A proof-theoretic analysis of collection

Lev D. Beklemishev*
Steklov Mathematical Institute
Gubkina 8, 117966 Moscow, Russia
e-mail: lev@bekl.mian.su

June 20, 1997

Abstract

By a result of Paris and Friedman, the collection axiom schema for Σ_{n+1} formulas, $B\Sigma_{n+1}$, is Π_{n+2} conservative over $I\Sigma_n$. We give a new proof-theoretic proof of this theorem, which is based on a reduction of $B\Sigma_n$ to a version of collection rule and a subsequent analysis of this rule via Herbrand's theorem. A generalization of this method allows us to improve known results on reflection principles for $B\Sigma_n$ and to answer some technical questions left open by Sieg [23] and Hájek [9]. We also give a new proof of independence of $B\Sigma_{n+1}$ over $I\Sigma_n$ by a direct recursion-theoretic argument and answer an open problem formulated by Gaifman and Dimitracopoulos [8].

1 Introduction

In this paper we shall deal with arithmetical theories containing Kalmar elementary arithmetic EA. EA is a first order theory formulated in the language with function symbols for all Kalmar elementary functions, and mathematical axioms of EA are (1) (open) defining equations for all these functions; (2) the schema of induction for open formulas. The fragment of EA in the language of PA is also known under the name $I\Delta_0 + \mathrm{Exp}$, and EA can be considered as a definitional extension of that theory. Bounded formulas in the language of EA are provably equivalent to quantifier free ones, and moreover, it is known that EA itself admits a purely universal (or quantifier free) axiomatization. Σ_n and Π_n formulas in the language of EA are prenex formulas obtained from the bounded ones by n alternating blocks of similar quantifiers, starting from '$\exists$' and '$\forall$', respectively.

Σ_n *collection principle* is the schema

$$B\Sigma_n: \qquad \forall x \leq a \exists y\, A(x,y) \to \exists z \forall x \leq a \exists y \leq z\, A(x,y),$$

for all Σ_n formulas $A(x,y)$, possibly containing other parameters. It can be interpreted as saying that every Σ_n definable function on a finite domain has a finite range. $B\Pi_n$ is defined similarly and is known to be equivalent to $B\Sigma_{n+1}$, in particular, $B\Sigma_1$ is equivalent to the quantifier free collection principle. It is also known that $B\Sigma_n$ is equivalent to the so-called *finite axiom of choice*:

$$\Sigma_n\text{-FAC}: \qquad \forall x \leq a \exists y\, A(x,y) \to \exists z \forall x \leq a\, A(x,(z)_x).$$

In summary, over EA the following schemata are equivalent: $B\Sigma_{n+1}$, $B\Pi_n$, Σ_{n+1}-FAC, Π_n-FAC (see [10]).

*The research described in this publication was made possible in part by the Russian Foundation for Fundamental Research (project 96-01-01222).

The study of restricted collection principles in arithmetic was initiated by Parsons [19], using proof-theoretic methods, and further developed in the 70s by Paris and his school, after the importance of collection principles for the model theory of arithmetic was realized. Parsons showed that the theories $EA + B\Sigma_n$ (for $n > 1$ simply denoted $B\Sigma_n$) form a proper hierarchy of fragments of PA that falls strictly between the levels of the hierarchy of restricted induction schemata:

$$EA \subset EA + B\Sigma_1 \subset I\Sigma_1 \subset B\Sigma_2 \subset I\Sigma_2 \subset \dots$$

Paris [16], and independently H. Friedman, discovered that, for all n, $B\Sigma_{n+1}$ is conservative over $I\Sigma_n$ w.r.t. Π_{n+2} sentences. This conservation result was obtained by model-theoretic methods. Because of its importance, and especially because various applications required a formalization of this result in a reasonably weak fragment of PA, later a few purely proof-theoretic proofs have been found [23, 5, 4].[1]

We suggest here a new and particularly clear proof-theoretic treatment of $B\Sigma_n$. Our analysis is based on a reduction of $B\Sigma_n$ to a natural version of Σ_n collection rule (by an application of the standard cut-elimination) and a subsequent independent analysis of this rule via Herbrand's theorem.

Σ_n collection rule does not seem to have been previously systematically studied in the literature. Its analysis allows for other useful applications, which are briefly described below. These results constitute the main content of this paper.

In contrast with the Paris and Friedman theorem, ever since the work of Parsons it has been known that $I\Sigma_n$ is a "much stronger" theory than $B\Sigma_n$, e.g., because the former proves the consistency of the latter. However, the precise strength of $I\Sigma_n$ w.r.t. $B\Sigma_n$ in terms of reflection principles was unknown. The best results in this direction are contained in [5], where it is shown that $I\Sigma_n$ proves the Π_{n+1} uniform reflection principle for $B\Sigma_n$. We show that this bound can be improved to Π_{n+2} reflection, and this result is best possible, because, in fact, $I\Sigma_n$ is equivalent to Π_{n+2} uniform reflection principle for $B\Sigma_n$.

This result relies on a generalization of the Paris and Friedman theorem that characterizes Π_{n+2} consequences of theories of the form $T + B\Sigma_n$, where T is any Π_{n+2} axiomatized extension of EA. This characterization is based on the analysis of Σ_{n+1} collection rule and, in our opinion, presents an independent interest.

As another application of the same techniques we characterize the closures of theories $B\Sigma_n$ under the induction rules of restricted arithmetical complexity. In particular, we answer a question formulated by Sieg [23] about the optimality of a certain conservation result for this hierarchy. We also give the affirmative answer to a question by Hájek [9], whether $PRA + B\Sigma_1$ is interpretable in PRA.

In the last section of this paper we construct a recursive function $f(x)$ with an elementary graph such that the function $\max_{i\leq x} f(i)$ does not belong to the closure of f and the class of elementary functions under composition. This allows us to construct sound Π_2 axiomatized theories that are not closed under the Σ_1 collection rule. In particular, this yields a new and easy recursion-theoretic proof of independence of $B\Sigma_1$ of the set of all true Π_2 sentences and provides an answer to an open question formulated by Gaifman and Dimitracopoulos [8]. Relativization of these results yields the independence of $B\Sigma_n$ of all true Π_{n+2} sentences, for $n > 1$, in a similar way. Our proof is to be compared with the original proof of that theorem due to Parsons [19], and with the proofs by Paris and Kirby [17], Adamowicz [1], and Kaye [11].

Acknowledgement. We thank A. Razborov for suggesting us a proof of Lemma 6.1, and V. Shavrukov for drawing to our attention an open problem in [8] and indicating

[1] The proof of this theorem given in [23] contains a gap that can hardly be repaired without a considerable reworking of the techniques of that paper. However, the treatment of relativization in our paper is reminiscent of [23] and can also be seen as a way of correcting that proof.

that an answer to it easily follows from our results.

2 Bootstrapping Σ_n collection rule

In this section we present some preparatory lemmata to the effect that some of the standard tricks used for the analysis of Σ_n collection schema work for Σ_n collection rule as well.

First, we adopt some very general terminology and notation concerning rules in arithmetic (cf also [3]). We say that a *rule* is a set of *instances*, that is, expressions of the form

$$\frac{A_1, \dots, A_n}{B},$$

where $A_1, \dots, A_n$ and B are formulas. Derivations using rules are defined in the standard way; $T + R$ denotes the closure of a theory T under a rule R and first order logic. $[T, R]$ denotes the closure of T under *unnested applications* of R, that is, the theory axiomatized over T by all formulas B such that, for some formulas $A_1, \dots, A_n$ derivable in T, $\frac{A_1,\dots,A_n}{B}$ is an instance of R. A rule R_1 is *reducible* to R_2 iff, for every theory T containing EA, $[T, R_1] \subseteq [T, R_2]$. R_1 and R_2 are *congruent* iff they are mutually reducible. We say that a rule R is *congruent to a set of axioms* U, iff R is congruent to U considered as a trivial rule (with the empty premise), or equivalently, iff $[T, R] \equiv T + U$ for any theory T extending EA. Notice that rules congruent to axiom schemata have trivial behaviour in the sense that they cannot be applied fruitfully more than once: nested applictions of such rules do not yield new theorems.

Collection rule is defined as follows:

$$\text{CR:} \qquad \frac{\forall x \exists y A(x,y)}{\forall x \exists y \forall u \leq x \exists v \leq y A(u,v)}$$

Σ_n-CR denotes the above rule with the restriction that A is a Σ_n formula, and similarly for Π_n. QF-CR is the collection rule for quantifier-free (or, equivalently, bounded) formulas in the language of EA. In general, we allow parameters to occur in A, however the following lemma is in order.

Lemma 2.1. *Σ_n-CR is reducible to parameter-free Σ_n-CR.*

Proof. An application of CR for a formula $\forall x \exists y A(x, y, a)$ can be reduced, using the standard pairing and projection functions, to the one for $\forall z \exists y A((z)_0, y, (z)_1)$, q.e.d.

