a left-continuous t-norm. Letting $\mathbf{S} = ([0,1], \odot, \Rightarrow, \leq, 0, 1)$, h embeds $\mathbf{A}$ into the standard MTL-algebra $\mathbf{S}$.

Hence the following holds.

THEOREM 3.7. *Every countable linearly ordered MTL-algebra can be embedded into a standard MTL-algebra.*

An immediate consequence of the previous result is the standard completeness theorem for MTL.

THEOREM 3.8. *MTL is complete with respect to its standard algebras. In other words, for every MTL formula φ, if MTL $\not\vdash \varphi$, there exists a left continuous t-norm $\odot : [0,1] \times [0,1] \to [0,1]$ and a valuation e into $([0,1], \odot, \Rightarrow, \leq, 0, 1)$ (where $\Rightarrow$ is the residuum of $\odot$) such that $e(\varphi) \neq 1$.*

It is worth noticing that the Jenei-Montagna method has been adapted to provide standard completeness for several other schematic extensions of the monoidal t-norm based logic MTL. In particular we recall Horčík's standard completeness for ΠMTL [57], and Esteva, Gispert, Godo and Montagna's standard completeness of SMTL and IMTL [36]. We also invite the interested reader to consult [106, §3.4] for an exhaustive presentation of the prominent MTL schematic extensions.

3.3. Further contributions to the algebraic analysis of many-valued logics

Franco Montagna's research activity on the algebraic analysis of many-valued logics touched many other topics.

One of his main interests regarded the expansion of MV-algebras with a binary connective $\cdot$ to be interpreted, in $[0,1]$, as the usual product between real numbers (cf. [85]). The resulting algebras are called *PMV$^+$-algebras* in [85][9] or *Montagna's PMV-algebras* in [69]. These structures were also investigated by Franco and Giovanni Panti in [97] and their development paved the way to further interesting research directions; see for instance [98], and the expansion of MV-algebras by a fix-point operator investigated by Luca Spada in [122, 123] and Enrico Marchioni and Luca Spada in [77].

[9] The name PMV$^+$ — and the subscript $^+$ in particular — was introduced to distinguish these structures from *PMV-algebras* introduced by A. Di Nola and A. Dvurečenskij [33].

In [39], in a joint work with Francesc Esteva and Lluis Godo, the logics ŁΠ and Ł$\Pi\frac{1}{2}$ were introduced[10]. These logics are expansions of Łukasiewicz logic (and hence of MV-algebras from the algebraic viewpoint) with a product conjunction (as in the case of PMV$^+$), a product residuation and, in the case of Ł$\Pi\frac{1}{2}$, a constant $\overline{\frac{1}{2}}$ which allows to define all rational constants from $[0, 1]$ within its language.

GBL-algebras (generalized BL-algebras) were introduced by Franco and Peter Jipsen in [61]. These structures are non-commutative (although finite GBL-algebras turn out to be commutative [61]) and non-prelinear generalizations of BL-algebras. The relevance of these structures lies on the fact that they can be regarded as many-valued versions of Heyting algebras, the equivalent algebraic semantics of Intuitionistic Logic.

Several other algebraic structures providing the equivalent algebraic semantics of fuzzy logic were introduced and studied by Montagna in a series of co-authored papers. Among them n-contractive BL-algebras [13] and weakly cancellative MTL-algebras [96].

In [90], Franco gave a detailed algebraic investigation of interpolation, Craig's interpolation and Beth's property in many-valued logics extending BL and studied their semantical counterparts, i.e., amalgamation and strong amalgamation for their corresponding varieties. The impact of Franco's seminal ideas on this topic is witnessed by the contributions that were published on the same subject after the appearance of [90]. Amongst them we recall the paper by Hitoshi Kihara and Hiroakira Ono [63] dealing with interpolation, Beth property and amalgamation for substructural logics, the paper [78], in which George Metcalfe, Costas Tsinakis and Montagna himself investigated amalgamation and interpolation for ordered algebras and the paper [76] by Marchioni and Metcalfe, where Craig interpolation is approached in the frame of semilinear substructural logics.

4. Probability, coherence and nonstandard analysis

Besides developing algebraic methods for the analysis of many-valued logics, Franco Montagna dedicated a considerable effort of his research in many-valued logics to investigating the foundations of probability theory on many-valued events and, in particular, de Finetti's betting game [28, 110].

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[10]The logic ŁΠ was introduced in [37] by Esteva and Godo, while the logic Ł$\Pi\frac{1}{2}$ was introduced by Montagna in [85]. In [39] the authors presented the two logics together and made important steps forward in the algebraic analysis of these formalisms. Thus, [39] can be reasonably considered as the basic reference for ŁΠ and Ł$\Pi\frac{1}{2}$.

In a nutshell, de Finetti's operative definition of (subjective) probability is given in terms of bets. The probability of an (unknown) event ϕ is the amount of money (or betting odd) α that a *fair* bookmaker would require to play the following game: A gambler chooses a real number λ, pays $\alpha\lambda$ to the bookmaker and receives from the bookmaker, in the possible world v, i.e. according to the valuation v, $\lambda v(\varphi)$. Note that λ, the gambler's stake, may be negative: paying $\lambda < 0$ is in fact the same as receiving $-\lambda$; in other words, for the gambler, betting a negative number corresponds to *reversing* her rôle with the bookmaker.

The only rationality criterion proposed by de Finetti is the following: suppose that the bookmaker accepts bets on the events $\phi_1, \ldots, \phi_n$ with betting odds $\alpha_1, \ldots, \alpha_n$ respectively. Then the assessment proposed by the bookmaker is *coherent* if there is no system of bets causing to him a sure loss. That is, there are no $\lambda_1, \ldots, \lambda_n$ that the gambler may propose such that in every possible world v, the total balance for the bookmaker is always strictly negative, i.e. $\sum_{i=1}^{n} \lambda_i(\alpha_i - v(\phi_i)) < 0$.

