# Another pathological well-ordering

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**Abstract.** For an any constructive ordinal $\lambda$ we exhibit a primitive recursive well-ordering of order type $\lambda$ such that the corresponding (full second order) transfinite induction principle is equiconsistent with $\mathsf{PRA}$. In particular, transfinite induction for this well-ordering does not prove the consistency of Peano arithmetic.

For a primitive recursive ordering relation $\prec$ on natural numbers, $\mathsf{WF}(\prec, X)$ will denote the formula $\forall x(\forall z \prec x X(z) \to X(x)) \to \forall x X(x)$, with a second order parameter $X$. $\mathsf{TI}(\prec, \Gamma)$ is the schema obtained from $\mathsf{WF}(\prec, X)$ by substituting for $X$ arbitrary formulas from a class $\Gamma$. $\mathsf{TI}(\prec)$ is short for $\mathsf{TI}(\prec, \Pi^1_\infty)$.

The *proof-theoretic ($\Pi^1_1$) ordinal* of a theory $T$ is usually defined as the supremum of order types of primitive recursive well-orders $\prec$ such that $T \vdash \mathsf{WF}(\prec, X)$. Of course, this definition only makes sense for theories whose language at least admits second order parameters. For first order theories the existing definitions of proof-theoretic ordinals heavily rely on the notions of "natural" ordinal notation systems and therefore lack comparable generality.

Attempts to define general notions of proof-theoretic ordinals relevant for arithmetical complexity classes such as $\Pi^0_2$ or $\Pi^0_1$ usually fail. This issue was treated by G. Kreisel [1, 2], who constructed a number of "pathological" counterexamples in order to demonstrate such failures. The most popular one from [2] exhibits, for any true $\Pi^0_1$ sentence $\pi$, a primitive recursive well-ordering $\prec$ of order type $\omega$ such that $\mathsf{PRA} + \mathsf{TI}(\prec, \Pi^0_0) \vdash \pi$ ($\Pi^0_0$ denotes the class of primitive recursive formulas). In fact, if $\pi$ has the form $\forall x \pi_0(x)$ with $\pi_0(x)$ primitive recursive, then one can define (cf. [3] for an elaboration)

$$x \prec y \iff (x < y \wedge \forall z < x\, \pi_0(z)) \vee (y < x \wedge \neg \forall z < x\, \pi_0(z)).$$

This example immediately shows the inadequacy of the naive definition of a proof-theoretic ordinal of a theory $T$ as the order type of the shortest primitive recursive well-ordering such that the corresponding transfinite induction (over $\mathsf{PRA}$) proves the consistency of $T$ (take the consistency assertion $\mathsf{Con}(T)$ for $\pi$).

In this note we give another pathological example, which is, in a sense, complementary to Kreisel's. Whereas Kreisel's well-ordering is pathologically strong in that by the short transfinite induction one can prove strong sentences, our well-orderings are long, but weak. More precisely, for an arbitrarily large constructive

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ordinal $\lambda$ we exhibit a primitive recursive well-ordering of order type $\lambda$ such that the corresponding (full second order) transfinite induction principle is equiconsistent with $\mathsf{PRA}$. In particular, transfinite induction for this well-ordering does not prove the consistency of Peano arithmetic.

Let $\mathsf{PRA}^2$ denote any reasonable version of the second-order primitive recursive arithmetic (one can take $\mathsf{RCA}_0$ for $\mathsf{PRA}^2$ as well). It is essential that $\mathsf{PRA}^2$ is a provably conservative extension of the first-order primitive recursive arithmetic $\mathsf{PRA}$, so that we have

$$\mathsf{PRA} \vdash \mathsf{Con}(\mathsf{PRA}) \to \mathsf{Con}(\mathsf{PRA}^2). \tag{1}$$