Lemma 2.2. *Σ_1-CR is reducible to QF-CR. Σ_{n+1}-CR is reducible to Π_n-CR.*

Proof. If $A(x, y)$ has the form $\exists z A_0(x, y, z)$, then an application of CR for $\forall x \exists y A(x, y)$ is reducible to CR for the formula $\forall x \exists y A_0(x, (y)_0, (y)_1)$, q.e.d.

Lemma 2.3. *$[EA, \Sigma_{n+1}\text{-CR}]$ contains $I\Sigma_n$.*

Proof. We shall derive an equivalent form of Σ_n induction, known as the *strong Σ_n collection principle*:

$$S\Sigma_n: \qquad \forall x \exists y \forall u \leq x\, (\exists z A(u, z, a) \rightarrow \exists z \leq y A(u, z, a)),$$

for $A \in \Sigma_n$. See [10] for an elementary proof of the equivalence of $I\Sigma_n$ and $S\Sigma_n$.

To prove $S\Sigma_n$ by Σ_{n+1}-CR, for a given Σ_n formula $A(x, y, a)$ we reason as follows. Obviously EA proves

$$\forall x \exists y\, (A(x, y, a) \vee \forall z \neg A(x, z, a)).$$

Applying Σ_{n+1}-CR we obtain

$$\forall x \exists y \forall u \leq x \exists v \leq y\, (A(u,v,a) \vee \forall z \neg A(u,z,a)),$$

and so, $[EA, \Sigma_{n+1}\text{-CR}]$ proves

$$\forall x \exists y \forall u \leq x\, (\exists z A(u,z,a) \rightarrow \exists v \leq y A(u,v,a)),$$

q.e.d.

Corollary 2.1. Σ_n-CR *is congruent to the following version of* choice rule:

$$\mathrm{CR}' : \qquad \frac{\forall x \exists y A(x,y)}{\forall x \exists z \forall i \leq x A(i,(z)_i)},$$

for Σ_n or, equivalently, Π_{n-1} formulas $A(x,y)$.

Proof. Σ_n-CR is obviously reducible to Σ_n-CR$'$. For the opposite reduction assume that a theory T proves $\forall x \exists y A(x,y)$, where A is Π_{n-1} (or quantifier-free if $n=1$). By Σ_n-CR we obtain

$$\exists z \forall i \leq x \exists y \leq z A(i,y),$$

whence

$$\exists z \forall i \leq x \exists y \leq (z)_i A(i,y).$$

By Lemma 2.3 $[T, \Sigma_n\text{-CR}]$ contains $I\Sigma_{n-1}$. By a theorem of Parsons ([20], see also [10]) the least element principle for $\Delta_0(\Sigma_{n-1})$ formulas, that is, for the formulas built up from Σ_{n-1} ones using boolean connectives and bounded quantifiers, is available in $I\Sigma_{n-1}$. It follows that we can derive that there is a minimal z such that

$$\forall i \leq x \exists y \leq (z)_i A(i,y).$$

Since the standard coding of finite sequences is monotonic in each argument, z is as required:

$$\forall i \leq x A(i,(z)_i),$$

q.e.d.

For technical purposes it will be convenient for us to have a version of CR in a sequential formulation. We adopt a variant of Tait calculus from [21] throughout the paper. In particular, sequents are finite sets of formulas, understood as disjunctions, negation is treated via de Morgan's laws, etc.

Our version of Σ_n collection (or, rather, Σ_n choice) rule is as follows:

$$\mathrm{CR}'' : \qquad \frac{\Gamma, \forall x \leq t \exists y A(x,y)}{\Gamma, \exists z \forall x \leq t A(x,(z)_x)},$$

where we require that $A \in \Sigma_n$ and Γ is a set of Π_{n+1} formulas. We call this rule Σ_n-CR$''$.

Lemma 2.4. Σ_n-CR *and* Σ_n-CR$''$ *are congruent.*

Proof. Σ_n-CR$'$ is obviously reducible to Σ_n-CR$''$ by letting t be a free variable, and letting Γ be empty. For the opposite reduction notice that by dropping the outer universal quantifiers in Γ (and introducing fresh free parameters) we may actually assume Γ to be Σ_n. Then the disjunction of Γ can be accumulated by the principal formula A of the choice rule, q.e.d.

Remark 2.1. Omitting the restriction that Γ is Π_{n+1} from the formulation of Σ_n-CR$''$ yields a rule which is congruent to $B\Sigma_n$ as an axiom schema.

3 Reduction of $B\Sigma_n$ to Σ_n collection rule

Here we show that the standard cut-elimination procedure can be seen as a reduction of $B\Sigma_n$ to Σ_n-CR. As a byproduct we derive a conservation result of Kaye, Paris and Dimitracopoulos [12] for $B\Sigma_n$ over the parameter-free Σ_n collection schema.

Theorem 1. *$B\Sigma_n$ is Π_{n+1} conservative over $EA + \Sigma_n$-CR.*

Proof. We shall consider cut-free derivations of sequents of the form $\neg B\Sigma_n, \Pi$, where Π is a set of Π_{n+1} formulas, and $\neg B\Sigma_n$ is a finite set of negated instances of Σ_n collection schema, each having the form

$$\exists z \exists u\, [\forall x \leq u \exists y A(x, y, z) \wedge \forall w \exists x \leq u \neg A(x, (w)_x, z)],$$

for some $A \in \Pi_{n-1}$. (Here we use the equivalence of $B\Sigma_n$ and Π_{n-1}-FAC over EA.) Let us call the formula in square brackets $B_A(z, u)$.

Every sequent Γ occurring in a cut-free proof of $\neg B\Sigma_n, \Pi$ consists of the formulas of the following two types: (i) formulas $\neg B\Sigma_n$, or those of the form $\exists u B_A(s, u)$ or $B_A(s, t)$ for some $A \in \Pi_{n-1}$ and for some terms s and t; (ii) subformulas of formulas from Π, or proper subformulas of $B_A(s, t)$. Let Γ^- denote the result of deleting all formulas of type (i) from such a sequent Γ. Notice that Γ^- consists entirely of Π_{n+1} formulas.[2]

Lemma 3.1. *If Γ is cut-free provable, then Γ^- is provable by Σ_n-CR$''$ (and first order logic).*

Proof. The proof goes by induction on the height of the derivation d of Γ. We only need to treat the case when a formula of type (i) is introduced by the last inference in d, and in this case essentially only formulas of the form $B_A(s, t)$ must be treated (in other cases, after applying the $(\cdot)^-$ operation, the premise and the conclusion of the inference will be the same). So, we assume the derivation d has the form

$$(\wedge) \qquad \frac{\forall x \leq t \exists y A(x, y, s), \Delta \qquad \forall w \exists x \leq t \neg A(x, (w)_x, s), \Delta}{B_A(s, t), \Delta},$$

where $A \in \Pi_{n-1}$. By the induction hypothesis we obtain Σ_n-CR$''$ derivations of

$$\forall x \leq t \exists y A(x, y, s), \Delta^- \tag{1}$$

and

$$\forall w \exists x \leq t \neg A(x, (w)_x, s), \Delta^-. \tag{2}$$

Since Δ^- consists of Π_{n+1} formulas, Σ_n-CR$''$ is applicable to (1) and we obtain a proof of

$$\exists w \forall x \leq t A(x, (w)_x, s), \Delta^-.$$

Applying the cut rule with the sequent (2) yields Δ^-, q.e.d.

The claim of Theorem 1 immediately follows from the above lemma, using the fact that EA has a purely universal axiomatization. We also obtain the following generalization.

Theorem 2. *If T is a Π_{n+2} axiomatized theory containing EA, then $T + B\Sigma_n$ is Π_{n+1} conservative over $T + \Sigma_n$-CR.*

[2] This is the reason we prefer to deal with the Π_{n-1}-FAC form of collection principle rather than with the $B\Pi_{n-1}$ form.

Proof. Notice that for Σ_{n+1} axiomatized theories T this statement obviously follows from Theorem 1 by the deduction theorem. For Π_{n+2} axiomatized theories we have to slightly generalize the proof of Theorem 1.

Now we consider cut-free derivations of sequents of the form $\neg T, \neg B\Sigma_n, \Pi$, where $\neg T$ is a set of negated axioms of T of the form $\exists x C(x)$, with $C \in \Pi_{n+1}$, and everything else is as before. We associate with the formulas $\exists x C(x)$ type (i), and their proper subformulas will have type (ii). As before, Γ^- will denote the result of deleting all formulas of type (i) from a sequent Γ. Then the analog of Lemma 3.1 is as follows.

Lemma 3.2. *If Γ is cut-free provable, then Γ^- is provable by Σ_n-CR$''$ and first order logic from the axioms of T (considered as initial sequents).*

Proof. We only have to treat the additional case, when the last inference of a given derivation has the form

$$(\exists) \qquad \frac{C(t), \Delta}{\exists x C(x), \Delta}.$$

By the induction hypothesis we obtain a proof of $C(t), \Delta^-$ from T using Σ_n-CR$''$ and first order logic. Then we derive $\exists x C(x), \Delta^-$, and apply the cut rule with the sequent $\forall x \neg C(x)$, which is an axiom of T, q.e.d.