De Finetti's theorem states that an assessment is coherent if and only if it can be extended to a finitely additive probability measure on the algebra spanned by the events $\phi_1, \ldots, \phi_n$, [110].

This approach has been firstly generalized to the case of many-valued events by Brunella Gerla [50], Jeff Paris [111] and Daniele Mundici [68, 103]. In this setting events are elements of an MV-algebra $\mathbf{A}$, rather than a boolean algebra, and they may take intermediate truth-values. The balance for the bookmaker is hence calculated by the same formula as above, but taking into account that possible worlds are MV-homomorphisms of $\mathbf{A}$ in the standard algebra $[0,1]_{MV} = ([0,1], \oplus, \neg, 0, 1)$. The rôle of finitely additive probability measures is hence played by MV-algebraic *states* [101, 104]. These are functions s mapping the carrier of an MV-algebra $\mathbf{A}$ in the real unit interval $[0,1]$ and further satisfying: $s(\top) = 1$, and $s(x \oplus y) = s(x) + s(y)$ whenever $x \odot y = \bot$. A state s is said to be *faithful* if $s(x) = 0$ implies $x = \bot$. The main theorem of [68] reads as follows:

THEOREM 4.1. *Let* $\mathbf{A}$ *be an MV-algebra, let* $a_1, \ldots, a_n \in A$ *and let* $a_1 \mapsto \alpha_1, \ldots, a_n \mapsto \alpha_n$ *be an assignment. Then the following are equivalent:*

- *The assignment is* coherent*, i.e. it avoids a sure loss for the bookmaker.*
- *There exists a state* s *of* $\mathbf{A}$ *such that, for all* $i = 1, \ldots, n$, $s(a_i) = \alpha_i$.

In the following subsections we will present two main contributions that Franco Montagna gave to the foundational aspects of generalized probability theory, namely, SMV-algebras and stable coherence.

4.1. Internal states and SMV-algebras

MV-algebras offer a suitable algebraic framework to treat states in a universal-algebraic setting. Indeed, the observation that the real unit interval $[0,1]$ serves both as the universe of the standard MV-chain (which is generic for the variety $\mathbb{MV}$ of MV-algebras) and as the range of states, inspired the paper [47] where Franco Montagna and Tommaso Flaminio introduced the notion of MV-algebraic *internal state*.

DEFINITION 4.2. An *MV-algebra with internal state* (*SMV-algebra* for short) is a system $\mathbf{A}_\sigma = (\mathbf{A}, \sigma)$ such that $\mathbf{A}$ is an MV-algebra and $\sigma : A \to A$ satisfies the following equations:

(σ1) $\sigma(0) = 0$,

(σ2) $\sigma(\neg x) = \neg\sigma(x)$,

(σ3) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$,

(σ4) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

An SMV-algebra $(\mathbf{A}, \sigma)$ is said to be *faithful* if it satisfies the following quasi-equation: $\sigma(x) = 0$ implies $x = 0$.

Obviously SMV-algebras form a variety denoted $\mathbb{SMV}$. While the first two equations (σ1) and (σ2) state that σ is normalized (i.e. $\sigma(1) = 1$), the equation (σ3) is a form of additivity for internal states, and (σ4) guarantees that σ is idempotent. Indeed, the equations (σ1)-(σ4) ensure σ to satisfy the basic properties of a state.

The intimate relation between states and internal states is witnessed by the following results (Theorem 4.3, Theorem 4.4 and Theorem 4.5).

Let $(\mathbf{A}, \sigma)$ be an SMV-algebra. [47, Lemma 3.3, (h)] shows that the image $\sigma(A)$ of A under σ is the universe of an MV-subalgebra of $\mathbf{A}$ that we will denote $\sigma(\mathbf{A})$. Therefore, if M is any maximal filter of $\sigma(\mathbf{A})$, the quotient $\sigma(\mathbf{A})/M$ is simple and hence it embeds into the standard MV-algebra $[0,1]_{MV}$ via a map η_M [23, Theorem 3.5.1]. Let ι_M be the canonical homomorphism of $\sigma(\mathbf{A})$ in $\sigma(\mathbf{A})/M$ and call s the map from A in $[0,1]$ defined in the following way: for every $x \in A$,

$$s(x) = \eta_M(\iota_M(\sigma(x))). \tag{2}$$

Then the following holds.

THEOREM 4.3 ([47]). *Let $(\mathbf{A}, \sigma)$ be an SMV-algebra and let $s : A \to [0,1]$ be defined as in (2). Then s is a state of $\mathbf{A}$.*

Conversely, let $\mathbf{A}$ be an MV-algebra. Using the MV-algebraic tensor product (cf. [102]) we let the MV-algebra $\mathbf{T} = [0,1]_{MV} \otimes \mathbf{A}$ which contains both $[0,1]_{MV}$ and $\mathbf{A}$ as MV-subalgebras. In particular, the maps $\Phi : [0,1]_{MV} \to \mathbf{T}$ such that $\Phi(\alpha) = \alpha \otimes 1$ and $\Psi : \mathbf{A} \to \mathbf{T}$ such that $\Psi(x) = 1 \otimes x$, are embeddings of $[0,1]_{MV}$ and $\mathbf{A}$ into $\mathbf{T}$ respectively.