**Theorem 1.** *For every recursive ordinal $\lambda$ there is a primitive recursive well-ordering $\prec$ of order type $\lambda$ such that*

$$\mathsf{PRA} + \mathsf{Con}(\mathsf{PRA}) \vdash \mathsf{Con}(\mathsf{PRA}^2 + \mathsf{TI}(\prec)).$$

*Proof.* Let $R(x,y)$ be a provably linear primitive recursive ordering relation of a given order type $\lambda$. Denote by $R \upharpoonright n$ the relation $R(x,y) \wedge \max(x,y) \leq n$. Obviously, for each $n$, $R \upharpoonright n$ is a finite linear ordering, therefore we obtain the following lemma.

**Lemma 1.** *For every $n$, $\mathsf{PRA}^2 \vdash \mathsf{TI}(R \upharpoonright \bar{n})$.*

The previous lemma is formalizable in $\mathsf{PRA}$ (the required $\mathsf{PRA}^2$-proof of an instance of the form $\mathsf{WF}(R \upharpoonright \bar{n}, A)$ is constructed primitively recursively from $n$ and the Gödel number $\ulcorner A \urcorner$). Let $(\mathsf{TI} \upharpoonright n)(R)$ denote the formula $\bigwedge_{\ulcorner A \urcorner \leq n} \mathsf{WF}(R, A)$.

**Lemma 2.** $\mathsf{PRA} \vdash \forall z \, \Box_{\mathsf{PRA}^2}(\mathsf{TI} \upharpoonright \dot{z})(R \upharpoonright \dot{z})$, *where $\Box_{\mathsf{PRA}^2}$ denotes a canonical provability predicate for $\mathsf{PRA}^2$.*

Define the required well-ordering $E(x,y)$ by the arithmetical fixed-point lemma in such a way that

$$\mathsf{PRA} \vdash E(x,y) \leftrightarrow R(x,y) \wedge \forall z < \max(x,y) \, \neg C(z, \ulcorner E \urcorner),$$

where $C(z,u)$ denotes a canonical primitive recursive formula expressing the predicate *"$u$ is the Gödel number of a formula $E(x,y)$, and $z$ codes a proof of a contradiction from the axioms of $\mathsf{PRA}^2 + \mathsf{TI}(E)$"*. Let $C_1(z,u)$ abbreviate $C(z,u) \wedge \forall w < z \neg C(w,u)$.

Obviously, a relation defined by $E$ as above is a primitive recursive linear order. We show that

$$\mathsf{PRA} + \mathsf{Con}(\mathsf{PRA}) \vdash \mathsf{Con}(\mathsf{PRA}^2 + \mathsf{TI}(E)).$$

By the definition of $E$

$$\mathsf{PRA} \vdash C_1(z, \ulcorner E \urcorner) \to (E(x,y) \leftrightarrow (R \upharpoonright z)(x,y)),$$

and using Löb's derivability conditions

$$\mathsf{PRA} \vdash \Box_{\mathsf{PRA}^2} C_1(\dot{z}, \ulcorner E \urcorner) \to \Box_{\mathsf{PRA}^2} \forall x,y \, (E(x,y) \leftrightarrow (R \upharpoonright \dot{z})(x,y)),$$

therefore

$$\begin{aligned}\mathsf{PRA} \vdash C_1(z, \ulcorner E \urcorner) &\to \Box_{\mathsf{PRA}^2} C_1(\dot{z}, \ulcorner E \urcorner)\\ &\qquad \text{since } C \text{ is primitive recursive}\\ &\to \Box_{\mathsf{PRA}^2} \forall x, y\, (E(x,y) \leftrightarrow (R \restriction \dot{z})(x,y))\\ &\to \Box_{\mathsf{PRA}^2} (\mathsf{TI} \restriction \dot{z})(E)\\ &\qquad \text{by Lemma 2.}\end{aligned}$$