Remark 3.1. The anonymous referee suggested that there is a relationship between the above Theorem 2 and Theorem 2.4 of Kaye, Paris and Dimitracopoulos [12] stating that, for $n > 0$, $B\Sigma_n$ is a Σ_{n+2} conservative extension of *parameter-free* Σ_n-collection schema $B\Sigma_n^-$. $B\Sigma_n^-$ is defined by the schema

$$\forall x \exists y\, A(x,y) \to \forall u \exists v \forall x \leq u \exists y \leq v\, A(x,y),$$

for Σ_n formulas $A(x,y)$ without any other free variables but x, y. This conservation result was proved model-theoretically.

By Lemma 2.1 $B\Sigma_n^-$ (and any extension of it) is closed under Σ_n-CR. However, as we shall see below, $EA + \Sigma_n$-CR is equivalent to $I\Sigma_{n-1}$, that is, is strictly weaker than $B\Sigma_n^-$. So, there does not seem to be a way of directly obtaining Theorem 2 from Kaye, Paris, and Dimitracopoulos' result. However, we can go the other way around and derive the conservation result for parameter-free collection schema from Theorem 2.

Corollary 3.1. *For $n > 0$, $B\Sigma_n$ is Σ_{n+2} conservative over $B\Sigma_n^-$.*

Proof. Assume $B\Sigma_n \vdash \phi$, where $\phi \in \Sigma_{n+2}$. Then, obviously, $\neg\phi + B\Sigma_n \vdash 0 = 1$. Since $\neg\phi \in \Pi_{n+2}$, by Theorem 2 we conclude $EA + \neg\phi + \Sigma_n$-CR $\vdash 0 = 1$. By Lemma 2.1 it follows that $B\Sigma_n^- + \neg\phi \vdash 0 = 1$ and $B\Sigma_n^- \vdash \phi$, q.e.d.

4 Analysing Σ_n collection rule

In order to show that $B\Sigma_{n+1}$ is Π_{n+2} conservative over $I\Sigma_n$ it remains us to prove the following two lemmas.

Lemma 4.1. *EA is closed under Σ_1-CR.*

Proof. We show that EA is closed under QF-CR. Suppose $EA \vdash \forall x \exists y A(x,y)$, with A quantifier-free. By Herbrand's theorem there are Kalmar elementary terms $t_1, \ldots, t_n$ such that

$$EA \vdash A(x, t_1(x)) \vee A(x, t_2(x)) \vee \ldots \vee A(x, t_n(x)).$$

Every elementary term is provably majorizable by a provably increasing one, so we obtain

$$EA \vdash \forall x \exists y \le t(x) A(x, y),$$

for a certain monotonic term t (this actually is a version of the so-called Parikh's theorem for EA). Provable monotonicity of t implies that

$$EA \vdash \forall x \exists y_{=t(x)} \forall u \le x \exists v \le y A(u, v),$$

q.e.d.

Corollary 4.1. *$EA + B\Sigma_1$ is conservative over EA for Π_2 formulas.*

Lemma 4.2. *$I\Sigma_n$ is closed under Σ_{n+1}-*CR.

Proof. This lemma is proved by a relativization of the previous argument. We shall only work out the details for the case $n = 1$; the proof for general n is similar. Our method is the same as in [3] and derives from [23, 15].

We shall work in a first order language obtained from that of EA by adding a new unary function symbol κ. $\Delta_0(\kappa)$ formulas are bounded formulas in the extended language, where κ is also allowed to occur in bounding terms. $\Sigma_1(\kappa)$ formulas are those of the form $\exists x_1 \ldots \exists x_n A$ with $A \in \Delta_0(\kappa)$, and similarly for other classes of the arithmetic hierarchy. Relativized version of Kalmar elementary arithmetic, EA^κ, is the theory axiomatized over EA by the schema of induction for $\Delta_0(\kappa)$ formulas.

Within EA^κ one can $\Delta_0(\kappa)$ define the graph of, and introduce a function symbol for, the function that encodes the sequence of the first $x + 1$ values of κ:

$$\bar{\kappa}(x) := \langle \kappa(0), \ldots, \kappa(x) \rangle.$$

For this, we first bound the values of κ from above, that is, define the function $m(x) := \max_{i \le x} \kappa(i)$. Consider an auxiliary function

$$t(x) := \mu z \le x.\ \forall i \le x\ \kappa(i) \le \kappa(z).$$

The graph of t is clearly $\Delta_0(\kappa)$, and since $t(x) \le x$, the totality of t is provable in EA^κ by $\Delta_0(\kappa)$ induction on x: For $x = 0$, one takes $t(0) = 0$. For $x = n + 1$, assuming $z = t(n) \le n$ exists, one takes $t(n+1) = z$, if $\kappa(n) \le \kappa(z)$, and $t(n+1) = n$, otherwise. Then one defines $m(x) = \kappa(t(x))$. Using $m(x)$ it is then easy to define and prove the totality of $\bar{\kappa}$ (cf. [3]).

Any function elementary in κ can be obtained from $\bar{\kappa}$ and elementary functions by composition. This fact is formalizable, hence all such functions are representable in EA^κ. For the same reason, every $\Delta_0(\kappa)$ formula turns out to be equivalent to an open formula in the language with a function symbol for $\bar{\kappa}$ and for all elementary functions. (Notice that $\kappa(x)$ itself can be expressed by the term $(\bar{\kappa}(x))_x$.) It is not difficult, although somewhat tedious, to see that EA^κ admits a purely universal axiomatization in this language. The reader interested in the details may consult [3] for an elaboration of the proofs of all these statements. These results obviously carry over to languages obtained from that of EA by adding more than one, and possibly more than unary, function symbol.

Notice that $\bar{\kappa}$ is a provably increasing function in EA^κ. Hence, any term in the language with $\bar{\kappa}$ and all elementary functions can be majorized by a provably increasing term, and thus we obtain a relativized version of Lemma 4.1.

Lemma 4.3. *EA^κ is closed under $\Sigma_1(\bar{\kappa})$-*CR. *The same holds for any extension of EA^κ by $\Pi_1(\bar{\kappa})$ axioms.*

Now we are ready to prove that $I\Sigma_1$ is closed under Π_1-CR (which is congruent to Σ_2-CR). We shall, once again, make use of the fact that $I\Sigma_1$ can be axiomatized by the strong collection schema $S\Sigma_1$ (which is equivalent to quantifier free strong collection schema).

Assume that

$$I\Sigma_1 \vdash \forall x \exists y \forall z \phi(x, y, z),$$

for ϕ quantifier-free. Then there are finitely many instances of strong collection

$$\forall x \exists y \forall u \leq x\, (\exists z A_i(u, z) \to \exists z \leq y A_i(u, z)), \tag{3}$$

for open formulas $A_i(u, z)$, $i = 1, \ldots, n$, that imply $\forall x \exists y \forall z \phi(x, y, z)$ over EA.[3] For each instance (3) we introduce a Skolem function $\nu_i(x)$, together with the axiom

$$S_i: \qquad \forall u \leq x\, (\exists z A_i(u, z) \to \exists z \leq \nu_i(x) A_i(u, z)).$$

Besides, we introduce a function $\kappa(x)$ together with a similar axiom

$$K: \qquad \forall u \leq x\, (\exists z \neg\phi((u)_0, (u)_1, z) \to \exists z \leq \kappa(x) \neg\phi((u)_0, (u)_1, z)).$$

Let T denote the theory $EA^{\kappa, \nu_1, \ldots, \nu_n}$ plus the axioms $S_1, \ldots, S_n$, and K.

Lemma 4.4. *T is interpretable in $I\Sigma_1$ by an interpretation which is identical on the formulas in the language of EA.*

Proof. Graphs of the functions ν_i will be defined by formulas $\nu_i(x) = y$ expressing that y is the least v such that

$$\forall u \leq x\, (\exists z A_i(u, z) \to \exists z \leq v A_i(u, z)).$$

Notice that these formulas are $\Delta_0(\Sigma_1)$. So, $I\Sigma_1$ proves $\forall x \exists! y\ \nu_i(x) = y$ (applying successively strong Σ_1 collection and $\Delta_0(\Sigma_1)$ least number principle). Besides, the functions ν_i thus introduced in $I\Sigma_1$ are all monotonic in the sense that $I\Sigma_1$ proves

$$\forall x, y\, (x \leq y \to \nu_i(x) \leq \nu_i(y)). \tag{4}$$

Now we recall a version of a result of Gaifman and Dimitracopoulos ([8], cf also [3]) stating that, for monotonic functions ν_i, $\Delta_0(\nu_i)$ induction schema is reducible over EA plus (4) to the induction schema for bounded formulas in the *graph* of ν_i, that is, for formulas built up from $\nu_i(x) = y$ and elementary ones by means of boolean connectives and quantifiers bounded by elementary functions. Since the graphs of ν_i are $\Delta_0(\Sigma_1)$, this means that $I\Sigma_1$ interprets $\Delta_0(\nu_i)$ induction.