Pick a state s of $\mathbf{A}$ and define $s_1 : \Psi(A) \to [0,1]$ by the stipulation $s_1(\Psi(x)) = s(x)$. Since $\Psi(A)$ is (the domain of) an MV-subalgebra of $\mathbf{T}$, by [67, Theorem 6] s_1 extends to a state $s_2 : T \to [0,1]$. Finally, define the map $\sigma : T \to T$ as follows: for every $z \in T$,

$$\sigma(z) = s_2(z) \otimes 1.$$

THEOREM 4.4 ([47]). *With the above notation and terminology, σ is well defined and $(\mathbf{T}, \sigma)$ is an SMV-algebra.*

Theorems 4.3 and 4.4 allow us to treat the coherence problem for a rational-valued assignment on formulas of Łukasiewicz logic inside the theory of SMV-algebras. Indeed Franco always considered de Finetti's coherence criterion as a logical principle, and since 2004 he pointed out that *coherence* should be characterized, in logical terms, as the *consistency* of a suitably defined theory in a modal probabilistic logic. In this sense, SMV-algebras offered him the right universal-algebraic environment for such characterization.[11]

As is well known, Łukasiewicz logic is *algebraizable* in the sense of Blok and Pigozzi (cf. [14]). Thus each formula ϕ in Łukasiewicz language can be regarded as a term in the language of MV-algebras. Let $\phi_1, \ldots, \phi_n$ be Łukasiewicz formulas, let us assume that $\frac{k_1}{m_1}, \ldots, \frac{k_n}{m_n}$ are rational numbers in $[0,1]$ and let $x_1, \ldots, x_n$ be fresh variables. Then, consider the following equations:

$$\epsilon_i : (m_i - 1)x_i = \neg x_i \text{ and } \delta_i : \sigma(\phi_i) = k_i x_i \text{ (for } i = 1, \ldots, n).$$ [12]

THEOREM 4.5 ([47]). *Let $\phi_1, \ldots, \phi_n$ be formulas of Łukasiewicz logic and let $\chi : \phi_1 \mapsto \frac{k_1}{m_1}, \ldots, \phi_n \mapsto \frac{k_n}{m_n}$ be a rational-valued assignment. Then the following are equivalent:*

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[11] It is worth pointing out that a similar, although weaker, result that characterizes rational-valued coherent assignments on classical formulas in terms of consistent theories of modal probabilistic logic was firstly published by Franco Montagna and Tommaso Flaminio in 2005 (cf. [46]). Later that approach has been extended and applied also to the case of classical conditional events [43, 75] and many-valued events [44].

[12] If $\mathbf{A}$ is an MV-algebra, then for every $n \in \mathbb{N}$ and every $x \in A$, nx stands for $x \oplus \ldots \oplus x$, n-times.

(i) *The assignment* χ *is coherent.*

(ii) *The equations* ϵ_i *and* δ_i *(for* $i = 1, \ldots, n$*) are satisfied in some non-trivial SMV-algebra.*

4.2. Stable coherence

There are at least two possible ways to approach conditional probability on many-valued events. The first one was introduced by T. Kroupa in [66]: Let $\mathbf{A}$ be an MV-algebra. By Di Nola's representation theorem [23, 32], there exists an ultrapower $^*[0,1]$ of $[0,1]$ and set X such that $\mathbf{A}$ embeds in the MV-algebra of functions $(^*[0,1])^X$. We will denote $^*\mathbf{A}$ the MV-algebra obtained by extending $\mathbf{A}$, regarded as an MV-subalgebra of $(^*[0,1])^X$, equipped with pointwise product. Then, if $s : {}^*A \to [0,1]$ is a state, we define a conditional state of $\mathbf{A}$ as the map $s' : A \times A \to [0,1]$ such that, for every $a \mid b \in A \times A$,

$$s'(a \mid b) = \tfrac{s(a \cdot b)}{s(b)},$$

whenever $s(b) > 0$, and undefined otherwise.

The second approach, proposed by D. Mundici [104, §15], regards the conditional probability of $\phi \mid \psi$ as the probability of ϕ in the theory axiomatized by ψ. Every free MV-algebra admits a conditional probability which satisfies the Rényi laws of conditional probability, is invariant under every automorphisms of the algebra, and is also independent, in the sense that if ϕ and ψ have no variable in common, then $P(\phi \mid \psi) = P(\phi)$.

Montagna's approach to conditional probability takes into account the fact that in a conditional event $\phi \mid \psi$, the antecedent ψ might not be completely true, but only partially true. Following de Finetti's seminal idea, Franco suggested the following foundational definition of conditional probability on many-valued events: The conditional probability of a conditional event $\phi \mid \psi$ is the amount α of euros that a rational bookmaker would assign to it in a game in which a gambler can choose a possibly negative real number λ, paying $\lambda\alpha$ to the bookmaker, and receiving, in the possible world v the amount of $\lambda(v(\phi)v(\psi) + \alpha(1 - v(\psi)))$. Hence the bookmaker's payoff is $\lambda v(\psi)(\alpha - v(\phi))$. In particular notice that when $v(\psi) = 0$ the bet is *called off*, when $v(\psi) = 1$ the bet is equivalent to a bet on ϕ, and when $0 < v(\psi) < 1$, then the bet is *partially valid*.

Franco used to explain and justify his approach by the following example.

EXAMPLE 4.6. When moving from classical to many-valued events, it is reasonable to assume that the truth value $v(\psi)$ of the antecedent of a conditional is neither 0 nor 1. Consider for instance the following: suppose that we are betting on the conditional event "The Barcelona soccer team will win the next match, provided that Messi plays". For convenience, let us denote by ϕ the event "the Barcelona soccer team will win" and by ψ the antecedent of the previous statement: "Messi will play", so that the above conditional event can be written as $\phi \mid \psi$. Assume that, during the soccer match (and hence in the possible world v), Messi plays the whole match except for the last 30 seconds. It would not make sense to completely invalidate the bet; instead it would be meaningful to think that the bet on this many-valued conditional event is true to the degree $v(\psi)$. Thus, if $v(\psi) = 1$, then the bet is completely valid. If $v(\psi) = 0$, then the bet is called off. In all the intermediate cases $0 < v(\psi) < 1$ the bet is partially valid with degree $v(\psi)$.

