Review of the paper
Fragments of Heyting-Arithmetic

by W. Burr

During the last quarter of the previous century a lot of research has been done on the subsystems of classical first-order Peano arithmetic PA. The progress has been witnessed in a number of monographs among which the work by P. Hájek and P. Pudlák [1] is perhaps the most comprehensive one. In comparison, the fragments of Heyting arithmetic HA — the intuitionistic counterpart of PA — largely remain an unexplored territory. The paper under review is an attempt to remedy this situation and to get an insight into this fascinating area.

Main fragments of PA are defined by restricting the formulas in the schemata of induction or collection to the classes Π_n or Σ_n of the arithmetical hierarchy. A difficulty in the intuitionistic case is that the prenex normal form theorem does not hold and, correspondingly, natural (exhaustive) hierarchies of arithmetical formulas are missing.

A memorable theorem of A. Visser, further improved by K. Wehmeier [2], states that the fragment of HA based on the induction schema for all prenex formulas is Π_2-conservative over the induction schema restricted to Π_2-formulas only. Thus, intuitionistically, prenex induction is much weaker than full induction, and hence it does not provide a meaningful classification of the fragments of HA. Rather, one would expect that a reasonable classification of arithmetical formulas by their logical complexity should take into account the nestings of implications on a par with the quantifier alternations.

To answer this concern W. Burr proposes a family of formula classes Φ_n as the proper intuitionistic analogues of the classes Π_n (for $n > 1$). These classes satisfy natural closure conditions, are classically equivalent to the classes Π_n, and $\bigcup_{n>1} \Phi_n$ exhausts all arithmetical formulas. Moreover, W. Burr shows that the induction schema restricted to the class Φ_n proves the same Π_2-sentences as the classical fragment of PA defined by the Π_n-induction schema, $I\Pi_n$. In other words, the Visser–Wehmeier theorem does not hold for the classes Φ_n. For his result W. Burr uses an interesting variation of the so-called Friedman–Dragalin translation due to T. Coquand and M. Hofmann. In my opinion, the isolation of the classes Φ_n is a very basic and important contribution of the paper to intuitionistic arithmetic.

It is worth noticing that the question of the intuitionistic analogues of the classes Σ_n is left open. In fact, even for the classes Π_n it is not excluded at present that there can be more than one natural counterpart of these classes in intuitionistic arithmetic. Only further research and 'experimentation' can settle this question. However, by the results of W. Burr we now have at least one good candidate.

In the remaining part of the paper some additional facts are established. The next result is an improvement of the Visser–Wehmeier theorem. One defines another hierarchy of classes of formulas Θ_n, essentially by closing the classes Φ_n under the existential (as well as the universal) quantification. This makes

the class Θ_2 already contain all prenex formulas. W. Burr shows, by means of Gödel's Dialectica interpretation, that the restriction of the induction schema to the class Θ_n proves the same Π_2-sentences as the classical fragment $I\Pi_n$. For $n = 2$ this statement implies the Visser–Wehmeier theorem.

Finally, collection principles in intuitionistic arithmetic are considered. For the standard formulation of the collection principle

$$\forall x \leq a \, \exists y \, \phi(x, y) \rightarrow \exists z \forall x \leq a \, \exists y \leq z \, \phi(x, y),$$

where ϕ is any arithmetical formula, the author shows, using a result of U. Kohlenbach, that the provably total functions are bounded by polynomials. This contrasts with the behaviour of this schema within the classical logic: classically, it is equivalent to the full induction. The author also verifies that the contrapositive formulation of the collection schema over intuitionistic logic proves the law of the excluded middle, that is, it implies not only HA, but the whole of PA.

The paper is written in a nice and clear style. Although it does not develop significant new 'methods', it does introduce an important new definition. It will predictably be often quoted in the future papers on intuitionistic arithmetic.

References

[1] P. Hájek and P. Pudlák. *Metamathematics of First Order Arithmetic.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[2] K. Wehmeier. *Semantical investigations in intuitionistic first-order arithmetic.* Ph.D. Thesis, Münster, 1996.