Similarly, we define the graph of κ by the formula $\kappa(x) = y$ expressing that y is the least v such that

$$\forall u \leq x\, (\exists z \neg\phi((u)_0, (u)_1, z) \to \exists z \leq v \neg\phi((u)_0, (u)_1, z)).$$

All the above considerations obviously apply to this formula, too, and, in fact, to all functions $\nu_1, \ldots, \nu_n, \kappa$ simultaneously, which means that all of the theory T is interpreted in $I\Sigma_1$, q.e.d.

Now we finish our proof of Lemma 4.2. First of all, notice that formulas S_i and K are $\Pi_1(\kappa, \nu_1, \ldots, \nu_n)$. Besides, axioms S_i logically imply all instances (3), so we have:

$$T \vdash \forall x \exists y \forall z \phi(x, y, z).$$

[3] W.l.o.g. we may assume that formulas A_i do not contain extra free parameters. Similarly to Remark 2.1 one can prove that the strong collection schema is equivalent to its parameter free version.

Now we observe that the function

$$\kappa_1(x,y) := \mu z \leq \kappa(\langle x,y\rangle).\ \neg\phi(x,y,z)$$

is elementary in κ, and hence it can be defined by a term in EA^{κ} (formulated in the language with a function symbol for $\bar{\kappa}$). Using axiom K we may then infer that T proves

$$\forall z\phi(x,y,z) \leftrightarrow \phi(x,y,\kappa_1(x,y)),$$

and hence

$$T \vdash \forall x\exists y\phi(x,y,\kappa_1(x,y)).$$

Now we are in a position to apply Lemma 4.3, as T is a $\Pi_1(\kappa,\nu_1,\ldots,\nu_n)$ axiomatized extension of $EA^{\kappa,\nu_1,\ldots,\nu_n}$. By $\Sigma_1(\bar{\kappa},\bar{\nu}_1,\ldots,\bar{\nu}_n)$-CR we conclude:

$$T \vdash \forall x\exists y\forall u \leq x\exists v \leq y\phi(u,v,\kappa_1(u,v)),$$

and thus

$$T \vdash \forall x\exists y\forall u \leq x\exists v \leq y\forall z\phi(u,v,z)).$$

By Lemma 4.4 the latter formula is derivable in $I\Sigma_1$, q.e.d.

Corollary 4.2. *$EA + \Sigma_{n+1}$-CR is equivalent to $I\Sigma_n$.*

Theorem 3. *$B\Sigma_{n+1}$ is Π_{n+2} conservative over $I\Sigma_n$.*

Proof: follows from Theorem 1 and the previous corollary, q.e.d.

Remark 4.1. Corollary 4.2 can also be easily derived from Theorem 3.

Theorem 3 has the following corollary: For any Σ_{n+2} axiomatized theory T extending EA, $T + B\Sigma_{n+1}$ is Π_{n+2} conservative over $T + I\Sigma_n$. This fact may not always hold for Π_{n+2} axiomatized theories, as we shall see later. Our goal here is to give a characterization of Π_{n+2} consequences of theories $T + B\Sigma_{n+1}$, for Π_{n+2} axiomatized theories T. It is to be compared with (and was partially inspired by) what is going on in Sections 5 and 9 of Visser [24].

Let π be a Π_{n+2} sentence of the form $\forall x\exists yA(x,y)$, where $A \in \Pi_n$ (or is quantifier free for $n=0$). Then $\bar{\pi}$ will denote the formula $\forall x\exists y\forall u \leq x\exists v \leq yA(u,v)$.

Theorem 4. *For any $\pi \in \Pi_{n+2}$, the closure of $EA+\pi$ under Σ_{n+1}-CR is equivalent to $I\Sigma_n + \bar{\pi}$ (or $EA + \bar{\pi}$, for $n = 0$).*

Proof. Clearly, one can derive both $I\Sigma_n$ and $\bar{\pi}$ in $[EA+\pi, \Sigma_{n+1}\text{-CR}]$. The crucial point is to demonstrate that $I\Sigma_n + \bar{\pi}$ is closed under Σ_{n+1}-CR. This fact is proved by a little generalization of our proof of Lemma 4.2.

Suppose we have a derivation of a Π_{n+2} formula $\forall x\exists y\phi(x,y)$ from EA, $\bar{\pi}$, and finitely many instances of $S\Sigma_n$. By introducing finitely many monotonic Skolem functions $\nu_1,\ldots,\nu_k$ in the same way as in Lemma 4.2 we can quantifier free represent these instances, as well as the formulas $\phi(x,y)$ and $A(x,y)$ (where $\pi = \forall x\exists yA(x,y)$). Then the theory $U := EA^{\nu_1,\ldots,\nu_k} + \forall x\exists y\forall u \leq x\exists v \leq yA_0(u,v)$+Skolem axioms for $\nu_1,\ldots,\nu_k$, where A_0 is the quantifier free translation of A, turns out to be interpretable in $I\Sigma_n + \bar{\pi}$. Now we introduce another Skolem function $f(x)$ together with the axiom

$$F: \qquad \forall u \leq x\exists v \leq f(x)A_0(u,v).$$

Since $\forall u \leq x\exists v \leq yA_0(u,v)$ is a bounded formula in the language of $EA^{\nu_1,\ldots,\nu_k}$, it is equivalent to a quantifier free formula in that theory, and hence axiom F is also equivalent to a quantifier free one. So, U admits a purely universal axiomatization in the language with f. We claim that U interprets $EA^{\nu_1,\ldots,\nu_k,f}$.

Indeed, under the interpretation

$$f(x) := \mu y.\ \forall u \leq x \exists v \leq y A_0(u,v),$$

U obviously proves the monotonicity of f, and the graph of f is $\Delta_0(\nu_1,\ldots,\nu_k)$. So, $\Delta_0(\nu_1,\ldots,\nu_k,f)$ induction is reducible to $\Delta_0(\nu_1,\ldots,\nu_k)$ induction, which is available in U. Now, by Lemma 4.3 we see that U is closed under $\Delta_0(\nu_1,\ldots,\nu_k,f)$ collection rule. Since ϕ is equivalent to a quantifier free formula in U, and U derives $\forall x \exists y \phi(x,y)$, U also proves

$$\forall x \exists y \forall u \leq x \exists v \leq y \phi(u,v).$$

By interpretability, the latter formula is provable in $I\Sigma_n + \bar{\pi}$, q.e.d.

Corollary 4.3. *If T is a Π_{n+2} axiomatizable extension of EA, then $[T, \Sigma_{n+1}\text{-CR}]$ is closed under Σ_{n+1}-CR.*

Corollary 4.4. *If T is a Π_{n+2} axiomatized extension of EA, then $T + B\Sigma_{n+1}$ is Π_{n+2} conservative over $I\Sigma_n + \overline{T}$, where $\overline{T}$ is the set of formulas $\bar{\pi}$ for all axioms π of T.*

Proof: follows from Theorems 2 and 4, q.e.d.

Remark 4.2. Theorem 4 gives us some sort of general description how Σ_n-CR works over an *arbitrary* theory T containing EA. Let $T \upharpoonright \Pi_n$ denote the theory axiomatized over EA by all Π_n theorems of T. Then by Theorem 4

$$[T, \Sigma_n\text{-CR}] \equiv T + [T \upharpoonright \Pi_{n+1}, \Sigma_n\text{-CR}] \equiv T + I\Sigma_{n-1} + \overline{T \upharpoonright \Pi_{n+1}}.$$

Notice that, even for T containing $I\Sigma_{n-1}$, the latter theory may have a larger Π_{n+1} fragment than $\overline{T \upharpoonright \Pi_{n+1}}$ (see Section 6). Further,

$$[[T, \Sigma_n\text{-CR}], \Sigma_n\text{-CR}] \equiv T + I\Sigma_{n-1} + \overline{(T + I\Sigma_{n-1} + \overline{T \upharpoonright \Pi_{n+1}}) \upharpoonright \Pi_{n+1}},$$

and in this way one continues, alternating $\upharpoonright \Pi_{n+1}$ and $\overline{(\cdot)}$ operations. Since the complexity of T can be higher than Π_{n+1}, in principle we may obtain a proper hierarchy.

5 Reflection principles

Clote, Hájek and Paris [5] gave a formalization of Paris and Friedman's conservation result in $I\Delta_0 + \text{Supexp}$. As a direct corollary they established that $I\Sigma_n$ proves the 1-consistency of $B\Sigma_n$. In fact, one can infer from their result a stronger statement: $I\Sigma_n$ proves the Π_{n+1} uniform reflection principle for $B\Sigma_n$, $\mathrm{RFN}_{B\Sigma_n}(\Pi_{n+1})$, where $\mathrm{RFN}_T(\Pi_n)$ denotes the schema

$$\forall x\ (\mathrm{Prov}_T(\ulcorner \phi(\dot{x}) \urcorner) \rightarrow \phi(x)), \qquad \phi(x) \in \Pi_n$$

asserting that every Π_n sentence provable in a theory T is true. Formalized Theorem 4 allows us to sharpen this result yet further.