The following result was proved in [91]:

THEOREM 4.7. *Consider an assignment on a finite set of conditional and unconditional events* $\Lambda : \phi_1 \mid \psi_1 \mapsto \alpha_1, \ldots, \phi_n \mid \psi_n \mapsto \alpha_n, \psi_1 \mapsto \beta_1, \ldots, \psi_n \mapsto \beta_n$ *on an MV-algebra* $\mathbf{A}$ *and such that* $\beta_i \neq 0$ *for all* $i = 1, \ldots, n$. *Then* Λ *avoids sure loss iff there is a state* s *of* ${}^*\mathbf{A}$ *such that, for all* $i = 1, \ldots, n$, $s(\psi_i) = \beta_i$, $s(\phi_i \cdot \psi_i) = \alpha_i \beta_i$.

When some β_i is 0, the assessment Λ may at the same time avoid sure loss and fail to be rational. In order to overcome this problem, Montagna, in a paper coauthored with Martina Fedel and Giuseppe Scianna [93], introduced the notion of *stable coherence* as follows: Consider again the following game: the bookmaker fixes a assessment $\Lambda : \phi_1 \mid \psi_1 \mapsto \alpha_1, \ldots, \phi_n \mid \psi_n \mapsto \alpha_n, \psi_1 \mapsto \beta_1, \ldots, \psi_n \mapsto \beta_n$. If some β_i is 0, the gambler can now force the bookmaker to change Λ by an infinitesimal in such a way that the betting odds of every antecedent ψ_i is strictly positive.

DEFINITION 4.8 ([93]). An assignment Λ is said to be *stably coherent* if there is a variant Λ' of Λ such that:

- Λ' avoids sure loss,
- all betting odds for the antecedents ψ_i's in Λ' are strictly positive,
- Λ and Λ' differ by an infinitesimal.

Clearly, as infinitesimals are directly involved in the definition of stable coherence, the probability measures (or states) that characterize of stable coherence will range on a non-trivial ultrapower ${}^*[0, 1]$ of the real unit interval.

REMARK 4.9. In the classical setting, the idea of using nonstandard probability measures to define conditional probabilities goes back to Krauss [65] and Nelson [105]. They first introduce a nonstandard probability measure P^* on an algebra of events such that only the impossible event may have probability zero, but a non-impossible event always take nonzero (possibly infinitesimal) probability. Then, the (standard) conditional probability of $a \mid b$, is defined as $St(P^*(a \wedge b)/P^*(b))$, where $St : {}^*\mathbb{R} \to \mathbb{R}$ denotes the *standard part* function.

Given an MV-algebra $\mathbf{A}$, a *hypervaluation* on $\mathbf{A}$ is a homomorphism h from ${}^*\mathbf{A}$ in ${}^*[0,1]$ such that, for every $\alpha \in {}^*[0,1]$, letting $f_\alpha \in {}^*A$ be the function constantly equal to α, one has $h(f_\alpha) = \alpha$.

Definition 4.10 ([93]). Let $\mathbf{A}$ be an MV-algebra. A *hyperstate* of ${}^*\mathbf{A}$ is a map s form ${}^*A$ into ${}^*[0,1]$ which is

(a) Additive: if $x \odot y = 0$, then $s(x \oplus y) = s(x) + s(y)$,

(b) Normalized: $s(1) = 1$,

(c) Homogeneous: for all $x \in {}^*A$ and for all $\alpha \in {}^*[0,1]$, $s(\alpha x) = \alpha s(x)$,

(d) Weakly faithful: if $x \in {}^*A$ and $s(x) = 0$, there is a hypervaluation v of $\mathbf{A}$ such that $v(x) = 0$.

It can be proved that every MV-algebra admits a faithful hyperstate (see [93, Theorem 4.2]). Furthermore, the following characterization theorem holds:

Theorem 4.11 ([93]). *Let $\Lambda : \phi_1 \mid \psi_1 \mapsto \alpha_1, \ldots, \phi_n \mid \psi_n \mapsto \alpha_n, \psi_1 \mapsto \beta_1, \ldots, \psi_n \mapsto \beta_n$ be a assignment on many-valued events, i.e., elements of an MV-algebra $\mathbf{A}$. Then Λ is stably coherent iff there is a faithful hyperstate s of ${}^*\mathbf{A}$ such that the following conditions hold:*

- *For all $i = 1, \ldots, n$, $s(\psi_i) - \beta_i$ is infinitesimal.*
- *For all $i = 1, \ldots, n$, $\alpha_i - \frac{s(\phi_i \cdot \psi_i)}{s(\psi_i)}$ is infinitesimal.*

4.3. Further contributions to uncertain reasoning on many-valued events

The algebraic structures we discussed in Section 4.1 provide the equivalent algebraic semantics for a modal probabilistic logic built on top of Łukasiewicz infinite-valued calculus. This formalism, called $SFP(\text{Ł}, \text{Ł})$ in [47] also admits a class of models which are generalized Kripke models with a state, called probabilistic Kripke models. In [48] an intriguing comparison between probabilistic Kripke models and SMV-algebras was investigated, relevant subclasses of SMV-algebras were introduced, and the SAT-problem for SMV-algebras was proved to be PSPACE. This latter result paved the way to the NP-completeness of the coherence problem for Łukasiewicz events proved in [16].

SMV-algebras have been quite intensively studied in the last years. A particular attention was devoted to the case in which the internal state of an MV-algebra **A** is an MV-endomorphism of **A**. These structures, called state-morphism MV-algebras (SMMV-algebras), were introduced by Di Nola and Dvurecěnskij in [34]. Subdirectly irreducible SMV-algebras and SMMV were fully characterized in [35] by Dvurečenskij, Kowalski, and Montagna. In the same paper, the authors described single generators of the variety of SMMV-algebras.

Franco's interest in the logical foundations of probability theory led him to criticize some aspects of de Finetti's ideas.

A first criticism arose by realizing that real bookmakers are non-reversible, i.e., they never accept negative stakes. In his paper [131] Walley showed that for non-reversible bookmakers coherence is fully captured by *imprecise probabilities*. In their paper [41] Fedel, Keimel, Montagna and Roth introduced a coherence criterion for non-reversible bookmaking in the realm of MV-algebras. Their characterization involves imprecise probabilities which are formulated in terms either of compact convex sets of probabilities or equivalently in terms of suitable sublinear functionals. In [40], Fedel, Hosni and Montagna further extended Walley's coherence criterion to MV-algebras, and provided a universal algebraic characterization of it. In [58], Hosni and Montagna considered stable coherence for the case of imprecise probabilities on MV-algebras.