Therefore by (1) we obtain

$$\begin{aligned}\mathsf{PRA} + \mathsf{Con}(\mathsf{PRA}) \vdash C_1(z, \ulcorner E \urcorner) &\to \Box_{\mathsf{PRA}^2} (\mathsf{TI} \restriction \dot{z})(E)\\ &\to \mathsf{Con}(\mathsf{PRA}^2 + (\mathsf{TI} \restriction \dot{z})(E)). \qquad (2)\end{aligned}$$

On the other hand, notice that by the monotonicity of the standard Gödel numbering, a proof of a contradiction from $\mathsf{PRA}^2 + \mathsf{TI}(E)$ coded by $z$ is simultaneously a proof of a contradiction from $\mathsf{PRA}^2 + (\mathsf{TI} \restriction \bar{z})(E)$, whence

$$\mathsf{PRA} \vdash C(z, \ulcorner E \urcorner) \to \neg \mathsf{Con}(\mathsf{PRA}^2 + (\mathsf{TI} \restriction \dot{z})(E)).$$

Therefore, by (2)

$$\mathsf{PRA} + \mathsf{Con}(\mathsf{PRA}) \vdash C_1(z, \ulcorner E \urcorner) \to \bot,$$

which means (by the primitive recursive least number principle) that

$$\begin{aligned}\mathsf{PRA} + \mathsf{Con}(\mathsf{PRA}) &\vdash \forall z \neg C(z, \ulcorner E \urcorner)\\ &\vdash \mathsf{Con}(\mathsf{PRA}^2 + \mathsf{TI}(E)).\end{aligned}$$

Since $\mathsf{Con}(\mathsf{PRA}^2 + \mathsf{TI}(E))$ is provable in $\mathsf{PRA} + \mathsf{Con}(\mathsf{PRA})$ and thereby true in the standard model, the predicate $\forall z < \max(x,y)\ \neg C(z, \ulcorner E \urcorner)$ is true for all $x$ and $y$. Hence $E(x,y)$ defines the same relation as $R(x,y)$, that is, $E$ has order type $\lambda$ in the standard model.

*Remark 1.* For the well-ordering $\prec$ satisfying the statement of Theorem 1 we have: $\Pi^0_1$-consequences of $\mathsf{PRA}^2 + \mathsf{TI}(\prec)$ are provable in $\mathsf{PA}$. In particular, $\mathsf{PRA}^2 + \mathsf{TI}(\prec) \nvdash \mathsf{Con}(\mathsf{PA})$.

*Proof.* Let $S$ denote the theory $\mathsf{PRA}^2 + \mathsf{TI}(\prec)$ and assume $S \vdash \pi$, where $\pi \in \Pi^0_1$. We obviously have

$$\mathsf{PRA} \vdash \mathsf{Con}(S) \to \mathsf{Con}(\mathsf{PRA} + \pi).$$

Hence

$$\begin{aligned}\mathsf{PA} &\vdash \mathsf{Con}(S)\\ &\vdash \mathsf{Con}(\mathsf{PRA} + \pi)\\ &\vdash \pi,\end{aligned}$$

by provable $\Sigma_1$-completeness.

*Remark 2.* Notice that the construction of $E$ essentially works for any, not necessarily well-founded, linear ordering $R$. The relation $E$ as defined in the proof of Lemma 2 will then be isomorphic to $R$. For example, if one starts with the ordering $R(x, y) \equiv y < x$, then one obtains a primitive recursive definition of the reverse ordering of natural numbers such that the corresponding (false) transfinite induction schema is consistent with $\mathsf{PRA}^2$.

*Remark 3.* Other types of pathologically weak well-orderings are known. In particular, for any constructive ordinal $\lambda$ there is a primitive recursive well-ordering $\prec$ such that $\mathsf{PA} \vdash \mathsf{TI}(\prec, \Pi^0_\infty)$. A similar result, however, fails for the intuitionistic arithmetic $\mathsf{HA}$ (cf. [4]).

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