Theorem 5. *For $n \geq 1$, $I\Sigma_n$ proves $\mathrm{RFN}_{B\Sigma_n}(\Pi_{n+2})$. In fact, over EA the following schemata are deductively equivalent: (a) $I\Sigma_n$; (b) $\mathrm{RFN}_{B\Sigma_n}(\Pi_{n+2})$; (c) $\mathrm{RFN}_{EA}(\Pi_{n+2})$.*

Proof. Notice that our proof of Theorem 4 is formalizable in $I\Delta_0 + \mathrm{Supexp}$, the only nonelementary step in that proof being an application of the Cut-elimination theorem. It is well-known that, for any theory T extending EA, $\mathrm{RFN}_T(\Pi_{n+1})$ is equivalent to $\mathrm{RFN}_T(\Sigma_n)$. So, we shall derive $\mathrm{RFN}_{B\Sigma_n}(\Sigma_{n+1})$ reasoning informally inside $I\Sigma_n$.

Assume that $B\Sigma_n \vdash \phi(k)$, for a sentence $\phi(k) \in \Sigma_{n+1}$, k being any numeral. Then $B\Sigma_n + \neg\phi(k)$ is an inconsistent theory. Since $\neg\phi(k) \in \Pi_{n+1}$, the theory $I\Sigma_{n-1} + \overline{\neg\phi(k)}$ must be inconsistent, too, by Corollary 4.4. So,

$$I\Sigma_{n-1} \vdash \neg\overline{\neg\phi(k)}. \tag{5}$$

By a well-known result of Leivant [14], $I\Sigma_n$ proves the Π_{n+2} uniform reflection principle for $I\Sigma_{n-1}$, so from (5) we may infer $\neg\overline{\neg\phi(k)}$ (as $\overline{\neg\phi(k)}$ is provably in $I\Sigma_{n-1}$ equivalent to a Π_{n+1} sentence). But $I\Sigma_n$ contains $B\Sigma_n$, and thus, within $I\Sigma_n$, $\overline{\neg\phi(k)}$ is provably equivalent to $\neg\phi(k)$, so we may finally infer $\phi(k)$. End of reasoning inside $I\Sigma_n$.

The second claim of Theorem 5 follows from the first one, together with the well-known fact that $\mathrm{RFN}_{EA}(\Pi_{n+2})$ implies $I\Sigma_n$ over EA, q.e.d.

Corollary 5.1. *For all $n \geq 1$ and $k \geq 2$, $\mathrm{RFN}_{B\Sigma_n}(\Pi_k)$ is deductively equivalent to $\mathrm{RFN}_{I\Sigma_{n-1}}(\Pi_k)$ over EA. For $k = 1$ the same equivalence holds over the stronger theory $I\Delta_0 + \mathrm{Supexp}$.*

Proof. Obvious for $k > n+2$, because $B\Sigma_n$ is Π_{n+2} axiomatizable. Follows from Theorem 5 for $k = n+2$, and from formalized Paris-Friedman conservation result for smaller k. For these n we also use the fact that $\mathrm{RFN}_{EA}(\Pi_2)$ implies Supexp over EA, and therefore formalizes our proof of Theorem 3, q.e.d.

As another corollary of this theorem and the results of [3] we shall characterize the closure of $B\Sigma_n$ under the induction rules. Recall that Σ_n induction rule, Σ_n-IR, is defined as follows:

$$\frac{A(0), \quad \forall x\,(A(x) \to A(x+1))}{\forall x A(x)},$$

for $A(x) \in \Sigma_n$. Π_n-IR is defined similarly. See [20, 23, 3] for basic information about the induction rules.

Sieg [23], p. 60, mentions as open the following question. By a result of Parsons, $I\Sigma_n$ is Π_{n+1} conservative over $EA + \Sigma_n$-IR. As $I\Sigma_n$ contains $B\Sigma_n$ and is closed under Σ_n-IR, $I\Sigma_n$ must also contain $B\Sigma_n + \Sigma_n$-IR and be Π_{n+1} conservative over it. The question is, whether this conservation result is best possible in terms of arithmetical complexity. The following proposition answers this question positively.

Proposition 5.1. *There is a Σ_{n+1} sentence, which is provable in $I\Sigma_n$ but not in $B\Sigma_n + \Sigma_n$-IR (nor in $B\Sigma_n + \Pi_{n+1}$-IR).*

Proof. Notice that $B\Sigma_n + \Pi_{n+1}$-IR can be axiomatized over $B\Sigma_n$ by a recursively enumerable set of Π_{n+1} sentences. By Theorem 5 we know that $I\Sigma_n$ proves the local Σ_{n+1} reflection principle for $B\Sigma_n$, that is, the schema

$$\mathrm{Prov}_{B\Sigma_n}(\ulcorner A \urcorner) \to A,$$

for all Σ_{n+1} *sentences* A. (This schema is usually denoted $\mathrm{Rfn}_{B\Sigma_n}(\Sigma_{n+1})$ and is strictly weaker than $\mathrm{RFN}_{B\Sigma_n}(\Sigma_{n+1})$.) By a well-known result of Kreisel and Lévy [13] (see also [2]), $\mathrm{Rfn}_T(\Sigma_{n+1})$ is not contained in any consistent Π_{n+1} axiomatized r.e. extension of a theory T. Therefore, there must exist an instance of $\mathrm{Rfn}_{B\Sigma_n}(\Sigma_{n+1})$ which is not provable in $B\Sigma_n + \Pi_{n+1}$-IR, q.e.d.

For a given theory T, let the theories $(T)^n_k$ be defined as follows:

$$(T)^n_0 := T; \quad (T)^n_{k+1} := (T)^n_k + \mathrm{RFN}_{(T)^n_k}(\Pi_n); \quad (T)^n_\omega := \bigcup_{k\geq 0}(T)^n_k.$$

The following theorem gives a precise characterization of the closure of $B\Sigma_n$ under Σ_k and Π_k induction rules in terms of iterated reflection principles.

Theorem 6. *1. $B\Sigma_n$ is closed under Σ_k-IR and Π_k-IR for $k < n$.*
2. $B\Sigma_n + \Pi_n\text{-IR} \equiv (I\Sigma_{n-1})^n_\omega + B\Sigma_n$ (for $n > 1$).
3. $B\Sigma_n + \Sigma_n\text{-IR} \equiv B\Sigma_n + \Pi_{n+1}\text{-IR} \equiv (I\Sigma_{n-1})^{n+1}_\omega + B\Sigma_n$.
4. $B\Sigma_n + \Sigma_k\text{-IR} \equiv B\Sigma_n + \Pi_{k+1}\text{-IR} \equiv (I\Sigma_{n-1})^{k+1}_\omega$, for $k > n$.

Proof. Most of the statements directly follow from formalized Theorem 3 and Theorems 1 and 2 of [3]. We only prove Statement 3. For this we need the following Lemma.

Lemma 5.1. *For any Π_{n+1} axiomatized theory T containing EA, $[T, \Sigma_n\text{-IR}]$ is closed under Σ_n-CR.*

Proof. We show that $[T, \Sigma_n\text{-IR}]$ can be axiomatized by Π_{n+1} formulas of the form $\bar{\pi}$. Suppose $\phi(x, y, a)$ is Π_{n-1} and

$$T \vdash \exists y\, \phi(0, y, a), \text{ and} \tag{6}$$
$$T \vdash \forall x\, (\exists y\, \phi(x, y, a) \to \exists y\, \phi(x+1, y, a)). \tag{7}$$

We let

$$\phi'(x, y) := \forall i \leq x\, \phi((i)_0, (y)_i, (i)_1).$$

Then it is not difficult to see that

$$T \vdash \exists y\, \phi'(0, y), \text{ and}$$
$$T \vdash \forall x\, (\exists y\, \phi'(x, y) \to \exists y'\, \phi'(x+1, y')).$$

We only prove the second statement, reasoning informally within T.

Assume $\forall i \leq x\, \phi((i)_0, (y)_i, (i)_1)$. If $(x+1)_0 = 0$, then by (6) there is a z such that $\phi(0, z, (x+1)_1)$, and one can take for y' the result of adding the element z to the end of the sequence y.

If $(x+1)_0 > 0$, then the code of the pair $p := \langle (x+1)_0 - 1, (x+1)_1 \rangle$ is strictly less than $x+1$, and so (by inductive assumption) there is a $z = (y)_p$ such that $\phi((x+1)_0 - 1, z, (x+1)_1)$. From (7) it follows that, for some z', there holds $\phi((x+1)_0, z', (x+1)_1)$. Consequently, one can take for y' the result of adding z' to the end of the sequence y. End of reasoning in T.