A second criticism regards *strict coherence*, which is specification of coherence in which the bookmaker never assigns value 0 to non-impossible events. By doing so, the bookmaker prevents the gambler from a sure win or a draw (if at least a draw would not be possible in some possible world, the book would not be coherent). The generalization of strict coherence to the case of many-valued events was one of the last topic of interest to Franco. His main results in this setting are contained in a yet unpublished manuscript (coauthored with Flaminio and Hosni) [45]. Here Franco is concerned with a characterization of strict coherence on semisimple MV-algebras by means of faithful states, so generalizing two results by Kemeny [62] and Shimony [118].

Acknowledgements. We would like to thank the anonymous referees for careful reading and precious suggestions. Furthermore, we would like to thank Sergei Artemov, Dick de Jongh, Albert Visser, Volodya Shavrukov, Luis Godo, Francesc Esteva, Sara Ugolini, Hykel Hosni for providing useful information that greatly contributed to drafting this paper.

The work of the first author was supported by Russian Foundation for Basic Research, project No. 15-01-09218a.

References

[1] Aglianò, P., I. M. A. Ferreirim, and F. Montagna, Basic Hoops: an Algebraic Study of Continuous t-norms. *Studia Logica* 87(1): 73–98, 2007.

[2] Aglianò, P., and F. Montagna, Varieties of BL-Algebras I: General Properties. *Journal of Pure and Applied Algebra* 181: 105–129, 2003.

[3] Aguzzoli, S., and S. Bova, The free n-generated BL-algebra. *Annals of Pure and Applied Logic* 161(9): 1144–1170, 2010.

[4] Artemov, S. N., *Extensions of arithmetic and modal logics.* PhD thesis, Steklov Mathematical Insitute, Moscow, 1979. In Russian.

[5] Artemov, S. N., Arithmetically complete modal theories. *Semiotika i Informatika* (14): 115–133. VINITI, Moscow, 1980. In Russian. English translation in: *American Mathematical Society Translations* 135(2): 39–54, 1987.

[6] Artemov, S. N., Nonarithmeticity of truth predicate logics of provability. *Doklady Akademii Nauk SSSR* 284(2): 270–271, 1985. In Russian. English translation in *Soviet Mathematics Doklady* 33: 403–405, 1985.

[7] Artemov, S. N., Numerically correct provability logics. *Doklady Akademii Nauk SSSR*, 290(6): 1289–1292, 1986. In Russian. English translation in *Soviet Mathematics Doklady* 34: 384–387, 1987.

[8] Artemov, S. N., and L. D. Beklemishev, On propositional quantifiers in provability logic. *Notre Dame Journal of Formal Logic* 34: 401–419, 1993.

[9] Baaz, M., P. Hájek, F. Montagna, and H. Veith, Complexity of t-tautologies. *Annals of Pure and Applied Logic* 113(1-3): 3–11, 2001.

[10] Beklemishev, L. D., On the classification of propositional provability logics. *Izvestiya Akademii Nauk SSSR, ser. mat.*, 53(5): 915–943, 1989. In Russian. English translation in *Math. USSR Izvestiya* 35: 247–275, 1990.

[11] Beklemishev, L. D., M. Pentus, and N. Vereshchagin, *Provability, complexity, grammars.* American Mathematical Society Translations, Series 2, 192, 1999.

[12] Berarducci, A., The interpretability logic of Peano Arithmetic. *The Journal of Symbolic Logic* 55: 1059–1089, 1990.

[13] Bianchi, M., and F. Montagna, n-Contractive BL-logics. *Archive for Mathematical Logic* 50(3-4): 257–285, 2011.

[14] Blok, W., and D. Pigozzi, *Algebraizable Logics.* Memoirs of the American Mathematical Society, 396 (77). Amer. Math Soc. Providence, 1989.

[15] Boolos, G., and V. McGee, The degree of the set of sentences of predicate provability logic that are true under every interpretation. *The Journal of Symbolic Logic* 52(1): 165–171, 1987.

[16] Bova, S., and T. Flaminio, The coherence of Łukasiewicz assessment is NP-complete. *International Journal of Approximate Reasoning* 51: 294–304, 2010.

[17] Bova, S., and F. Montagna, The consequence relation in the logic of commutative GBL-algebras is PSPACE-complete. *Theoretical Computer Science* 410(12-13): 1143–1158, 2009.

[18] Busaniche, M., and F. Montagna, Hájek Logic BL and BL-algebras. In P. Cintula, P. Hájek, and C. Noguera, (eds.), *Handbook of Mathematical Fuzzy Logic* - Volume 1: 355–447. Volume 37 of Studies in Logic, Mathematical Logic and Foundations. College Publications, London, 2011.

[19] Carlson, T., Modal logics with several operators and provability interpretations. *Israel Journal of Mathematics* 54: 14–24, 1986.

[20] Chang, C. C., Algebraic Analysis of Many-valued Logics. *Transactions of the American Mathematical Society* 88: 467–490, 1958.

[21] Ciabattoni, A., G. Metcalfe, and F. Montagna, Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions. *Fuzzy Sets and Systems* 161(3): 369–389, 2010.

[22] Ciabattoni, A., and F. Montagna, Proof theory for locally finite many-valued logics: Semi-projective logics. *Theoretical Computer Science* 480: 26–42, 2013.

[23] Cignoli, R., I. M. L. D'Ottaviano, and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*. Kluwer, Dordrecht, 2000.

[24] Cignoli, R., F. Esteva, L. Godo, and A. Torrens, Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing* 4(2): 106–112, 2000.