The formula $\exists y\, \phi'(x, y)$ is Σ_n, so by Σ_n-IR we may infer $\forall x \exists y\, \phi'(x, y)$, which implies $\forall a \forall x \exists y\, \phi(x, y, a)$ and (within $I\Sigma_{n-1}$) has the form $\bar{\pi}$. A similar (and, in fact, easier) argument shows that for each axiom π of T, $\bar{\pi}$ is also provable in $[T, \Sigma_n\text{-IR}]$. The result follows by Theorem 4, because $[T, \Sigma_n\text{-IR}]$ obviously contains $I\Sigma_{n-1}$, q.e.d.

Remark 5.1. Notice that our proof of the above lemma, in particular, shows that Σ_n-IR is reducible to parameter-free Σ_n-IR.

Corollary 5.2. *For any k, the theory $(I\Sigma_{n-1})^{n+1}_k + B\Sigma_n$ is Π_{n+1} conservative over $(I\Sigma_{n-1})^{n+1}_k$.*

Proof. By Corollary 8.1 of [3] $(I\Sigma_{n-1})^{n+1}_k$ coincides with the closure of $I\Sigma_{n-1}$ under k nested applications of Σ_n-IR. Hence, by the previous lemma, $(I\Sigma_{n-1})^{n+1}_k$ is closed under Σ_n-CR. The result follows by Theorem 2, q.e.d.

Now we complete the proof of Theorem 6 (Statement 3). To prove the equivalence

$$B\Sigma_n + \Sigma_n\text{-IR} \equiv (I\Sigma_{n-1})^{n+1}_\omega + B\Sigma_n,$$

by Theorem 3 of [3] we only have to establish that, for each k, $(I\Sigma_{n-1})^{n+1}_k$ is the maximal Π_{n+1} axiomatized subtheory of $(I\Sigma_{n-1})^{n+1}_k + B\Sigma_n$. This follows from the previous corollary.

For the equivalence $B\Sigma_n + \Pi_{n+1}\text{-IR} \equiv (I\Sigma_{n-1})^{n+1}_\omega + B\Sigma_n$, by Theorem 2 of [3], we have to show that for each k, over $I\Sigma_{n-1}$,

$$\mathrm{RFN}_{(I\Sigma_{n-1})^{n+1}_k + B\Sigma_n}(\Pi_{n+1}) \equiv \mathrm{RFN}_{(I\Sigma_{n-1})^{n+1}_k}(\Pi_{n+1}).$$

This immediately follows from formalized Corollary 5.2, q.e.d.

We summarize the results of Theorem 6 in the diagram shown on Fig. 1. Here, vertical lines correspond to partially conservative extensions of theories, and horizontal lines correspond to extensions by ω times iterated reflection principles (of indicated complexity).[4] All the arithmetical complexity bounds are optimal.

Hájek [9] formulates as an open problem the question whether $PRA + B\Sigma_1$ is relatively interpretable in PRA. We give an obvious answer to this question.

Proposition 5.2. *$PRA + B\Sigma_1$ is relatively interpretable in PRA.*

Proof. PRA proves the consistency of each finite subtheory of $PRA + B\Sigma_1$. Indeed, any finite subtheory of this theory is contained in a subtheory of the form $(EA)^2_k + B\Sigma_1$, where $(EA)^2_k$ is the fragment of PRA with function symbols for functions of the $k+3$ class of the Grzegorczyk hierarchy, or, equivalently, the extension of EA by k times iterated uniform Π_2 reflection. By Corollary 5.2 or, alternatively, by the obvious analogy of Corollary 4.1, $(EA)^2_k + B\Sigma_1$ is Π_2 conservative over $(EA)^2_k$. It is obvious (alternatively, well-known, see [23]) that PRA proves $\mathrm{RFN}_{(EA)^2_k}(\Pi_2)$ for each k, hence, by formalized conservativity,

$$PRA \vdash \mathrm{RFN}_{(EA)^2_k + B\Sigma_1}(\Pi_2).$$

The result follows by Feferman's theorem [7], as both PRA and $PRA + B\Sigma_1$ are reflexive theories, q.e.d.

Remark 5.2. We note, in passing, that the negative answer to Problem (2) in Hájek [9] is contained in the work of Visser [25].

6 Independence of collection rule

Σ_1-CR inputs a Π_2 formula and outputs another Π_2 formula. However, so far we have only encountered theories that were closed under Σ_1-CR. Here, for every sound finite Π_2 axiomatized extension T of EA, we construct a true Π_2 sentence π such that $T + \pi$ is not closed under Σ_1-CR. As a direct corollary we obtain a new recursion-theoretic proof of independence of $B\Sigma_1$ of the set of all true Π_2 sentences.

[4] A small singularity occurs twice at the bottom level of this hierarchy. Namely, $EA + \Pi_1$-IR is equivalent to ω times iterated Π_1 reflection for EA w.r.t. the proofs of *bounded cut-rank*, and similarly for the closure of $EA + B\Sigma_1$ under Π_1-IR. This deviation is due to the fact that $EA + \Pi_1$-IR does not prove the full Cut-elimination theorem for first order logic. See [3] for a discussion.

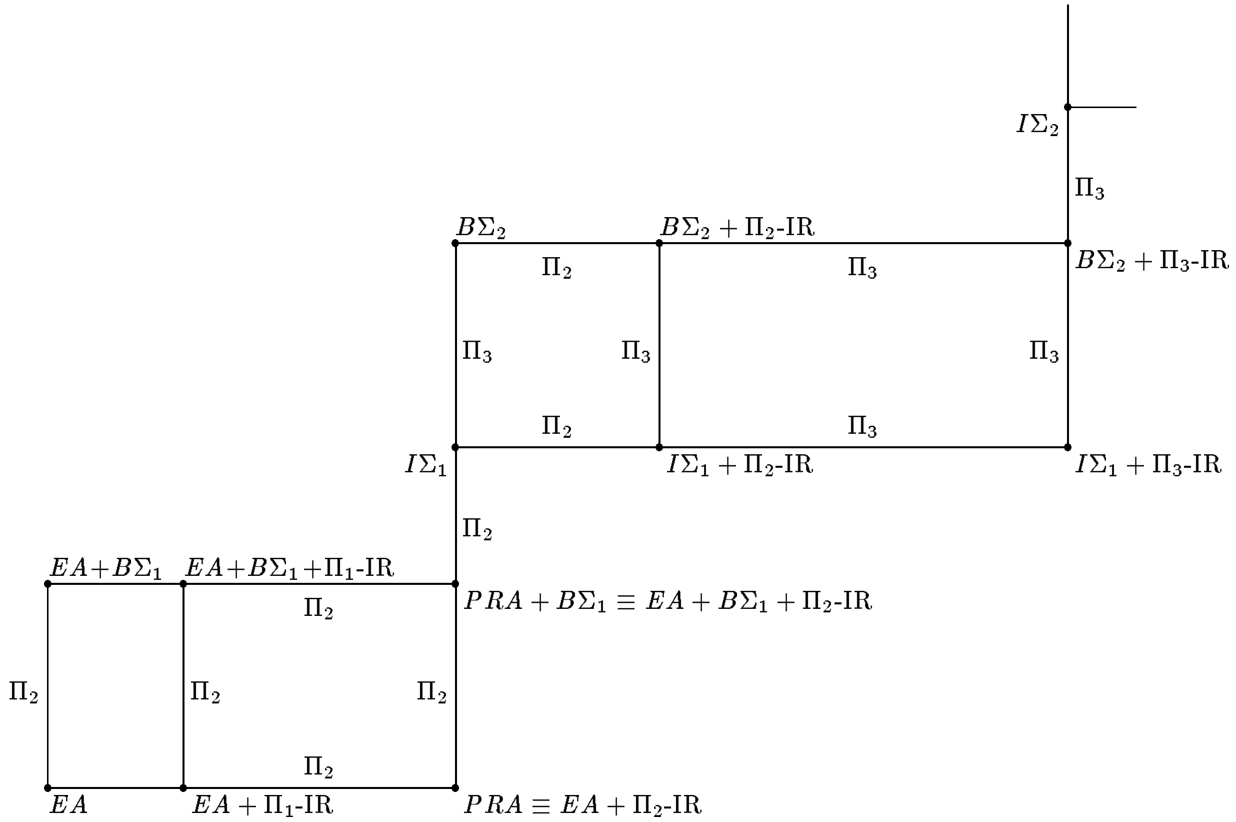

Figure 1: Collection axioms and induction rules.