[25] Cintula, P., P. Hájek, and C. Noguera (eds.). *Handbook of Mathematical Fuzzy Logic* (Vol 1). volume 37 of Studies in Logic, Mathematical Logic and Foundations. College Publications, London, 2011.

[26] Cintula, P., Hájek P., and C. Noguera (eds.). *Handbook of Mathematical Fuzzy Logic* (Vol 2). volume 38 of Studies in Logic, Mathematical Logic and Foundations. College Publications, London, 2011.

[27] Corsi, E. A., and F. Montagna, The Rényi-Ulam games and many-valued logics. *Fuzzy Sets and Systems*, in print. DOI: 10.1016/j.fss.2015.09.006.

[28] de Finetti, B., Sul significato soggettivo della probabilità. *Fundamenta Mathematicae* 17: 298–329, 1931. Translated into English as "On the subjective meaning of probability", in P. Monari and D. Cocchi (eds.), *Probabilità e Induzione*, Clueb, Bologna: 291–321, 1993.

[29] de Jongh, D., M. Jumelet, and F. Montagna, On the proof of Solovay's theorem. *Studia Logica* 50(1): 51–70, 1991.

[30] de Jongh, D., and F. Veltman, Provability logics for relative interpretability. In P. P. Petkov (ed.), *Mathematical Logic*: 175–208. Plenum Press, New York, 1990.

[31] de Jongh, D., A. Visser, Explicit fixed points in interpretability logic. *Studia Logica* 50(1): 39–49, 1991.

[32] Di Nola, A., Representation and reticulation by quotients of MV-algebras. *Ricerche di Matematica* (Naples) 40: 291–297, 1991.

[33] Di Nola, A., and A. Dvurečenskij, Product MV-algebras. *Multiple-Valued Logics* 5: 193–215, 2001.

[34] Di Nola, A., and A. Dvurečenskij, State-morphism MV-algebras, *Annals of Pure and Applied Logic*, Festschrift on the occasion of Franco Montagna's 60th birthday 161: 161–173, 2009.

[35] Dvurečenskij, A., T. Kowalski, and F. Montagna, State morphism MV-algebras, *International Journal of Approximate Reasoning* 52: 1215–1228, 2011.

[36] Esteva, F., J. Gispert, L. Godo, and F. Montagna, On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica* 71(2): 199–226, 2002.

[37] Esteva, F., and L. Godo, Putting Together Łukasiewicz and Product Logics. *Mathware and Soft Computing* 6: 219–234, 1999.

[38] Esteva, F., and L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* 124: 271–288, 2001.

[39] Esteva, F., L. Godo, and F. Montagna, The ŁΠ and ŁΠ$\frac{1}{2}$ logics: two complete fuzzy systems joining Łukasiewicz and Product Logics. *Archive for Mathematical Logic* 40(1): 39–67, 2001.

[40] Fedel, M., H. Hosni, and F. Montagna, A logical characterization of coherence for imprecise probabilities. *International Journal of Approximate Reasoning* 52(8): 1147–1170, 2011.

[41] Fedel, M., K. Keimel, F. Montagna, and W. Roth, Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic. *Forum Mathematicum* 25: 405–441, 2013.

[42] Feferman, S., Arithmetization of metamathematics in a general setting. *Fundamenta Mathematicae* 49: 35–92, 1960.

[43] Flaminio, T., NP-Containment for the Coherence Tests of Assessment of Conditional Probability: a Fuzzy-Logical Approach. *Archive for Mathematical Logic* 46(3-4): 301–319, 2007.

[44] Flaminio, T., and L. Godo, A logic for reasoning on the probability of fuzzy events. *Fuzzy Sets and Systems* 158: 625–638, 2007.

[45] Flaminio, T., H. Hosni, and F. Montagna, A characterization of strict coherence for infinite-valued events. *Manuscript*.

[46] Flaminio, T., and F. Montagna, A logical and algebraic treatment of conditional probability. *Archive for Mathematical Logic* 44: 245–262, 2005.

[47] Flaminio, T., and F. Montagna, MV-algebras with internal states and probabilistic fuzzy logics. *International Journal of Approximate Reasoning* 50(1): 138–152, 2009.

[48] Flaminio, T., and F. Montagna, Models for many-valued probabilistic reasoning. *Journal of Logic and Computation* 21(3): 447–464, 2011.

[49] Friedman, S.-D., M. Rathjen, and A. Weiermann, Slow consistency. *Annals of Pure and Applied Logic* 164(3): 382–393, 2013.

[50] Gerla, B., MV-algebras, multiple bets and subjective states. *International Journal of Approximate Reasoning* 25: 1–13, 2000.

[51] Gottwald, S., *A Treatise on Many-valued Logics*, Studies in Logic and Computation 9, Research Studies Press Ltd., Baldock, UK, 2001.

[52] Goris, E., and J. J. Joosten, A new principle in the interpretability logic of all reasonable arithmetical theories. *Logic Journal of the IGPL* 19: 1–17, 2011.

[53] Hájek, P., *Metamathematics of Fuzzy Logic*, Kluwer, 1998.

[54] Hájek, P., and F. Montagna, The logic of Π_1-conservativity. *Archive for Mathematical Logic* 30(2): 113–123, 1990.

[55] Hájek, P., and F. Montagna, The logic of Π_1-conservativity continued. *Archive for Mathematical Logic* 32: 57–63, 1992.

[56] Hájek, P., and F. Montagna, A note on the first-order logic of complete BL-chains. *Mathematical Logic Quarterly* 54(4): 435–446, 2008.

[57] Hŏrcík, R., Standard completeness theorem for ΠMTL. *Archive for Mathematical Logic* 44(4): 413–424, 2005.

[58] Hosni, H., and F. Montagna, Stable Non-standard Imprecise Probabilities. Proceedings of IPMU2014 (A. Laurent, et al. eds.), *Communication in Computer and Information Science* 444: 436–445, 2014.

[59] Japaridze, G. K., *Modal-logical means of investigation of provability.* PhD Thesis (Diss. kand. philos. nauk), Moscow, Moscow State University, 1986.