We shall deal with various classes of number-theoretic functions. The basic class is the class of elementary functions $\mathcal{E}$. For a class K, $\mathbf{C}(K)$ denotes the closure of $K \cup \mathcal{E}$ under composition. $\mathbf{E}(K)$ is the elementary closure of K, that is, the class of functions obtained from $K \cup \mathcal{E}$ by composition and bounded sums and products. It is well-known (cf [18, 3]) that, for a unary function f, $\mathbf{E}(f) = \mathbf{C}(\bar{f})$, where $\bar{f}(x) := \langle f(0), \ldots, f(x) \rangle$. For a function oracle κ, relativized classes are defined as follows:

$$
\begin{aligned}
\mathcal{E}^{\kappa} &:= \mathbf{E}(\kappa) \\
\mathbf{C}^{\kappa}(K) &:= \mathbf{C}(\mathcal{E}^{\kappa} \cup K) = \mathbf{C}(K \cup \{\bar{\kappa}\}) \\
\mathbf{E}^{\kappa}(K) &:= \mathbf{E}(\mathcal{E}^{\kappa} \cup K) = \mathbf{E}(K \cup \{\kappa\})
\end{aligned}
$$

A number-theoretic function $f(x)$ is called *provably total recursive* in a theory T iff the graph of f can be represented by a Σ_1 formula $\psi(x, y)$ such that

$$T \vdash \forall x \exists ! y\, \psi(x, y).$$

The class of p.t.r.f.s of a theory T is denoted $\mathcal{D}(T)$. Obviously, $\mathcal{D}(T)$ is closed under composition and contains $\mathcal{E}$ for T containing EA.

We shall rely on the following two easy results from [3]. The first proposition essentially follows from Herbrand's theorem, and the second one follows from the first and the fact that $\mathbf{E}(f) = \mathbf{C}(\bar{f})$.

Proposition 6.1. *Let π be a true Π_2 sentence of the form $\forall x \exists ! y \phi(x, y)$ with ϕ quantifier free. Let $f : \mathbf{N} \to \mathbf{N}$ be a function such that $\mathbf{N} \models \forall x \phi(x, f(x))$. Then $\mathcal{D}(EA + \pi) = \mathbf{C}(f)$.*

Proposition 6.2. *If a sound theory T containing EA is closed under Σ_1-CR, then $\mathcal{D}(T)$ is elementarily closed.*

By this proposition, the construction of the required Π_2 sentence will be reduced to the following lemma.

Lemma 6.1. *There exists a function f with an elementary graph such that the class $\mathbf{C}(f)$ is not elementarily closed. More specifically, the function $\max_{i\le x} f(i)$ (as well as $\bar{f}(x)$) does not belong to $\mathbf{C}(f)$.*

Proof. To see that our construction of f is natural, it would be helpful for the reader to have in mind the following interpretation of the classes $\mathbf{C}(f)$ and $\mathbf{E}(f)$. $\mathbf{E}(f)$ is the class of all functions computable in elementary time on Turing machines with a function oracle f. $\mathbf{C}(f)$ is the class of functions computable in elementary time, but using only bounded number of oracle queries.[5]

It is intuitively clear that, for a very irregular function f, there is no better way to compute $\max_{i\le x} f(i)$ than to ask the oracle all $x+1$ initial values of f. This means that any bounded number of queries will not be enough.

Now we proceed to the formal construction; its idea was suggested to us by A. Razborov. Every unary function of $\mathbf{C}(f)$ is represented by a term composed of f and elementary functions. We fix some natural elementary enumeration of such terms, which we may also assume to possess the additional property that the n-th term t_n in the enumeration has at most n occurrences of the symbol f.

The function f will be defined diagonally by stages. At n-th stage we shall guarantee that $\max_{i\le x} f(i) \neq t_n(x)$ for some x. The construction of f will be recursive and, in fact, can be viewed as a process of computation of f. (To evaluate $f(x)$ one must proceed with the construction until $f(x)$ is defined, and then output the value of f that is found.) So, we shall really describe a certain Turing machine rather than just a function. We may assume without loss of generality that the Turing machine computing f has an internal clock, that is, a reserved tape that always contains the current time (= the number of steps) written in unary. Our construction will have the property that, whenever $f(x)$ is being defined, it is defined to be equal to the current value of the clock. This guarantees that the graph of f is elementary (because the size of the full protocol of the computation of f is polynomial in the number of steps).

Let us now inductively assume that f is defined on a certain finite subset $D_n \subseteq \mathbf{N}$ by the beginning of Stage n of the construction, and let us describe Stage n. (D_0 is empty.)

Pick the least number x such that there are at least $n+1$ different elements of $\mathbf{N} \setminus D_n$ smaller than x. Start computing the value $t_n(x)$ in accordance with the following rule: If the subroutine for t_n asks for a value of $f(a)$ which is not yet defined, at that moment set the value of $f(a)$ to be equal to the current time, add a to D_n, and proceed further with the computation of $t_n(x)$. In this way a certain value y of $t_n(x)$ is eventually computed. Then, if necessary, run idle for a while, until the current time gets strictly "later" than y.

Now we notice that, since t_n used no more than n oracle queries, by the pigeonhole principle there must exist an element $a < x$ where f is not yet defined. We pick the least such a and define $f(a)$ to be equal to the current time. This will guarantee that

$$\max_{i\le x} f(i) \geq f(a) > y = t_n(x).$$

End of Stage n of the construction.

[5] As we deal here with function oracles, the time complexity measure should be treated with some care: there is no canonical definition of elementary time (as well as polytime) computable functional. For a discussion see Cook [6].

The following properties of our construction are easy to verify:

- f is eventually everywhere defined;
- f is a recursive function with an elementary graph;
- $\forall n \exists x \ \max_{i \le x} f(i) \neq t_n(x)$.

This completes the proof of Lemma 6.1, q.e.d.

Remark 6.1. The function constructed above is not only recursive, but can be easily seen to belong to the class $\mathcal{E}_4$ of the Grzegorczyk hierarchy (because the universal function for $\mathcal{E}$ does).

Clearly, the proof of Lemma 6.1 relativizes, so we obtain

Lemma 6.2. *Let κ be a recursive oracle function. Then there is a recursive function f with an elementary graph such that $\mathbf{C}^{\kappa}(f)$ is not elementarily closed.*

Proof. We only notice that, if the function κ is recursive, then the universal function for the class $\mathcal{E}^{\kappa}$ is recursive as well. Therefore, our construction of f remains recursive and f has an elementary graph, q.e.d.

Theorem 7. *For any sound finite Π_2 axiomatized extension T of EA there is a true Π_2 sentence π such that $T + \pi$ is not closed under Σ_1-CR.*

Proof. Let T be axiomatized over EA by a Π_2 sentence $\forall x \exists y \phi(x, y)$ with ϕ quantifier free. W.l.o.g. we may assume that T is closed under Σ_1-CR and that ϕ is functional, that is, EA proves

$$\phi(x, y_1) \wedge \phi(x, y_2) \to y_1 = y_2.$$

(Otherwise take $\phi'(x, y) := \phi(x, y) \wedge \forall z < y \neg \phi(x, z)$ instead of ϕ.) Since T is sound, there is a recursive function κ such that $\mathbf{N} \vDash \forall x \phi(x, \kappa(x))$, so by Propositions 6.1 and 6.2

$$\mathcal{D}(T) = \mathbf{C}(\kappa) = \mathbf{E}(\kappa).$$

Now let f be as in Lemma 6.2, and let $\psi(x, y)$ represent the (elementary) graph of f. Again, we may assume ψ functional. We define:

$$\pi := \forall x \exists y \psi(x, y).$$

By Proposition 6.1

$$\mathcal{D}(T + \pi) = \mathbf{C}^{\kappa}(f).$$

If $T + \pi$ were closed under Σ_1-CR, $\mathcal{D}(T + \pi)$ would be elementarily closed, by Proposition 6.2. This contradicts the definition of f, q.e.d.

Corollary 6.1. *$B\Sigma_1$ is not derivable from the set of all true Π_2 sentences.*

Proof. Since $B\Sigma_1$ is finitely axiomatizable over EA, if it were derivable, there would exist a finite Π_2 axiomatized extension T of EA containing $B\Sigma_1$. Then every extension of T would be closed under Σ_1-CR, contradicting Theorem 7, q.e.d.

Gaifman and Dimitracopoulos [8] prove a theorem stating, in particular, that if a function f has an elementary definable graph and a theory T extending EA proves the totality of f together with a relevant instance of Σ_1 collection schema:

$$\forall x \exists y \, f(x) = y \to \forall x \exists y \forall u < x \exists v < y \, f(u) = v,$$

then the theory T contains the induction schema for all $\Delta_0(f)$ formulas. They ask whether the latter condition is necessary, in particular, if one only requires the closure of T under the induction rule for $\Delta_0(f)$ formulas.

We answer both variants of their question negatively.

Corollary 6.2. *There is a function f with an elementary graph such that $EA + \forall x \exists y f(x) = y$ does not contain $\Delta_0(f)$ induction (either in the form of schema or a rule).*

Proof. Take f as in Lemma 6.1. Notice that $\Delta_0(f)$ induction over EA easily proves the totality of the function $\bar{f}$ (we have already used this fact in the development of EA^f, see also [3], Proposition 8.1). Hence, if $\Delta_0(f)$ induction were available in T, the class $\mathcal{D}(T)$ would be elementarily closed, contradicting the choice of f. Obviously, $\Delta_0(f)$ induction schema is equivalent to $\Delta_0(f)$ induction rule over EA — this answers the second part of the question by Gaifman and Dimitracopoulos, q.e.d.