[60] Jenei, S., and F. Montagna, A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica* 70: 183–192, 2002.

[61] Jipsen, P., and F. Montagna, On the structure of generalized BL-algebras. *Algebra Universalis* 55: 226–237, 2006.

[62] Kemeny, J. G., Fair Bets and Inductive Probabilities. *The Journal of Symbolic Logic* 20(3): 263–273, 1955.

[63] Kihara, H., and H. Ono, Interpolation properties, Beth definability properties and amalgamation properties for substructural logics. *Journal of Logic Computation* 20 (4): 823–875, 2010.

[64] Klement, E. P., R. Mesiar, and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[65] Krauss, P. H., Representation of conditional probability measures on boolean algebras. *Acta Mathematica Academiae Scientiarum Hungaricae* Tomus 19 (3-4): 229–241, 1968.

[66] Kroupa, T., Conditional probability over MV algebras, *Fuzzy Sets and Systems* 149(2): 369–384, 2005.

[67] Kroupa, T., Representation and extension of states on MV-algebras. *Archive for Mathematical Logic* 45: 381–392, 2006.

[68] Kühr, J., and D. Mundici, De Finetti Theorem and Borel states in $[0,1]$-valued Algebraic Logic. *International Journal of Approximate Reasoning* 46(3):605–616, 2007.

[69] Lapenta, S., *MV-algebras with product: connecting the Pierce-Birkhoff conjecture with Łukasiewicz logic*. Ph. D. Thesis, Università deli Studi della Basilicata, 2015.

[70] Lindström, P., and V. Shavrukov, The $\forall\exists$-theory of Peano Σ_1-sentences. *Journal of Mathematical Logic* 8(2): 251–280, 2008.

[71] Löb, M. H., Solution of a problem of Leon Henkin. *The Journal of Symbolic Logic* 20: 115–118, 1955.

[72] Łukasiewicz, J., and A. Tarski, Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie* cl. III, 23(iii): 30–50, 1930.

[73] Macintyre, A., and H. Simmons, Gödel's diagonalization technique and related properties of theories. *Colloquium Mathematicae* 28: 165–180, 1973.

[74] Magari, R., The diagonalizable algebras (the algebraization of the theories which express Theor.; II). *Bollettino della Unione Matematica Italiana,* Serie 4 12:117–125, 1975.

[75] Marchioni, E., and L. Godo, A logic for reasoning about coherent conditional probability: a fuzzy modal logic approach. *Lecture Notes in Artificial Intelligence*, 9th European Conference on Logics in Artificial Intelligence JELIA'04. J. J. Alferes and J. Leite (eds.), 3229: 213–225, 2004.

[76] Marchioni, E., and G. Metcalfe, Craig interpolation for semilinear substructural logics. *Mathematical Logic Quarterly* 58(6): 468–481, 2012.

[77] Marchioni, E., and L. Spada, Advances in the theory of $\mu Ł\Pi$ algebras. *Logic Journal of the IGPL* 19(3): 476–489, 2011.

[78] Metcalfe, G., F. Montagna, and C. Tsinakis, Amalgamation and interpolation in ordered algebras. *Journal of Algebra* 402: 21–82, 2014.

[79] Montagna, F., On the algebraization of a Feferman's predicate. *Studia Logica* 37: 221–236, 1978.

[80] Montagna, F., On the diagonalizable algebra of Peano arithmetic. *Bollettino della Unione Matematica Italiana,* B (5), 16: 795–812, 1979.

[81] Montagna, F., Interpretations of the first-order theory of diagonalizable algebras in Peano arithmetic. *Studia Logica* 39: 347–354, 1980.

[82] Montagna, F., Undecidability of the first-order theory of diagonalizable algebras. *Studia Logica* 39: 355–359, 1980.

[83] Montagna, F., The predicate modal logic of provability. *Notre Dame Journal of Formal Logic* 25: 179–189, 1987.

[84] Montagna, F., Provability in finite subtheories of PA. *The Journal of Symbolic Logic* 52(2): 494–511, 1987.

[85] Montagna, F., An algebraic approach to propositional fuzzy logic. *Journal of Logic, Language, and Information* 9: 91–124, 2000.

[86] Montagna, F., The Free BL-Algebra on One Generator. *Neural Network World* 5: 837–844, 2000.

[87] Montagna, F., Storage Operators and Multiplicative Quantifiers in Many-valued Logics. *Journal of Logic and Computation* 14(2): 299–322, 2004.

[88] Montagna, F., On the predicate logics of continuous t-norm BL-algebras. *Archive for Mathematical Logic* 44(1): 97–114, 2005.

[89] Montagna, F., Subreducts of MV-algebras with product and product residuation. *Algebra Universalis* 53: 109–137, 2005.

[90] Montagna, F., Interpolation and Beth's property in propositional many-valued logics: A semantic investigation. *Annals of Pure and Applied Logic* 141(1-2): 148–179, 2006.

[91] Montagna, F., A notion of coherence for books on conditional events in many-valued logic, *Journal of Logic and Computation* 21(5): 829–850, 2011.

[92] Montagna, F., Partially Undetermined Many-Valued Events and Their Conditional Probability. *Journal of Philosophical Logic* 41(3): 563–593, 2012.

[93] Montagna, F., M. Fedel, and G. Scianna, Non-standard probability, coherence and conditional probability on many-valued events. *International Journal of Approximate Reasoning* 54: 573–589, 2013.

[94] Montagna, F., C. Marini, and G. Simi, Product logic and probabilistic Ulam games. *Fuzzy Sets and Systems* 158(6): 639–651, 2007.

[95] Montagna, F., and C. Noguera, Arithmetical complexity of first order predicate fuzzy logics over distinguished semantics. *Journal of Logic and Computation* 20: 399–424, 2010.

[96] Montagna, F., C. Noguera, and R. Hŏrcík, On Weakly Cancellative Fuzzy Logics. *Journal of Logic and Computation* 16(4): 423–450, 2006.

[97] Montagna, F., and G. Panti, Adding structure to MV-algebras. *Journal of Pure and Applied Algebra* 164(3): 365–387, 2001.

[98] Montagna, F. and L. Spada, Continuous approximations of product implication in MV-algebras with product. *Soft Computing* 9(3): 149–154, 2005.

[99] Montagna, F., and C. Tsinakis, Ordered groups with a conucleus. *Journal of Pure and Applied Algebra* 214(1): 71–88, 2010.

[100] Montagna, F., and S. Ugolini, A Categorical Equivalence for Product Algebras. *Studia Logica* 103(2): 345–373, 2015.

[101] Mundici, D., Averaging the Truth-value in Łukasiewicz Logic. *Studia Logica* 55(1): 113–127, 1995.

[102] Mundici, D., Tensor Products and the Loomis-Sikorski Theorem for MV-Algebras. *Advances in Applied Mathematics* 22: 227–248, 1999.

[103] Mundici, D., Bookmaking over Infinite-valued Events. *International Journal of Approximate Reasoning* 43: 223–240, 2006.

[104] Mundici, D., *Advanced Łukasiewicz calculus and MV-algebras*, Trends in Logic 35, Springer 2011.

[105] Nelson, E., *Radically Elementary Probability Theory*. Annals of Mathematics Studies, Princeton University Press, 1988.

[106] Noguera, C., *Algebraic study of axiomatic extensions of triangular norm based fuzzy logics*. Ph. D. Thesis, University of Barcelona, 2006.

[107] Pakhomov, F. N., Undecidability of the elementary theory of the semilattice of GLP-words. *Sbornik Mathematics* 203(8): 1211–1229, 2012.

[108] Pakhomov, F. N., On elementary theories of GLP-algebras. Technical report, 2014. ArXiv:1412.4439 [math.LO].

[109] Pakhomov, F. N., On elementary theories of ordinal notation systems based on reflection principles. *Proceedings of the Steklov Institute of Mathematics* 289: 194–212, 2015.

[110] Paris, J. B., *The uncertain reasoner's companion. A Mathematical Perspective*. Cambridge University Press, 1994.

[111] Paris, J. B., A Note on the Dutch Book Method. In *Proceedings of the Second International Symposium on Imprecise Probabilities and their Applications* (ISIPTA), 2001, pp. 301–306.

[112] Segerberg, K., *An essay in classical modal logic*. Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, Uppsala, 1971.

[113] Shavrukov, V. Yu, The logic of relative interpretability over Peano arithmetic. *Preprint, Steklov Mathematical Institute*, Moscow, 1988. In Russian.

[114] Shavrukov, V. Yu, Subalgebras of diagonalizable algebras of theories containing arithmetic. *Dissertationes Mathematicae* 323, 1993.

[115] Shavrukov, V. Yu, A smart child of Peano's. *Notre Dame Journal of Formal Logic* 35: 161–185, 1994.

[116] Shavrukov, V. Yu, Undecidability in diagonalizable algebras. *The Journal of Symbolic Logic* 62(1): 79–116.

[117] Shavrukov, V. Yu, Effectively inseparable Boolean algebras in lattices of sentences. *Archive for Mathematical Logic* 49(1): 69–89, 2010.

[118] Shimony, A., Coherence and the Axioms of Confirmation, *The Journal of Symbolic Logic* 20(1): 1–28, 1955.

[119] Smoryński, C., The finite inseparability of the first order theory of diagonalizable algebras. *Studia Logica* 41: 347–349, 1982.

[120] Smoryński, C., Arithmetical analogues of McAloon's unique Rosser sentences. *Archive for Mathematical Logic* 28: 1–21, 1989.

[121] Solovay, R. M., Provability interpretations of modal logic. *Israel Journal of Mathematics* 28: 33–71, 1976.

[122] Spada, L., μMV-algebras: An approach to fixed points in Łukasiewicz logic. *Fuzzy Sets and Systems* 159(10): 1260–1267, 2008.

[123] Spada, L., ŁΠ logic with fixed points. *Archive for Mathematical Logic* 47(7-8): 741–763, 2008.

[124] Švejdar, V., Modal analysis of generalized Rosser sentences. *The Journal of Symbolic Logic* 48: 986–999, 1983.

[125] Vardanyan, V. A., Arithmetic complexity of predicate logics of provability and their fragments. *Doklady Akademii Nauk SSSR* 288(1): 11–14, 1986. In Russian. English translation in *Soviet Mathematics Doklady* 33: 569–572, 1986.

[126] Visser, A., Peano's smart children. A provability logical study of systems with built-in consistency. *Notre Dame Journal of Formal Logic* 30: 161–196, 1989.

[127] Visser, A., Interpretability logic. In P. P. Petkov, (ed.), *Mathematical Logic*, Plenum Press, New York, 1990, pp. 175–208.

[128] Visser, A., An overview of interpretability logic. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakhariaschev, (eds.), *Advances in Modal Logic, v.1, CSLI Lecture Notes, No. 87*: 307–359. CSLI Publications, Stanford, 1998.

[129] Visser, A., The Second Incompleteness Theorem and bounded interpretations. *Studia Logica* 100(1): 399–418, 2012.

[130] Visser, A., and M. de Jonge, No escape from Vardanyan's theorem. *Archive for Mathematical Logic* 45(5): 539–554, 2006.

[131] Walley, P., *Statistical Reasoning with Imprecise Probabilities*, Volume 42 of Monographs on Statistics and Applied Probability, Chapman and Hall, London 1991.

[132] Zambella, D., Shavrukov's theorem on the subalgebras of diagonalizable algebras for theories containing $I\Delta_0 + \exp$. *Notre Dame Journal of Formal Logic* 35: 147–157, 1994.

[133] Zambella, D., *Chapters on Bounded Arithmetic & Provability Logic*. PhD thesis, Universiteit van Amsterdam, September 1994.

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