In order to generalize Theorem 7 to Σ_n-CR for $n > 1$ we must relativize Lemma 6.2 in full generality.

Lemma 6.3. *Let κ be a function oracle. Then there is a recursive in κ function f such that the graph of f is elementary in κ and $\mathbf{C}^\kappa(f)$ does not contain $\max_{i \le x} f(i)$.*

Proof. That the construction of the function f is recursive in κ is clear. We only have to be a little more careful in order to see that the graph of f is elementary in κ. We use the well-known result, essentially due to Kleene, stating that for a Turing machine M with a function oracle κ the predicate $P_M(x, y, z)$ expressing that z is a protocol of a computation of M of length y on input x can be expressed in the form $Q(\bar{\kappa}(z), x, y, z)$, where Q is a Kalmar elementary predicate (see, e.g., [22], §7.2). By the construction of f this means that the graph of f, being elementary in P_M for the concrete M, is elementary in κ, q.e.d.

Obviously, the previous lemma also holds for any finite number of function oracles.

Theorem 8. *For any sound finite Π_{n+2} axiomatized extension T of $I\Sigma_n$ there is a true Π_{n+2} sentence π such that $T + \pi$ is not closed under Σ_{n+1}-CR.*

Proof (sketch). Rather than relativizing Propositions 6.1 and 6.2, we give a simpler direct argument. First of all, w.l.o.g. we may assume that T is closed under Σ_{n+1}-CR. Then, exactly as in the proof of Theorem 4, we extend the language of T by finitely many monotonic Skolem functions $\kappa_1, \ldots, \kappa_n$, so that T attains a purely unversal axiomatization in the extended language. (Here we rely upon the fact that $I\Sigma_n$ is finitely axiomatizable.) Using the fact that $\Delta_0(\Sigma_n)$ formulas are provably Δ_{n+1} in $I\Sigma_n$ (cf [10], p.68), it is not difficult to see that graphs of all these functions, as well as the graphs of $\bar{\kappa}_1, \ldots, \bar{\kappa}_n$, are Δ_{n+1} in $I\Sigma_n$. Therefore, so are (the interpretations of) arbitrary $\Delta_0(\bar{\kappa}_1, \ldots, \bar{\kappa}_n)$ formulas: such formulas are equivalent to quantifier free ones, and the latter are Δ_1 in the graphs of $\bar{\kappa}_1, \ldots, \bar{\kappa}_n$.

Now we construct f as in Lemma 6.3, let $\psi_0(x, y)$ quantifier free (in the language with $\bar{\kappa}_1, \ldots, \bar{\kappa}_n$) represent the graph of f, and define $\pi_0 := \forall x \exists y \psi_0(x, y)$. The required Π_{n+2} sentence π is defined as the interpretation of π_0 in the original language of EA, that is, $\pi := \forall x \exists y \psi(x, y)$, where $\psi(x, y)$ is the natural Δ_{n+1} interpretation of $\psi_0(x, y)$. Extending the language of T by yet another function symbol for f, together with the axiom $\forall x \psi_0(x, f(x))$, we obatin a theory which is equivalent to $T + \pi$ and has a purely universal axiomatization.

Applying Σ_{n+1}-CR to π yields a sentence $\forall x \exists y \forall u \le x \exists v \le y\, \psi(x, y)$ which is equivalent to $\forall x \exists y \forall u \le x \exists v \le y\, \psi_0(x, y)$ in T and implies that the function $\max_{i \le x} f(i)$ is total (because T contains $EA^{\kappa_1, \ldots, \kappa_n}$). Since the graph of $\max_{i \le x} f(i)$ is Δ_0 in the graph of f, the graph of $\max_{i \le x} f(i)$ is $\Delta_0(\bar{\kappa}_1, \ldots, \bar{\kappa}_n)$ definable, and therefore quantifier free definable in T. By Herbrand's theorem, since the class $\mathbf{C}^{\kappa_1, \ldots, \kappa_n}(f)$ is provably closed under definitions by cases, this implies that there is a

$\mathbf{C}^{\kappa_1,\ldots,\kappa_n}(f)$ term witnessing $\max_{i\leq x} f(i)$. But this is impossible by the construction of f, q.e.d.

Corollary 6.3. *$B\Sigma_{n+1}$ is not derivable from the set of all true Π_{n+2} sentences.*

References

[1] Z. Adamowicz. A recursion-theoretic characterization of instances of $B\Sigma_n$ provable in $\Pi_{n+1}(\mathrm{N})$. *Fundamenta mathematicae*, 129:231–236, 1988.

[2] L.D. Beklemishev. Notes on local reflection principles. Logic Group Preprint Series 133, University of Utrecht, 1995.

[3] L.D. Beklemishev. Induction rules, reflection principles, and provably recursive functions. *Annals of Pure and Applied Logic*, 85(3):193–242, 1997.

[4] S. Buss. The witness function method and fragments of Peano arithmetic. In *Proceedings of the ninth International Congress on Logic, Methodology and Philosophy of Science; Uppsala, Sweden, August 7–14, 1991*, pages 29–68. Elsevier, North-Holland, Amsterdam, 1994.

[5] P. Clote, P. Hájek, and J. Paris. On some formalized conservation results in arithmetic. *Archive for Mathematical Logic*, 30:201–218, 1990.

[6] S.A. Cook. Computational complexity of higher type functions. In *Proceedings of the international congress of math., Kyoto, Japan, 1990*, pages 55–69. Math. Society of Japan, 1991.

[7] S. Feferman. Arithmetization of metamathematics in a general setting. *Fundamenta Mathematicae*, 49:35–92, 1960.

[8] H. Gaifman and C. Dimitracopoulos. Fragments of Peano's arithmetic and the MDRP theorem. In *Logic and algorithmic (Zurich, 1980), (Monograph. Enseign. Math., 30)*, pages 187–206. Genève, University of Genève, 1982.

[9] P. Hájek. Interpretability and fragments of arithmetic. In P. Clote and J. Krajíček, editors, *Arithmetic, Proof Theory, and Computational Complexity*, pages 185–196. Oxford University Press, Oxford, 1992.

[10] P. Hájek and P. Pudlák. *Metamathematics of First Order Arithmetic*. Springer-Verlag, Berlin, Heidelberg, New-York, 1993.

[11] R. Kaye. Using Herbrand-type theorems to separate strong fragments of arithmetic. In P. Clote and J. Krajíček, editors, *Arithmetic, Proof Theory, and Computational Complexity*, pages 238–246. Oxford University Press, Oxford, 1992.

[12] R. Kaye, J. Paris, and C. Dimitracopoulos. On parameter free induction schemas. *Journal of Symbolic Logic*, 53(4):1082–1097, 1988.

[13] G. Kreisel and A. Lévy. Reflection principles and their use for establishing the complexity of axiomatic systems. *Zeitschrift f. math. Logik und Grundlagen d. Math.*, 14:97–142, 1968.

[14] D. Leivant. The optimality of induction as an axiomatization of arithmetic. *Journal of Symbolic Logic*, 48:182–184, 1983.

[15] H. Ono. Reflection principles in fragments of Peano Arithmetic. *Zeitschrift f. math. Logik und Grundlagen d. Math.*, 33(4):317–333, 1987.

[16] J. Paris. A hierarchy of cuts in models of arithmetic. In L. Pacholsky et al., editor, *Model theory of algebra and arithmetic. Proceedings, Karapacz, Poland, 1979. Lecture Notes in Mathematics, v. 834*, pages 312–337. Springer-Verlag, Berlin, 1980.

[17] J. Paris and L. Kirby. Σ_n-collection schema in arithmetic. In *Logic Colloquium '77*, pages 199–209. North Holland, Amsterdam, 1978.

[18] C. Parsons. Hierarchies of primitive recursive functions. *Zeitschrift f. math. Logik und Grundlagen d. Math.*, 14(4):357–376, 1968.

[19] C. Parsons. On a number-theoretic choice schema and its relation to induction. In Kino, Myhill, and Vessley, editors, *Intuitionism and Proof Theory*, pages 459–473. North Holland, Amsterdam, 1970.

[20] C. Parsons. On n-quantifier induction. *Journal of Symbolic Logic*, 37(3):466–482, 1972.

[21] H. Schwichtenberg. Some applications of cut-elimination. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 867–896. North Holland, Amsterdam, 1977.

[22] J.R. Shoenfield. *Mathematical Logic.* Addison-Wesley Publishing Company, 1967.

[23] W. Sieg. Fragments of arithmetic. *Annals of Pure and Applied Logic*, 28:33–71, 1985.

[24] A. Visser. The formalization of interpretability. *Studia Logica*, 50(1):81–106, 1991.

[25] A. Visser. Unprovability of small inconsistency. *Archive for Math. Logic*, 32:275–298, 1993.